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Algebras intertwining compact operators

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The main result of this note is that if $\mathcal{A}$ is a norm-closed algebra of (bounded) operators on a (complex) Banach space $\mathfrak{X}$, if K is a nonzero compact operator on $\mathfrak{X}$, and if $\mathcal{A}K=K\mathcal{A}$, then $\mathcal{A}$ has a non-trivial (closed) invariant subspace. This is an extension of Lomonosov's theorem that every compact operator has a non-trivial hyperinvariant subspace. For injective compact operators we shall prove stronger results.

Lomonosov's result [4] quoted above implies that if $AK=KA$ for every A in the algebra $\mathcal{A}$, then $\mathcal{A}$ has invariant subspaces. It is very easy to construct uniformly closed algebras $\mathcal{A}$ with $\mathcal{A}K=K\mathcal{A}$ for some compact operator K, where $\mathcal{A}$ has members not commuting with K. (See Remark (i) below.) In Section 2 we shall mention other contrasts with the case of Lomonosov's result.

In what follows $\mathfrak{X}$ and $\mathfrak{Y}$ will always denote Banach spaces, $\mathcal{B}(\mathfrak{X})$ the algebra of all operators on $\mathfrak{X}$, and $\mathcal{A}$ a subalgebra of $\mathcal{B}(\mathfrak{X})$; subalgebras are not assumed to have identities.

1. Main Results. We start with the following lemma which may be of some independent interest. (See also Remark (iv) in Section 2.)

Lemma 1. *Let K be a compact operator on $\mathfrak{X}$ and let S be any operator on $\mathfrak{Y}$. Let T be a bounded linear transformation of $\mathfrak{Y}$ into $\mathfrak{X}$ such that $KTS=T$. Then T has finite rank.*

Proof. Let C be the circle of radius $r>0$ centered at the origin in the complex plane; assume that (i) r is sufficiently small so that $1-\lambda S$ is invertible for λ inside C; and (ii) C does not intersect the spectrum of K. Let P be the Riesz projection

$$P=\frac{1}{2\pi i}\int_C (K-\lambda)^{-1}\,d\lambda.$$

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(See, e.g., [7, p. 31].) We show that $T\mathfrak{Y}$ is contained in the finite-dimensional space $(1-P)\mathfrak{X}$. Let y be any vector in $\mathfrak{Y}$ and let $x=Ty$. Then, for λ inside C,

$$Px = (K-\lambda)PTS(1-\lambda S)^{-1}y = (K_P-\lambda)PTS(1-\lambda S)^{-1}y,$$

where $K_P=PK|P\mathfrak{X}$.

Since $\sigma(K_P)$ lies inside C, it follows that $(K_P-\lambda)^{-1}Px$ has an analytic extension to the entire plane and thus $Px=0$. Hence $x\in(1-P)\mathfrak{X}$.

Lemma 2. *Let A and B be bounded linear transformations from $\mathfrak{X}$ into $\mathfrak{Y}$. If $A\mathfrak{X}\subseteq B\mathfrak{X}$ and if B is injective, then there exists an operator S on $\mathfrak{X}$ such that $A=BS$.*

Proof. The proof given in [2] for a stronger theorem on Hilbert spaces works for this lemma also: just observe that $B^{-1}A$ is a closed operator.

Theorem 3. *Let $\mathscr{A}$ be a norm-closed subalgebra of $\mathscr{B}(\mathfrak{X})$ and let K be an injective, non-quasinilpotent compact operator on $\mathfrak{X}$ such that $\mathscr{A}K\subseteq K\mathscr{A}$. Then $\mathscr{A}$ has a nonzero finite-dimensional invariant subspace.*

