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On a representation of a C*-algebra in a Lorentz algebra

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§ 1. **Introduction.** Let $\mathscr{K}$ be a Hilbert space with the usual inner product (,) and let J be an hermitian unitary aperator on $\mathscr{K}$. A *Lorentz algebra* on $\{\mathscr{K}, J\}$ is defined as a Banach subalgebra of the full operator algebra $\mathfrak{B}(\mathscr{K})$ on $\mathscr{K}$, invariant under the involution $a \to Ja^*J$ [3]. A non-zero closed subspace $\mathscr{M}$ of $\mathscr{K}$ is said to be J-uniformly positive if there is a constant $\lambda \in (0, 1]$ with $\lambda \|x\|^2 \leqq (Jx, x)$ for all x in $\mathscr{M}$.

In this paper we shall study, for a given C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ and ist derivation δ, a certain representation π_δ (defined in § 3) of $\mathfrak{A}$ on $\{\mathscr{H} \oplus \mathscr{H}, J_0\}$ with $\pi_\delta(a^*) = J_0 \pi_\delta(a)^* J_0$ for all a in $\mathfrak{A}$. In Section 2 we shall show that there is a bijective correspondence between the set $\mathscr{M}(J_0)$ of all maximal $\mathscr{J}$-uniformly positive subspaces of $\mathscr{H} \oplus \mathscr{H}$ and a certain class of operators on $\mathscr{H}$. In Section 3 we shall investigate the relationship between globally $\pi_\delta(\mathfrak{A})$-invariant elements of $\mathscr{M}(J_0)$ and derivations of $\mathfrak{A}$.

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§ 2. **The set $\mathscr{M}(J_0)$.** Let $\mathscr{H}$ be a Hilbert space and J_0 the operator on $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$ defined by $J_0(\xi \oplus \eta) = \eta \oplus \xi$ for all $\xi, n \in \mathscr{H}$. Then $J_0 = J_0^* = J_0^{-1}$. For an operator S on $\mathscr{H}$ we denote by $G(S)$ the graph of S in $\tilde{\mathscr{H}}$, i.e., the set of all $\xi \oplus S\xi \in \tilde{\mathscr{H}}$ with $\xi \in D(S)$, where $D(S)$ denotes the domain of S.

Let J_1 be the hermitian unitary operator on $\tilde{\mathscr{H}}$ defined by $J_1(\xi \oplus \eta) = \xi \oplus (-\eta)$ for all $\xi, \eta \in \mathscr{H}$ and let $\mathscr{M}(J_1)$ be the set of all maximal J_1-uniformly positive subspaces of $\tilde{\mathscr{H}}$. The following lemma is shown in [2].

Lemma 2.1. *G is a bijection from the set of all $S \in \mathfrak{B}(\mathscr{H})$ with $\|S\| < 1$ onto $\mathscr{M}(J_1)$.*

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Theorem 2.2. *G is a bijection from the set of all $T \in \beta(\mathscr{H})$ with*

(1) $$(T+1)^{-1} \in \mathfrak{B}(\mathscr{H}) \quad \text{and} \quad \|(T-1)(T+1)^{-1}\| < 1$$

onto $\mathscr{M}(J_0)$. In this case $T^{-1} \in \mathfrak{B}(\mathscr{H})$.

Proof. We shall show that the following correspondences

$$\begin{array}{ccccccc} \{T: (1)\} & \xrightarrow{(i)} & \{S: \|S\| < 1\} & \xrightarrow{(ii)} & \mathscr{M}(J_1) & \xrightarrow{(iii)} & \mathscr{M}(J_0) \\ T & \mapsto & S & \mapsto & \mathscr{M}_1 & \mapsto & \mathscr{M} \end{array}$$

are bijections given by (i) $S=(T-1)(T+1)^{-1}$, (ii) $G(S)=\mathscr{M}_1$ (by Lemma 2.1), (iii) $\mathscr{M}=u\mathscr{M}_1$; and that $G(T)=\mathscr{M}$, where $u=2^{-1/2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$.

It is clear that (i) and (ii) are bijections. We shall show that (iii) is also a bijection. If a closed subspace $\mathscr{M}_1$ of $\tilde{\mathscr{H}}$ is J_1-uniformly positive, then there exists a constant $\lambda \in (0, 1]$ such that

$$\lambda \|ux\|^2 = \lambda \|x\|^2 \leqq (J_1 x, x) = (J_0 ux, ux)$$

for all $x \in \mathscr{M}_1$. Hence $u\mathscr{M}_1$ is J_0-uniformly positive. Conversely, if $\mathscr{M}$ is a J_0-uniformly positive closed subspace of $\tilde{\mathscr{H}}$, $u^*\mathscr{M}$ is J_1-uniformly positive. As the correspondence preserves the order of set inclusion, if $\mathscr{M}_1$ is maximal, so is $u\mathscr{M}_1$ and vice versa.

