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Approximation by unitary and essentially unitary operators

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Introduction. In [9] P. R. HALMOS formulated the problem of normal spectral approximation in the algebra of bounded linear operators on a Hilbert space. One special case of this problem is the problem of unitary approximation; this case has been studied in [3], [7, Problem 119], and [13]. The main purpose of this paper is to continue this study of unitary approximation and some related problems.

In Section 1 we determine the distance (in the operator norm) from an arbitrary operator on a separable infinite-dimensional Hilbert space to the set of unitary operators in terms of familiar operator parameters. We also study the problem of the existence of unitary approximants. Several conditions are given that are sufficient for the existence of a unitary approximant, and it is shown that some operators fail to have a unitary approximant. This existence problem is solved completely for weighted shifts and compact operators.

Section 2 studies the problem of approximation by two sets of essentially unitary operators. It is shown that both the set of compact perturbations of unitary operators and the set of essentially unitary operators are proximinal; this latter fact is shown to be equivalent to the proximinality of the unitary elements in the Calkin algebra.

Notation. Throughout this paper H will denote a fixed separable infinite-dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For an arbitrary operator T, we write $\|T\|=\sup\{\|Tf\|:f \text{ in } H \text{ and } \|f\|=1\}$ and $m(T)=\inf\{\|Tf\|:f \text{ in } H \text{ and } \|f\|=1\}$. The spectral radius of T is $r(T)$. We write $|T|=(T^*T)^{1/2}$, and $E(\cdot)$ is the spectral measure for $|T|$.

The index of an operator T is defined by $\operatorname{ind}(T)=\dim\ker(T)-\dim\ker(T^*)$ if at least one of these numbers is finite, and we use the convention that $\operatorname{ind}(T)=0$ if both number are $\aleph_0$.

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The ideal of compact operators is denoted by $K(H)$, and π is the canonical homomorphism from $B(H)$ onto the Calkin algebra $C(H)=B(H)/K(H)$. The operator T is Fredholm if $\pi(T)$ is invertible in $C(H)$. The spectrum of $\pi(T)$ is $\sigma_e(T)$ with spectral radius $r_e(T)$; the complement of $\sigma_e(T)$ is denoted by $\varrho_e(T)$. We write $\|T\|_e=\|\pi(T)\|$ and $m_e(T)=$ the infimum of $\sigma_e(|T|)$. The unilateral weighted shift of multiplicity one with weight sequence $(\alpha_1, \alpha_2, \dots)$ is denoted shift $(\alpha_1, \alpha_2, \dots)$. If $\mathcal{M}$ is a set of operators, then an operator X_0 in $\mathcal{M}$ is an *$\mathcal{M}$-approximant* of the operator T if $\|T-X_0\|=\inf\{\|T-X\|:X \text{ in } \mathcal{M}\}$. The set $\mathcal{M}$ is *proximinal in* $B(H)$ (or simply *proximinal*) if every operator T has an $\mathcal{M}$-approximant.

1. Unitary operators. We shall frequently use the following theorem. It appears in [5, Theorem 2.2] for the case that T_1 is a Fredholm operator; a slightly different proof is given below for completeness.

1.1. Theorem. *If T_1 and T_2 are in $B(H)$ and if $\|T_1-T_2\|_e<m_e(T_1)$, then* $\operatorname{ind}(T_1)=\operatorname{ind}(T_2)$.

Proof. Consider first the case $T_1=1$. If $\|1-T_2\|_e<1$, then there exists K in $K(H)$ such that $\|1-T_2-K\|<1$. Hence T_2+K is invertible, so clearly $\operatorname{ind}(T_2+K)=$ $=0$ Thus T_2 is Fredholm and $\operatorname{ind}(T_2)=\operatorname{ind}(T_2+K)=0$ [2, Lemma 5.20]. Thus $\operatorname{ind}(T_2)=\operatorname{ind}(1)=0$.

For the general case, we can assume that $m_e(T_1)>0$. Then there exists L in $B(H)$ such that $\|L\|_e=1/m_e(T_1)$ and LT_1 is a compact perturbation of the identity (this can be seen by looking at the polar decomposition of T_1). Then $\|1-LT_2\|_e=$ $=\|LT_1-LT_2\|_e\leqq\|L\|_e\cdot\|T_1-T_2\|_e<1$. Hence LT_2 is Fredholm of index 0 by the above result.

Consequently T_1 is Fredholm if and only if T_2 is Fredholm, and in this case $\operatorname{ind}(T_1)=-\operatorname{ind}(L)=\operatorname{ind}(T_2)$ by the additivity of the index for Fredholm operators [2, Theorem 5.36].

If both T_1 and T_2 fail to be Fredholm, then $\dim\ker T_1^*=\aleph_0=\dim\ker T_2^*$. This follows because both LT_1 and LT_2 are Fredholm, which implies that both T_1 and T_2 have closed range and finite-dimensional kernel [2, proof of Theorem 5.17]. Hence both T_1 and T_2 are Fredholm unless $\dim\ker T_1^*=\aleph_0=\dim\ker T_2^*$. Thus in this one remaining case it follows that $\operatorname{ind}(T_1)=\operatorname{ind}(T_2)=-\aleph_0$.