Proof. Assume, with no loss of generality, that there exists a nonzero x_0 in $\mathfrak{X}$ with $Kx_0=x_0$. Then $\mathscr{A}x_0\subseteq K(\mathscr{A}x_0)$. The linear manifold $\mathscr{A}x_0$ of $\mathfrak{X}$ is the range of the linear transformation T of $\mathscr{A}$ (considered as a Banach space) into $\mathfrak{X}$ defined by

$$T(A) = Ax_0.$$

Clearly T is bounded, and $T(\mathscr{A})\subseteq KT(\mathscr{A})$. Let $\mathscr{A}_0$ be the null space of T and let $\mathfrak{Y}$ be the quotient space $\mathscr{A}/\mathscr{A}_0$. Then $\hat{T}\mathfrak{Y}\subseteq K\hat{T}\mathfrak{Y}$, where $\hat{T}$ is the induced injective linear transformation from $\mathfrak{Y}$ into $\mathfrak{X}$. Since K is injective, so is $K\hat{T}$. Thus there exists, by Lemma 2, an operator S on $\mathfrak{Y}$ with $\hat{T}=K\hat{T}S$. It follows from Lemma 1 that $\hat{T}$ has finite rank. But

$$\hat{T}(\mathfrak{Y}) = T(\mathscr{A}) = \mathscr{A}x_0,$$

and thus $\mathscr{A}x_0$ is a finite-dimensional invariant subspace for $\mathscr{A}$. (If $\mathscr{A}x_0=0$, then x_0 generates a 1-dimensional invariant subspace.)

The following lemma is a special case of a result of Foiaş [3].

Lemma 4. *Let $\mathscr{A}K\subseteq K\mathscr{A}$, where $\mathscr{A}$ is norm-closed and K is injective (and not necessarily compact). Then the map φ on $\mathscr{A}$ defined by*

$$AK = K\varphi(A).$$

is a continuous algebra homomorphism.

Proof. The map φ is clearly a homomorphism. To prove that φ is continuous, it suffices to show that it is a closed map: if $A=\lim A_n$ and $B=\lim \varphi(A_n)$, then

$$AK=\lim A_n K=\lim K\varphi(A_n)=KB,$$

and thus $B=\varphi(A)$.

Theorem 5. *Let $\mathcal{A}K\subseteq K\mathcal{A}$, where $\mathcal{A}$ is norm-closed and K is injective, compact, and quasinilpotent. Then $\mathcal{A}$ has a non-trivial invariant subspace.*

Proof. The main idea in the proof is that used in the simple, elegant proof given by H. M. Hilden for Lomonosov's result quoted above (cf. [7], p. 165).

We start, as in Lomonosov's proof, by assuming with no loss of generality that $\|K\|=1$, and that $\mathcal{A}$ is transitive if possible. Fix x_0 in $\mathfrak{X}$ such that $\|Kx_0\|>1$ (and thus also $\|x_0\|>1$), and let $\mathfrak{S}$ be the open ball of radius 1 centered at x_0. It follows from the transitivity of $\mathcal{A}$ that the open sets $A^{-1}(\mathfrak{S}), A\in\mathcal{A}$, cover $\mathfrak{X}\setminus\{0\}$, and thus they also cover the compact set $\overline{K\mathfrak{S}}$. Hence there is a finite subset $\mathcal{F}$ of $\mathcal{A}$ such that

$$\overline{K\mathfrak{S}}\subseteq \bigcup_{A\in\mathcal{F}} A^{-1}(\mathfrak{S}).$$

Let $r=\max\{\|A\|:A\in\mathcal{F}\}$. Given any positive integer n, one can inductively obtain $A_1,\ldots,A_n$ in $\mathcal{F}$ such that

$$A_nKA_{n-1}K\ldots A_2KA_1Kx_0\in\mathfrak{S}.$$

But, again by induction,

$$A_nKA_{n-1}K\ldots A_2KA_1K=K^n\varphi\big(\ldots(\varphi(\varphi(\varphi(A_n)A_{n-1})A_{n-2})\ldots A_1\big).$$

Therefore

$$\|A_nKA_{n-1}K\ldots A_1K\|\leqq\|K^n\|\cdot\|\varphi\|^n\cdot r^n=\big\|(r\|\varphi\|\cdot K)^n\big\|.$$

Since $r\|\varphi\|K$ is quasinilpotent, the vector $A_nK\ldots A_1Kx_0$ in $\mathfrak{S}$ would be arbitrarily small for sufficiently large n. This contradicts the fact that $\|x_0\|>1$. Thus $\mathcal{A}$ cannot be transitive.

Theorem 6. *Let $\mathcal{A}$ be a norm-closed subalgebra of $\mathcal{B}(\mathfrak{X})$ which intertwines a nonzero compact operator. Then $\mathcal{A}$ has a non-trivial invariant subspace.*

Proof. Let $\mathcal{A}K=K\mathcal{A}$, where K is compact and nonzero. If K is injective, the assertion follows from Theorems 3 and 5; otherwise the null space of K is a non-trivial subspace invariant under $\mathcal{A}$.