We shall show that $G(T)=\mathscr{M}$. Since $(1-S)\mathscr{H}=\mathscr{H}$, it follows that

$$G(T) = G((1+S)(1-S)^{-1}) = \{(1-S)\xi \oplus (1+S)\xi : \xi \in \mathscr{H}\} =$$

$$= u\{\xi \oplus S\xi : \xi \in \mathscr{H}\} = uG(S) = u\mathscr{M}_1 = \mathscr{M}.$$

Finally, since $T=(1+S)(1-S)^{-1}$, it is clear that $T^{-1} \in \mathfrak{B}(\mathscr{H})$. The proof is complete.

§ 3. Representations of C*-algebras in a Lorentz algebra on $\{\tilde{\mathscr{H}}, J_0\}$. Let $\mathfrak{A}$ be a C^*-algebra acting on $\mathscr{H}$ and δ be a $*$-derivation on $\mathfrak{A}$ $(\delta(a^*)=\delta(a)^*$ for all $a \in \mathfrak{A})$. We shall define a mapping π_0 of $\mathfrak{A}$ into $\mathfrak{B}(\tilde{\mathscr{H}})$ by

$$\pi_\delta(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$$

for all a in $\mathfrak{A}$. Then it is easily seen that π_δ is a faithful representation of $\mathfrak{A}$ on $\tilde{\mathscr{H}}$ with $\pi_\delta(a^*)=J_0\pi_\delta(a)^*J_0$ for $a \in \mathfrak{A}$, and $\pi_\delta(\mathfrak{A})$ is a Lorentz algebra on $\{\tilde{\mathscr{H}}, J_0\}$. Therefore $\pi_\delta(\mathfrak{A})$ is *C^*-equivalent* (that is, isomorphic to some C^*-algebra as an involutive Banach algebra).

Lemma 3.1. *Let $\mathscr{K}$ be a Hilbert space with dim $\mathscr{K}=2$ and π the natural representation of $\mathfrak{A}$ onto $\mathfrak{A}\otimes 1_{\mathscr{K}}$.*

(i) *π_δ is similar to π.*

(ii) *π_δ is similar to $\pi_{\delta'}$ for any *-derivation δ'.*

Proof. (i) Since δ is implemented by some $k\in\mathfrak{B}(\mathscr{H})$ [6], we have

$$\pi_\delta(a) = \begin{bmatrix} a & ka-ak \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^{-1}$$

for all $a\in\mathfrak{A}$. Hence π_δ is similar to π.

(ii) Clear from (i).

Theorem 3.2. *Let T be an operator with (1) of Theorem 2.2 and $\mathscr{M}=G(T)$. Then $\mathscr{M}$ is invariant under $\pi_\delta(\mathfrak{A})$ if and only if $\delta(a)=[T^{-1}, a]$ for all a in $\mathfrak{A}$.*

Proof. Since we have

$$\pi_\delta(a)(\xi\oplus T\xi) = (a\xi+\delta(a)T\xi)\oplus aT\xi = (aT^{-1}+\delta(a))T\xi\oplus aT\xi \tag{2}$$

for all $a\in\mathfrak{A}$ and $\xi\in\mathscr{H}$, it follows that the invariance of $\mathscr{M}=G(T)$ under $\pi_\delta(\mathfrak{A})$ is equivalent to the fact that $\delta(a)=[T^{-1}, a]$ for all $a\in\mathfrak{A}$, which completes the proof.

Any operator T with $\mathscr{M}=G(T)$ in Theorem 3.2 (or Theorem 2.2) can not be skew-adjoint. Otherwise $S=(T-1)(T+1)^{-1}$ is unitary, which is impossible by Theorem 2.2.

In our previous paper [3], we have shown that a Lorentz algebra with identity with respect to J is C^*-equivalent if it has a maximal J-uniformly positive invariant subspace. Even in the case of a Lorentz algebra without identity, we remark, this statement holds by [4; Corollary 12] and the same way as a proof of [3; Theorem 3.5]. The converse holds whenever the C^*-equivalent Lorentz algebra with identity is commutative by [5; Theorem 6.1]. As for Lorentz algebras $\pi_\delta(\mathfrak{A})$, we have

Corollary 3.3. *$\pi_\delta(\mathfrak{A})$ always has a maximal J_0-uniformly positive invariant subspace.*

Proof. Since δ is a *-derivation, there exists an invertible skew-adjoint operator k in the double commutant $\mathfrak{A}''$ of $\mathfrak{A}$ implementing δ, [6]. If we set $T=(k+\varepsilon 1)^{-1}$ for any constant $\varepsilon>0$, then T satisfies (1) of Theorem 2.2 and $\delta(a)=[T^{-1}, a]$ for all $a\in\mathfrak{A}$. Therefore $\mathscr{M}=G(T)\in\mathscr{M}(J_0)$ by Theorem 2.2 and it is invariant under $\pi_\delta(\mathfrak{A})$ by Theorem 3.2. This completes the proof.