1.2. Corollary. *If* $\operatorname{ind}(T)<0$ *and U is a unitary operator, then* $\|T-U\|\geqq$ $\geqq 1+m_e(T)$.

Proof. Clearly $\|T-U\|=\|U^*T-1\|\geqq r(U^*T-1)$. Assertion: Each number in the open ball $\{\zeta:|\zeta|<m_e(T)\}$ is an eigenvalue of T^*U. To see this, let $|\zeta|<m_e(T)$ and apply Theorem 1.1 to the operators $T_1=U^*T$ and $T_2=U^*T-\zeta$; notice

that $m_e(T)=m_e(U^*T)$ and $\mathrm{ind}(U^*T)=\mathrm{ind}(T)$. Hence $\mathrm{ind}(U^*T-\zeta)=\mathrm{ind}(U^*T)<0$. This proves the assertion, and the assertion implies 1.2.

We can now determine the distance from an arbitrary operator T to the set of unitary operators. Write $u(T)=\inf\{\|T-U\|:U$ a unitary operator$\}$.

1.3. Theorem.

(i) *If* $\mathrm{ind}(T)=0$, *then* $u(T)=\max\{\|T\|-1, 1-m(T)\}$.

(ii) *If* $\mathrm{ind}(T)<0$, *then* $u(T)=\max\{\|T\|-1, 1+m_e(T)\}$.

(The case $\mathrm{ind}(T)>0$ follows from (ii) by considering the adjoint of T).

Proof. Assertion (i) is true also in finite dimensions [3] and is proved here in a similar manner. The main point is that it is possible to find a unitary operator U such that $T=U|T|$ by enlarging the partial isometry in the polar decomposition of T (if necessary). Then $\|T-U\|=\||T|-1\|$, and it is easy to see that $\||T|-1\|=\max\{\|T\|-1, 1-m(T)\}$. That this maximum is a lower bound for $u(T)$ is also easy to see by using the triangle inequality. This proves assertion (i).

To prove assertion (ii), let $E(\cdot)$ be the spectral measure for $|T|$, and for $\varepsilon>0$ let E_ε denote the projection $E([0, m_e(T)+\varepsilon])$. Then $\dim E_\varepsilon(H)=\aleph_0$ since $m_e(T)$ has the equivalent definition $m_e(T)=\inf\{x\geqq 0:\dim E([0,x])H=\aleph_0\}$ (see [4, p. 185]).

Because $\mathrm{ind}(T)<0$, there exists a (non-unitary) isometry S such that $T=S|T|$. Because $E_\varepsilon(H)$ and $\ker S^*\oplus SE_\varepsilon(H)$ have equal dimension and co-dimension, there exists an isometry V_ε in $B(H)$ that maps $E_\varepsilon(H)$ onto $\ker S^*\oplus SE_\varepsilon(H)$. Define the operator $U_\varepsilon=V_\varepsilon E_\varepsilon+S(1-E_\varepsilon)$.

Assertion: *U_ε is a unitary operator.*

Proof. It is easy to see that U_ε is an isometry; that U_ε is onto follows since

$$U_\varepsilon(E_\varepsilon(H)) = \ker S^*\oplus SE_\varepsilon(H)$$

and

$$U_\varepsilon(H\ominus E_\varepsilon(H)) = S(H\ominus E_\varepsilon(H)).$$

Assertion: $\|T-U_\varepsilon\|\leqq\max\{\|T\|-1, 1+m_e(T)+\epsilon\}$.

Proof. Clearly $\|T-U_\varepsilon\|=\|U_\varepsilon^*T-1\|$; we examine the operator U_ε^*T. It is not difficult to see from the definition of U_ε that $E_\varepsilon(H)$ reduces $U_\varepsilon^*T=U_\varepsilon^*S|T|$. With respect to the decomposition $H=E_\varepsilon(H)\oplus(1-E_\varepsilon)(H)$, if follows that $U_\varepsilon^*T=X_\varepsilon\oplus Y_\varepsilon$ with $\|X_\varepsilon\|\leqq m_e(T)+\epsilon$ and $Y_\varepsilon=$ restriction of $|T|$ to the (reducing) subspace $(1-E_\varepsilon)(H)$.

Thus $\|U_\varepsilon^*T-1\|=\max\{\|X_\varepsilon-1\|, \|Y_\varepsilon-1\|\}$. Clearly $\|X_\varepsilon-1\|\leqq 1+m_e(T)+\epsilon$ and $\|Y_\varepsilon-1\|\leqq\||T|-1\|$. The fact that $\max\{1+m_e(T)+\epsilon, \||T|-1\|\}=\max\{1+m_e(T)+\epsilon, \|T\|-1\}$ follows easily. This proves $u(T)\leqq\max\{1+m_e(T), \|T\|-1\}$.