2. Remarks.

(i) Let $\mathfrak{H}_1$ be a finite-dimensional subspace of the infinite-dimensional Hilbert space $\mathfrak{H}$. Let K_1 be an invertible, non-scalar operator on $\mathfrak{H}_1$, and let K_2 be an injective compact operator in $\mathfrak{H}\ominus\mathfrak{H}_1$. Let $K=K_1\oplus K_2$, and let $\mathcal{A}=\mathcal{B}(\mathfrak{H}_1)\oplus\mathcal{A}_2$, where $\mathcal{A}_2$ is the commutant of K_2. Then $\mathcal{A}K=K\mathcal{A}$, but not every member of $\mathcal{A}$ commutes with K. (In this example $\mathcal{A}$ is also weakly closed.)

A less trivial example can be constructed as follows. Let K be any nonzero compact operator on $\mathfrak{H}$ and let $\mathcal{A}$ be the weakly closed algebra of all operators of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ on $\mathfrak{H}\oplus\mathfrak{H}$, where A and C commute with K but B is arbitrary. Then $\mathcal{A}$ intertwines the operator $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ but is not contained in the commutant of *any* nonscalar operator.

(ii) There are examples of $\mathcal{A}$ and K as in Theorem 3 with $\mathcal{A}K$ *properly* contained in $K\mathcal{A}$. Let $\mathcal{A}=\mathcal{B}^*$, where $\mathcal{B}$ is the algebra of all analytic Toeplitz operators on a Hilbert space $\mathfrak{H}$, and represent $\mathcal{A}$ as an algebra of uppertriangular matrices. Let K be the compact operator represented by a diagonal matrix $\mathrm{Diag}\{\lambda^n\}_{n=0}^{\infty}$ with $|\lambda|<1$. Then it can be verified that $\mathcal{A}K\subsetneqq K\mathcal{A}$.

(iii) In contrast with the case of Lomonosov's Theorem, it is essential in our results that $\mathcal{A}$ be closed. Let, for instance, $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathfrak{H}$, and let $\mathcal{A}$ be the algebra of all operators on $\mathfrak{H}$ whosè matrices elative to $\{e_i\}_{i=1}^{\infty}$ have only finitely many nonzero entries. Then $\mathcal{A}$ is clearly (topologically) transitive, but $\mathcal{A}K=K\mathcal{A}$ for any injective operator (compact or not) whose matrix relative to $\{e_i\}_{i=1}^{\infty}$ is diagonal.

(iv) Using properties of decomposable operators [1, pp. 30—31] one can prove another version of Lemma 1 as follows:

Lemma 1′. *Let K be a non-invertible, decomposable operator on $\mathfrak{X}$ and let S be any operator on $\mathfrak{Y}$. Let T be a bounded linear transformation of $\mathfrak{Y}$ into $\mathfrak{X}$ such that $KTS=T$. Then the range of T is not dense in $\mathcal{K}$.*

A corresponding version of Theorem 3 follows.

Theorem 3′. *Let $\mathcal{A}$ be a norm-closed subalgebra of $\mathcal{B}(\mathfrak{X})$ and let K be an injective, non-invertible, decomposable operator with a nonzero eigenvalue, such that $\mathcal{A}K\subseteq K\mathcal{A}$. Then $\mathcal{A}$ has a non-trivial invariant subspace.*

Lemma 1′ and Theorem 3′ remain true if "decomposable" is replaced by "hyponormal" or by "subspectral". (Use [5, Lemma 1] and [6, Proposition 1].)

(v) We conjecture that if a norm-closed algebra $\mathcal{A}$ leaves invariant the range of a compact operator K, then it has a non-trivial invariant subspace. The hypothesis is equivalent to the inclusion $\mathcal{A}K \subseteqq K\mathcal{B}(\mathfrak{X})$.

A weaker version of the conjecture is obtained by assuming $\mathcal{A}$ to be closed in the strong operator topology. Validity of this version would follow from that of the transitive-algebra conjecture [7, p. 138]: every strongly closed transitive algebra of operators on $\mathfrak{X}$ is $\mathcal{B}(\mathfrak{X})$.

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