Example. For any fixed non-zero skew-adjoint operator k and a given C^*-

algebra $\mathfrak{A}$ acting on $\mathscr{H}$, we define an algebra on $\tilde{\mathscr{H}}$ as follows;

$$\tilde{\mathfrak{A}}_k = \left\{ \begin{bmatrix} a & kb-ak \\ 0 & b \end{bmatrix} : a, b \in \mathfrak{A} \right\}.$$

Then it is a Lorentz algebra on $\{\tilde{\mathscr{H}}, J_0\}$, but it has no maximal J_0-uniformly positive invariant subspace. In fact, if $\tilde{\mathfrak{A}}_k$ has a maximal J_0-uniformly positive invariant subspace $\mathscr{M}=G(T)$, by the same computation as the proof of Theorem 3.2, $kb-ak=T^{-1}b-aT^{-1}$ for every $a, b\in\mathfrak{A}$, which implies $k=T^{-1}$ since a C^*-algebra has an approximately identity. This is a contradiction to the non-skew-adjointness of T.

Let $\mathrm{Rep}\,\mathfrak{A}$ be the set of all *-representations of $\mathfrak{A}$ on Hilbert spaces (in a usual sense) and $\sim$ the unitary equivalence in $\mathrm{Rep}\,\mathfrak{A}$. Let $\mathrm{Rep}_\delta\,\mathfrak{A}$ be the subset of all $\pi\in\mathrm{Rep}\,\mathfrak{A}$ similar to π_0. If $\pi\in\mathrm{Rep}_\delta\,\mathfrak{A}$ then there exists an intertwining operator A such that $\pi(a)=A\pi_\delta(a)A^{-1}$ for all $a\in\mathfrak{A}$. Then we have

Theorem 3.4. *There is a bijection of $\pi\in\mathrm{Rep}_\delta\,\mathfrak{A}/\sim$ onto the set of all positive operators B on $\tilde{\mathscr{H}}$ with $B^{-1}\in B(\tilde{\mathscr{H}})$ and $J_0B\in\pi_\delta(\mathfrak{A})'$ (the commutant of $\pi_\delta(\mathfrak{A})$) by the condition $B=A^*A$, where A denotes the intertwining operator mentioned above. Furthermore if we put $\langle x, y\rangle_B=(Bx, y)$ for $x, y\in\tilde{\mathscr{H}}$ then π_δ is a *-representation on a Hilbert space $\{\tilde{\mathscr{H}}, \langle\ ,\ \rangle_B\}$.*

Proof. If $\pi\in\mathrm{Rep}_\delta\,\mathfrak{A}$, there exists an invertible operator A such that $\pi(a)=$ $=A\pi_\delta(a)A^{-1}$ for all $a\in\mathfrak{A}$. Since we have

$$J_0A^*A\pi_\delta(a)A^{-1}A^{*-1}J_0 = J_0A^*\pi(a)A^{*-1}J_0 = \left(J_0A^{-1}\pi(a^*)AJ_0\right)^* = \\ = \left(J_0\pi_\delta(a^*)J_0\right)^* = \pi_\delta(a).$$

for all $a\in\mathfrak{A}$, it follows that $JA^*A\in\pi_\delta(\mathfrak{A})'$.

Suppose that $\pi'(a)=A'\pi_\delta(a)A'^{-1}$ and $A'^*A'=A^*A$. Then we have $\pi'(a)=$ $=U'U^{-1}\pi(a)\,(U'U^{-1})^{-1}$ for all $a\in\mathfrak{A}$, where $A=U|A|$ and $A'=U'|A'|$ are the polar decompositions of A and A' respectively. Thus $\pi'\sim\pi$ and hence bijectivity follows.

On the other hand, since $J_0B\in\pi_\delta(\mathfrak{A})'$ we have

$$\langle\pi_\delta(a^*)x, y\rangle = \left(BJ_0\pi_\delta(a^*)J_0x, y\right) = \left(\pi_\delta(a)^*Bx, y\right) = \langle x, \pi_\delta(a)y\rangle$$

for all $a\in\mathfrak{A}$ and $x, y\in\tilde{\mathscr{H}}$. Therefore π_δ is a *-representation of $\mathfrak{A}$ on $\{\tilde{\mathscr{H}}, \langle\ ,\ \rangle_B\}$. This completes the proof.

Remark. The above proof shows that the result is valid for any representation ψ of $\mathfrak{A}$ on $\{\mathscr{K}, J\}$ with $\psi(a^*)=J\psi(a)^*J$. Therefore a C^*-equivalent Lorentz algebra $\psi(\mathfrak{A})$ on $\{\mathscr{K}, J\}$ has a maximal J-uniformly positive invariant subspace if and only if ψ is similar to some $*$-representation of $\mathfrak{A}$ on a Hilbert space by [5; the proof of Theorem 6.1 and Remark 1].

On the other hand, as easily seen from Theorem 3.4, a representation π of a C^*-algebra $\mathfrak{A}$ on $\{\mathscr{K}, J\}$ with $\pi(a^*)=J\pi(a)^*J$ $(J\neq 1, -1)$ is not similar to any irreducible $*$-representation of $\mathfrak{A}$.

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