The reverse inequality follows from Corollary 1.2 and the triangle inequality. This proves Theorem 1.3.

In [8] it was shown that every operator has a positive approximant that is in the C*-algebra generated by the identity and the operator. For approximation by unitary operators, however, the situation is considerably different.

1.4. Theorem.

(i) *If* $\operatorname{ind}(T)=0$, *then* $T=U|T|$ *for some unitary approximant* U.

If the index of T *is non-zero, then* $u(T)\geqq 1$; *we consider the following two cases.*

(ii) *If* $\operatorname{ind}(T)\neq 0$ *and* $u(T)=1$, *then* T *fails to have a unitary approximant.*

(iii) *If* $\operatorname{ind}(T)<0$ *and* $u(T)>1$, *then each one of the following conditions is sufficient for* T *to have a unitary approximant:*

(a) $$\|T\|-1>1+m_e(T),$$

(b) $$\dim E([0, m_e(T)])(H)=\aleph_0,$$

(c) $m_e(T)$ *is a cluster point of eigenvalues of* $|T|$.

(The case $\operatorname{ind}(T)>0$ and $u(T)>1$ follows from (iii) by considering the adjoint of T).

Proof. Assertion (i) follows easily from the proof of Theorem 1.3 (i).

Assertion (ii) is a consequence of [14, p. 408]. For if $\operatorname{ind}(T)\neq 0$ and U is a unitary operator such that $\|U-T\|=u(T)=1$, then $\|1-U^*T\|=1$ and hence [14] implies $\operatorname{ind}((1-U^*T)-1)=0=\operatorname{ind}(-U^*T)$. It is easy to see, however, that $\operatorname{ind}(-U^*T)=\operatorname{ind}(T)$. Hence no such unitary operator U exists.

For the proof of (iii) (a), choose $\epsilon>0$ such that $m_e(T)+1+\epsilon\leqq\|T\|-1=u(T)$. Then the unitary operator U_ε constructed in the proof of Theorem 1.3 (ii) is shown by that proof to be a unitary approximant of T.

If (iii) (b) holds, then the construction of U_ε can be carried out in exactly the same way as above with $\epsilon=0$; again, this can be seen from the proof of Theorem 1.3 (ii).

If (iii) (c) holds, the construction is as follows. If $m_e(T)=0$, then (iii) (a) gives a unitary approximant since $\|T\|-1=u(T)>1$ by hypothesis (iii). If $m_e(T)>0$, then we use the following lemma to construct a unitary approximant of T; after this lemma is proved, the proof of (iii) (c) is straightforward.

1.5. Lemma. *If* $\alpha>0$, *then there exists a sequence* $\{\alpha_k\}$ *of real numbers such that* $\alpha_k>\alpha$ *for all* k *and such that* $\|1+\operatorname{shift}(\alpha_1, \alpha_2, \dots)\|=1+\alpha$.

Proof. Notation: Let $\{e_1, e_2, \dots\}$ be an orthonormal basis that is shifted. For any sequence $\{\alpha_k\}$, let A_n be the compression of the operator $|1+\operatorname{shift}(\alpha_1, \alpha_2, \dots)|^2$ to the span of $\{e_1, \dots, e_n\}$.

We prove below that there is some choice of $\{\alpha_k\}$ such that $\alpha_k>\alpha$ for all k and $A_n<(1+\alpha)^2$ for all n (where $<$ is the usual partial order for Hermitian operators). Since the norm is weakly lower semicontinuous, this proves $\|1+\text{sihft}(\alpha_1, \alpha_2, \dots)\| \leqq 1+\alpha$; the reverse inequality follows from Theorem 1.3 (ii) since $m_e(\text{shift}(\alpha_1, \alpha_2, \dots)) \geqq \alpha$.

The n-by-n matrix A_n is a tridiagonal matrix, about which the following two facts are known [11, p. 180]:

(1) For the characteristic polynomials $p_0(x)=1$ and $p_n(x)=\det(A_n-x)$, $n=1, 2, \dots,$ there are recursion relations $p_{n+1}(x)=\{1+\alpha_{n+1}^2-x\}p_n(x)-\alpha_n^2 p_{n-1}(x)$, $n=1, 2, \dots$.

(2) For any real number x, the number of eigenvalues of A_n that are less than x is equal to the number of sign changes between consecutive terms of the sequence $\{p_0(x), p_1(x), \dots, p_n(x)\}$.

By (2) we shall prove 1.5 if we show there exists some choice of $\{\alpha_k\}$ such that $\alpha_k>\alpha$ and $\text{sign } p_n((1+\alpha)^2) \neq \text{sign } p_{n+1}((1+\alpha)^2)$ for all n. This is because there will be n sign changes (with $x=(1+\alpha)^2$), and hence all n (positive) eigenvalues of A_n will be less than $(1+\alpha)^2$. Write $q_n=p_n((1+\alpha)^2)$. It thus suffices to define $\{\alpha_k\}$ such that $\alpha_k>\alpha$ and such that for all integers n we have $q_n/q_{n+1}<0$.

We define such a sequence $\{\alpha_k\}$ by induction.

To begin, choose $\alpha_1>\alpha$ such that $\alpha_1^2<\alpha^2((\alpha+2)/(\alpha+1))$.

We shall use the fact that this upper bound on α_1^2 implies the pair of inequalities $-1<(\alpha_1^2 q_0)/\alpha q_1)<0$.

To see this, note that $\alpha_1^2(\alpha+1)-\alpha(\alpha^2+2\alpha)<0$ so that $\alpha(\alpha_1^2-\alpha^2-2\alpha)<-\alpha_1^2$ or $\alpha(\alpha_1^2-\alpha^2-2\alpha)/\alpha_1^2<-1$ and thus $(\alpha q_1)/(\alpha_1^2 q_0)<-1$ since $q_0=1$ and $q_1=\alpha_1^2-\alpha^2-2\alpha$. The desired pair of inequalities now follow by inverting the above inequality.

Next, assume that $\alpha_1, \dots, \alpha_k$ have been chosen $>\alpha$ such that for $j=1, \dots, k$, there are the pair of inequalities $-1<(\alpha_j^2 q_{j-1})/(\alpha q_j)<0$.

Choose $\alpha_{k+1}>\alpha$ such that $\alpha_{k+1}^2<\alpha^2\{\alpha+2+((\alpha_k^2 q_{k-1})/(\alpha q_k))\}/(\alpha+1)$.

Assertion. *This upper bound on* α_{k+1}^2 *implies the pair of inequalities*

$$-1 < (\alpha_{k+1}^2 q_k)/(\alpha q_{k+1}) < 0.$$

Proof. The upper bound clearly implies that $\alpha_{k+1}^2(\alpha+1)<\alpha^2(\alpha+2)+ +((\alpha q_{k-1}\alpha_k^2)/q_k)$ and thus $\alpha(\alpha_{k+1}^2-\alpha^2-2\alpha)-((\alpha q_{k-1}\alpha_k^2)/q_k)<-\alpha_{k+1}^2$ so that $\alpha\{(\alpha_{k+1}^2-\alpha^2-2\alpha)q_k-\alpha_k^2 q_{k-1}\}/q_k<-\alpha_{k+1}^2$. Thus by (1) $(\alpha q_{k+1})/q_k<-\alpha_{k+1}^2$ or $(\alpha q_{k+1})/(q_k\alpha_{k+1}^2)<-1$.

By inverting the above inequality, the assertion follows.

Thus we can define by induction a sequence $\{\alpha_j\}$ such that for all j both $\alpha_j>\alpha$ and

$$(\alpha_j^2 q_{j-1})/(\alpha q_j) < 0.$$

This clearly implies sign $(q_{j-1}) \neq$ sign (q_j) for all j, and completes the proof of Lemma 1.5.

1.6 Remark. If $0 \leqq \beta_k \leqq \alpha_k$, with α and α_k as in 1.5, then $\|1+\text{shift}(\beta_1, \beta_2, \ldots)\| \leqq \leqq 1+\alpha$. Proof: Write $\beta_k = (1/2)(\alpha_k' + \alpha_k'')$ with $|\alpha_k'| = |\alpha_k''| = \alpha_k$. Then shift$(\beta_1, \beta_2, \ldots)$ is the average of two shifts each unitarily equivalent to shift $(\alpha_1, \alpha_2, \ldots)$ (see [7], Problem 75); this is sufficient to prove above inequality.

By using Lemma 1.5, it is straightforward to complete the proof of Theorem 1.4 (iii) (c). Write $\alpha = m_e(T) > 0$ and choose $\{\alpha_k\}$ as in 1.5. Choose a strictly decreasing sequence $\{a_k\}$ of eigenvalues of $|T|$ such that $\alpha < a_k \leqq \alpha_k$ for $k=1, 2, \ldots$ (if it is possible to choose eigenvalues a_k with $0 \leqq a_k \leqq \alpha$ for all k, then Theorem 1.4 (iii) (b) gives a unitary approximant). Let $\{f_k\}$ be a sequence of (orthogonal) unit vectors such that $|T| f_k = a_k f_k$ and put $M = \text{span}\{f_1, f_2, \ldots\}$. Because ind $(T) < 0$, there exists an isometry S such that $T = S|T|$ and $-\dim\ker S^* = \text{ind}(T)$.

If the index of T is finite, proceed as follows. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\ker S^*$, where $n = \dim\ker S^*$. Define an operator U by $Uf_k = -e_k$ for $k=1, \ldots, n$ and $Uf_k = -Sf_{k-n}$ for $k=n+1, n+2, \ldots$ and $Ug = Sg$ for g in $H \ominus M$. It is not difficult to see that U is a unitary operator.

Assertion. *U is a unitary approximant of T.*

Proof. Define $M_k = \text{span}\{f_j : j \equiv k (\text{mod } n)\}$, $k=1, \ldots, n$; clearly $M = M_1 \oplus \ldots \ldots \oplus M_n$. It is straightforward to verify that each M_k reduces U^*T and the part of U^*T on M_k is $-\text{shift}(a_k, a_{n+k}, a_{2n+k}, \ldots)$. It is also straightforward to verify that the part of U^*T on the (reducing) subspace $H \ominus M$ is the restriction of $|T|$ to this (reducing) subspace. Since $\{a_j\}$ is a strictly decreasing sequence, the norm of the identity plus shift $(a_k, a_{n+k}, a_{2n+k}, \ldots)$ is $\leqq 1+\alpha$ (cf. Remark 1.6); it follows that $\|T-U\| = \|U^*T - 1\| \leqq \max\{1+\alpha, \||T|-1\|\}$. It is not difficult to see that this maximum equals $\max\{1+\alpha, \|T\|-1\}$, which is $u(T)$.

If the index of T is $-\aleph_0$, proceed as follows. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\ker S^*$. Define the operator U by $Uf_{2^{k-1}} = -e_k$ and $Uf_{2^{k-1}(2n+1)} = -Sf_{2^{k-1}(2n-1)}$ for $k, n = 1, 2, \ldots$ and $Ug = Sg$ for g in $H \ominus M$. It is not difficult to see that U is a unitary operator.

Assertion. *U is a unitary approximant of T.*

Proof. For $k=1, 2, \ldots$ define the subspace $M_k = \text{span}\{f_j : j = 2^{k-1}(2n+1), n=0, 1, 2, \ldots\}$; then $M = M_1 \oplus M_2 \oplus \ldots$. Each M_k reduces U^*T, and the restriction of U^*T to M_k is $-\text{shift}(a_{2^{k-1}}, a_{3 \cdot 2^{k-1}}, a_{5 \cdot 2^{k-1}}, \ldots)$. Since $\{a_j\}$ is a strictly decreasing sequence, the norm of the identity plus each of these shifts is $\leqq 1+\alpha$. The part of U^*T on $H \ominus M$ is the part of $|T|$ on this reducing subspace, and we can conclude as before that

$$\|T-U\| = \|U^*T-1\| = \max\{1+\alpha, \|T\|-1\} = u(T).$$

This completes the proof of Theorem 1.4.

Theorem 1.4 implies the following result, which applies in particular to weighted shifts and compact operators.

1.7. Theorem. *If T is an operator such that $m_e(T)$ is a cluster point of eigenvalues of $|T|$, then T has a unitary approximant if and only if* $\operatorname{ind}(T)=0$ *or* $u(T)>1$.

Proof. If $u(T)>1$ and $\operatorname{ind}(T)\neq 0$, then Theorem 1.4 (iii) (c) applied to T (or else T^*) gives a unitary approximant; if $\operatorname{ind}(T)=0$, then 1.4 (i) gives an approximant. The one remaining case is covered by 1.4 (ii).

1.8. Example. The compact operator shift $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ has index -1 and is at distance 1 from the unitary operators by Theorem 1.3 (ii). Hence, by Theorem 1.4 (ii), it fails to have a unitary approximant.

1.9. Example. If S is the (unweighted) unilateral shift and 0 is the zero operator on H, then the operator $S\oplus 0$ on $H\oplus H$ does not have a unitary approximant that is in the von Neumann algebra it generates. Proof: By Theorem 1.3 (i), $S\oplus 0$ is at distance 1 from the unitary operators, and, by Theorem 1.4 (i), it has an approximant. The von Neumann algebra generated by $S\oplus 0$ and the identity on $H\oplus H$ is $\{T\oplus\zeta : T$ in $B(H)$ and ζ a complex number$\}$, and the unitary operators in this algebra are $\{U\oplus\zeta_1 : U$ a unitary operator in $B(H)$ and $|\zeta_1|=1\}$. It follows from [7, Problem 119] that $\|(S\oplus 0)-(U\oplus\zeta_1)\|=2$; hence the algebra fails to contain a unitary approximant of $S\oplus 0$.

1.10. Remark. Theorem 1.4 does not describe all operators that have unitary approximants. For example, if S is the (unweighted) unilateral shift and $0<x<1$, then the operator $S+x$ has index -1 [2, Theorem 7.26], fails to satisfy (a), (b), or (c) of 1.4 (iii) and has the identity as a unitary approximant. A similar anyalysis works for the operator S^2+x and fails for the operator $S(S+x)$; the existence of a unitary approximant for $S(S+x)$ is apparently not known.

2. Essentially unitary operators. We shall use the following theorem to prove two results on approximation by essentially unitary operators (*i.e.* operators whose image in $C(H)$ is a unitary element).

2.1. Theorem. *If T is any operator and W is a maximal partial isometry such that* $\operatorname{ind}(W)\neq\operatorname{ind}(T)$, *then* $\|W-T\|_e\geq 1+m_e(T)$.

Proof. It is sufficient to prove this result only for $\operatorname{ind}(T)\leq 0$, since in this case $m_e(T)\geq m_e(T^*)$. Hence we assume $\operatorname{ind}(T)\leq 0$.

If $m_e(T)=0$, then $m_e(T^*)=0$; since $\pi(W)$ or $\pi(W^*)$ is an isometry in $C(H)$, this implies $\|\pi(W)-\pi(T)\|\geqq 1$. Thus we can and do assume $m_e(T)>0$.

With these two assumptions, the proof is divided into four cases depending on whether T or W is Fredholm. Write $\mathcal{O}=\{\zeta\colon |\zeta|<m_e(T)\}$.

Case (i). If both T and W are Fredholm, then $\pi(W)$ is a unitary element in $C(H)$ and $\|W-T\|_e=\|\pi(W)-\pi(T)\|=\|1-\pi(TW^*)\|$.

Assertion. *The set $\mathcal{O}$ is included in a bounded component of $\varrho_e(TW^*)$.*

Proof. Because $\pi(T)$ is invertible and $\pi(W)$ is a unitary element, it follows that $m_e(T)=m_e(T^*)=m_e(TW^*)=m_e(WT^*)$. Hence if $|\zeta|<m_e(T)$, then both $\pi(TW^*-\zeta)$ and $\pi(WT^*-\bar{\zeta})$ are bounded below by $m_e(T)-|\zeta|>0$; this implies $\pi(TW^*-\zeta)$ is invertible, *i.e.* $\mathcal{O}$ is included in $\varrho_e(TW^*)$. Note that $\operatorname{ind}(TW^*)\neq 0$ by the additivity of the index for Fredholm operators. Since the index is constant on components of $\varrho_e(TW^*)$ and is zero on the unbounded component, it follows that $\mathcal{O}$ is included in a bounded component. This assertion implies that $r_e(1-TW^*)\geqq \geqq 1+m_e(T)$; hence $\|W-T\|_e\geqq 1+m_e(T)$.

In each of the three remaining cases we prove that $\mathcal{O}$ is included in $\sigma_e(TW^*)$ because the index is $-\aleph_0$ in $\mathcal{O}$.

Case (ii). If T is Fredholm and W is not Fredholm, then either $\dim\ker W^*= =\aleph_0$ or $\dim\ker W=\aleph_0$.

Assume $\dim\ker W=\aleph_0$. Then W^* is an isometry and hence $\|\pi(W)-\pi(T)\|\geqq \geqq\|1-\pi(TW^*)\|$. Note that $m_e(TW^*)\geqq m_e(T)$ since WT^*TW^* is unitarily equivalent to the compression of T^*T to the range of W^*. Thus Theorem 1.1 implies that if ζ is in $\mathcal{O}$, then $\operatorname{ind}(TW^*-\zeta)=\operatorname{ind}(TW^*)$.

Assertion. $\operatorname{ind}(TW^*)=-\aleph_0$.

Proof. $\dim\ker TW^*<\aleph_0$ since W^* is an isometry and $\dim\ker T<\aleph_0$. The fact that $\dim\ker WT^*=\aleph_0$ follows since the kernel of W has dimension $\aleph_0$ and the range of T^* is a closed subspace of finite co-dimension (since T^* is Fredholm); the intersection of any two such closed subspaces has dimension $\aleph_0$. This proves the assertion.

Since $\operatorname{ind}(TW^*-\zeta)=-\aleph_0$ for each ζ in $\mathcal{O}$, it follows that $\pi(TW^*-\zeta)$ is not invertible; hence $\mathcal{O}$ is included in $\sigma_e(TW^*)$. This implies $r_e(1-TW^*)\geqq \geqq 1+m_e(T)$, and hence $\|W-T_e\|\geqq 1+m_e(T)$.

If $\dim\ker W^*=\aleph_0$, then $\|W-T\|_e\geqq 1+m_e(T)$ follows by symmetry since the proof above used only that T^* is Fredholm and that $m_e(T)=m_e(T^*)$, but not the hypothesis $\operatorname{ind}(T)\leqq 0$.

Case (iii). If T fails to be Fredholm and W is Fredholm, then $\dim\ker T^*=\aleph_0$ (because if $\ker T^*$ is finite-dimensional, then the assumptions $\operatorname{ind}(T)\leqq 0$ and $m_e(T)>0$ imply T is Fredholm). Hence $\operatorname{ind}(TW^*)=-\aleph_0$ since $\dim\ker TW^*\leqq$

$\leqq \dim\ker T + \dim\ker W^* < \aleph_0$ and $\dim\ker WT^* = \aleph_0$. Note that $m_e(TW^*) = m_e(T)$ since $\pi(W^*)$ is a unitary element of $C(H)$. Again, Theorem 1.1 implies that $\operatorname{ind}(TW^* - \zeta) = -\aleph_0$ for ζ in $\mathcal{O}$, and consequently $\mathcal{O} \subset \sigma_e(TW^*)$. Thus $\|W - T\|_e = \|1 - TW^*\|_e \geqq r_e(1 - TW^*) \geqq 1 + m_e(T)$.

Case (iv). If both T and W fail to be Fredholm, then $\dim\ker T^* = \aleph_0$ (for the same reasons as in Case (iii)) and $\dim\ker W = \aleph_0$ (since $\operatorname{ind}(W) \neq \operatorname{ind}(T)$). Thus $\operatorname{ind}(TW^*) = -\aleph_0$ since $\dim\ker TW^* \leqq \dim\ker T < \aleph_0$ (since W^* is an isometry and $\operatorname{ind}(T) \leqq 0$) and $\dim\ker WT^* = \aleph_0$. Furthermore, $m_e(TW^*) \geqq m_e(T)$ since WT^*TW^* is unitarily equivalent to the compression of T^*T to the range of W^*. Again, Theorem 1.1 implies that $\mathcal{O} \subset \sigma_e(TW^*)$. Hence $\|W - T\|_e \geqq \|1 - TW^*\|_e \geqq r_e(1 - TW^*) \geqq 1 + m_e(T)$. This completes the proof of Theorem 2.1.

2.2 Corollary [12]. *If T_1 and T_2 are isometries such that $\|T_1 - T_2\| < 2$, then* $\dim\ker(T_1^*) = \dim\ker(T_2^*)$.

Proof. For any isometry T, $\dim\ker(T^*) = -\operatorname{ind}(T)$ since $\ker(T) = \{0\}$, and $m_e(T) = 1$ since $T^*T = 1$. Thus if $\dim\ker(T_1^*) \neq \dim\ker(T_2^*)$, then Theorem 2.1 asserts $\|T_1 - T_2\| \geqq \|T_1 - T_2\|_e \geqq 2$; this proves the corollary.

The next theorem follows from Theorem 2.1 and the results of Section 1.

2.3 Theorem. *The set* $\{U + K : U$ *a unitary operator and* K *a compact operator*$\}$ *is a proximinal subset of* $B(H)$. *For* T *in* $B(H)$, *write* $v(T)$ *for the distance from* T *to this set; there are two cases:*

(i) *If* $\operatorname{ind}(T) = 0$, *then* $v(T) = \max\{\|T\|_e - 1, 1 - m_e(T)\}$

(ii) *If* $\operatorname{ind}(T) < 0$, *then* $v(T) = \max\{\|T\|_e - 1, 1 + m_e(T)\}$.

(The case $\operatorname{ind}(T) > 0$ follows from (ii) by considering the adjoint of T).

Proof. We prove the distance assertion and show that each distance is attained, which proves the proximinality assertion.

To prove (i), Let U be a unitary operator such that $T = U|T|$; let K_1 be the compact operator $E[0, m_e(T)) \cdot (|T| - m_e(T)) + E(\|T\|_e, \|T\|] \cdot (|T| - \|T\|_e)$, where $E(\cdot)$ is the spectral measure of $|T|$. Then $\|T - U - UK_1\| = \||T| - 1 - K_1\| = \||T| - 1\|_e$, and it is easy to see that this number is equal to $\max\{\|T\|_e - 1, 1 - m_e(T)\}$. That this maximum is a lower bound for $v(T)$ is easy to see by using the triangle inequality. This proves (i).

To prove (ii), let S be an isometry such that $T = S(T)$. We shall obtain a lower bound for $v(T)$ and prove it is attained. Since $|T|$ is the sum [2, Exercise 5.17] of a diagonal operator and a compact operator, there exists a compact operator K_2 such that $|T| - K_2 \geqq 0$, $\sigma(|T| - K_2) \subset [m_e(T), \|T\|_e]$ and $m_e(T)$ is an eigenvalue of $|T| - K_2$ of multiplicity $\aleph_0$. Then $\|S(|T| - K_2)\| = \|T\|_e$ and $m(S(|T| - K_2)) = m_e(S(|T| - K_2)) = m_e(T)$.

If $m_e(T) > 0$, then T is semi-Fredholm and $\operatorname{ind}(T - SK_2) = \operatorname{ind}(T) < 0$.

Theorem 1.3 (ii) and Theorem 1.4 (iii) (b) then imply that there is a unitary operator U_0 such that $\|T-SK_2-U_0\|=u(T-SK_2)=\max\{\|T\|_e-1, 1+m_e(T)\}$.

That this maximum is a lower bound for $v(T)$ is easy to see: $v(T)\geqq\|T\|_e-1$ by the triangle inequality, and $v(T)\geqq 1+m_e(T)$ from Theorem 2.1. Thus U_0+SK_2 is an approximant of T and $v(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$ if $m_e(T)>0$.

If $m_e(T)=0$, then $\dim\ker(T-SK_2)=\dim\ker(T-SK_2)^*=\aleph_0$ and hence $\operatorname{ind}(T-SK_2)=0$. Theorem 1.3 (i) and Theorem 1.4 (i) imply that there is a unitary operator U_0 such that $\|T-SK_2-U_0\|=u(T-SK_2)=\max\{\|T\|_e-1, 1\}$ (since $m_e(T-SK_2)=0$).

That this maximum is a lower bound for $v(T)$ is again easy to see. Hence U_0+SK_2 is an approximant of T, and $v(T)=\max\{\|T\|_e-1, 1\}=\max\{\|T\|_e-1, 1+m_e(T)\}$. This completes the proof of Theorem 2.3.

The set of compact perturbations of unitary operators is precisely the set of essentially unitary operators of index zero [1], and the previous theorem shows that this is a proximinal subset of $B(H)$. The next theorem shows that the same is true of the set of all essentially unitary operators.

2.4. Theorem. *The set* $\{W \text{ in } B(H): \pi(W) \text{ a unitary element of } C(H)\}$ *is a proximinal subset of* $B(H)$. *For T in* $B(H)$, *write* $u_e(T)$ *for the distance from T to this set; there are two cases:*

(i) *If* $\operatorname{ind}(T)$ *is finite, then* $u_e(T)=\max\{\|T\|_e-1, 1-m_e(T)\}$

(ii) *If* $\operatorname{ind}(T)=-\aleph_0$, *then* $u_e(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$.

(The case $\operatorname{ind}(T)=+\aleph_0$ follows from (ii) by considering the adjoint of T).

Proof. (i) If the index of T is finite, then T can be written $T=W|T|$ with W a maximal partial isometry such that $\operatorname{ind}(W)=\operatorname{ind}(T)$; then $\pi(W)$ is a unitary element in $C(H)$. Let K_1 be the compact operator in the proof of Theorem 2.3 (i). Then $W+WK_1$ is an essentially unitary operator, and by the definition of K_1, $\|T-W-WK_1\|\leqq\||T|-1\|_e=\max\{\|T\|_e-1, 1-m_e(T)\}$; that this maximum is a lower bound for $u_e(T)$ is easy to see. This proves part (i).

To prove (ii), note that Theorem 2.1 implies $\|T-W\|\geqq 1+m_e(T)$ for every essentially unitary operator W, and clearly $\|T-W\|\geqq\|T\|_e-1$. By Theorem 2.3 (ii) there exists a unitary operator U and a compact operator K such that $\|T-U-K\|=$ $=v(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$. Hence $U+K$ is also an essentially unitary approximant of T, and $u_e(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$. This proves Theorem 2.4.

Theorem 2.4 together with the following observation shows that the set of unitary elements in $C(H)$ is a proximinal subset in $C(H)$.

2.5. Proposition. *If* $\mathscr{S}$ *is a non-empty subset of* $C(H)$ *and T is in* $B(H)$, *then* $\operatorname{dist}(T, \pi^{-1}(\mathscr{S}))=\operatorname{dist}(\pi(T), \mathscr{S})$, *and* $\pi(T)$ *has an* $\mathscr{S}$*-approximant if and only if T has a* $\pi^{-1}(\mathscr{S})$*-approximant.*

Proof. The equality of distances is basically a consequence of the definition of the norm in $C(H)$:$\inf\{\|T-S'\|:S'$ in $\pi^{-1}(\mathscr{S})\}=\inf\{\|T-S-K\|:\pi(S)$ in $\mathscr{S}$ and K in $K(H)\}=\inf\{\|\pi(T)-s\|:s$ in $\mathscr{S}\}$.

If $\pi(T)$ has an $\mathscr{S}$-approximant s, then $s=\pi(S)$ for some S in $B(H)$. Since the set of compact operators (the case $\mathscr{S}=\{0\}$) is proximinal in $B(H)$ ([6], [10]), there exists a compact operator K such that $\|T-S-K\|=\|\pi(T-S)\|$. Then $S+K$ is in $\pi^{-1}(\mathscr{S})$ and

$$\|T-(S+K)\|=\|\pi(T)-\pi(S)\| = \operatorname{dist}(\pi(T),\mathscr{S}) = \operatorname{dist}(T,\pi^{-1}(\mathscr{S})).$$

Conversely, let S be a $\pi^{-1}(\mathscr{S})$-approximant of T. Then $\pi(S)$ is an $\mathscr{S}$-approximant of $\pi(T)$ since $\operatorname{dist}(\pi(T),\mathscr{S})\leqq\|\pi(T)-\pi(S)\|\leqq\|T-S\|=\operatorname{dist}(T,\pi^{-1}(\mathscr{S}))$. This proves the proposition.

If in the above proposition $\mathscr{S}$ is the set of unitary elements in $C(H)$, then Theorem 2.4 shows that the set of unitary elements in $C(H)$ is proximinal.

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