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Hyperinvariant subspaces of operators of class $C_0(N)$

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1. Let Θ be an $n\times n$ matrix over the Hardy space H^∞ on the circle, n being a fixed natural number. Such a matrix is called *inner* if $\Theta(e^{it})$ is unitary a.e.t. Associated with an inner matrix Θ are a Hilbert space $\mathfrak{H}(\Theta)$ and an operator $S(\Theta)$ defined by $\mathfrak{H}(\Theta)=H_n^2\ominus\Theta H_n^2$ and $S(\Theta)h=P_\Theta(\chi h)$ $(h\in\mathfrak{H}(\Theta))$, where H_n^2 is the Hardy space of n dimensional (column) vector valued functions, P_Θ is the projection from H_n^2 onto $\mathfrak{H}(\Theta)$, and $\chi(e^{it})=e^{it}$. Any contraction T of class $C_0(m)$ with $m\leqq n$ (i.e. $T^k\to 0$, $T^{*k}\to 0$ as $k\to\infty$, and rank $((1-T^*T)^{1/2})=\text{rank}\,((1-TT^*)^{1/2})=$ $=m$) is unitarily equivalent to $S(\Theta)$ with a suitable inner $n\times n$ matrix Θ (see [5]).

A subspace $\mathfrak{L}$ of a Hilbert space $\mathfrak{H}$ is said to be *hyperinvariant* for an operator T on $\mathfrak{H}$ if it is invariant for all operators on $\mathfrak{H}$ that commute with T. Operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ are said to be *quasi-similar* if there are quasi-affinities (i.e. operators with zero kernel and dense range) X from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and Y from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ such that $XT_1=T_2X$ and $T_1Y=YT_2$.

Theorem 1. *Let Θ and Φ be inner matrices over H^∞. If $S(\Theta)$ and $S(\Phi)$ are quasi-similar, then there exist quasi-affinities X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ and Y from $\mathfrak{H}(\Phi)$ to $\mathfrak{H}(\Theta)$ such that*

(i) $XS(\Theta)=S(\Phi)X$ *and* $S(\Theta)Y=YS(\Phi)$,

(ii) *the correspondences $\varphi\colon\mathfrak{L}\to\overline{X\mathfrak{L}}$ and $\psi\colon\mathfrak{M}\to\overline{Y\mathfrak{M}}$ establish an isomorphism from the lattice $\mathcal{I}_\Theta$ of hyperinvariant subspaces for $S(\Theta)$ onto the lattice $\mathcal{I}_\Phi$ for $S(\Phi)$, and its inverse, $\psi=\varphi^{-1}$.*

Proof. The hypothesis of quasi-similarity implies that for $\mathfrak{L}\in\mathcal{I}_\Theta$

$$\varphi(\mathfrak{L})=\bigvee_Z\{Z\mathfrak{L}\,|\,ZS(\Theta)=S(\Phi)Z\} \tag{1}$$

belongs to $\mathcal{I}_\Phi$ (cf. [3], p. 108). By one of the MOORE-NORDGREN theorems ([1], [2]) the quasi-similarity of $S(\Theta)$ and $S(\Phi)$ implies that there exist matrices $\Delta, \Delta', \Lambda,$

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and Λ' each of whose determinants is relatively prime to the determinants of Θ and Φ, and such that

(2) $$\Delta\Theta = \Phi\Lambda \quad \text{and} \quad \Theta\Lambda' = \Delta'\Phi.$$

Define the operator X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ and Y from $\mathfrak{H}(\Phi)$ to $\mathfrak{H}(\Theta)$ by

(3) $$Xh = P_{\Phi}\Delta h \quad (h\in\mathfrak{H}(\Theta)) \quad \text{and} \quad Yg = P_{\Theta}\Delta' g \quad (g\in\mathfrak{H}(\Phi)).$$

Relation (2) guarantees condition (i), and X, Y are quasi-affinities (see [2]). Take an arbitrary $\mathfrak{L}$ in the lattice $\mathcal{I}_\Theta$ and let $\mathfrak{L}'=\varphi(\mathfrak{L})$. By a well-known theorem ([5]) the (hyper-) invariance of $\mathfrak{L}$ and $\mathfrak{L}'$ implies the existence of inner matrices Θ_1, Θ_2, Φ_1 and Φ_2 over H^∞ satisfying

(4) $$\Theta = \Theta_2\Theta_1 \quad \text{and} \quad \Phi = \Phi_2\Phi_1,$$

and

(5) $$\mathfrak{L} = \Theta_2(H_n^2\ominus\Theta_1 H_n^2) \quad \text{and} \quad \mathfrak{L}' = \Phi_2(H_n^2\ominus\Phi_1 H_n^2).$$

By the definition (1) of $\varphi(\mathfrak{L})$ we have $X\mathfrak{L}\subseteqq\varphi(\mathfrak{L})=\mathfrak{L}'$. On the other hand, since YZ commutes with $S(\Theta)$ for every Z occurring in (1), hyperinvariance of $\mathfrak{L}$ for $S(\Theta)$ implies $YZ\mathfrak{L}\subseteqq\mathfrak{L}$, and therefore $Y\mathfrak{L}'=Y\varphi(\mathfrak{L})\subseteqq\mathfrak{L}$. Now the inclusions $\overline{X\mathfrak{L}}\subseteqq\mathfrak{L}'$ and $Y\mathfrak{L}'\subseteqq\mathfrak{L}$, and relations (2)-(5) imply $\Delta\Theta_2H_n^2\subseteqq\Phi_2H_n^2$ and $\Delta'\Phi_2H_n^2\subseteqq$ $\subseteqq\Theta_2H_n^2$; whence we deduce the exisctence of matrices A and B over H^∞ such that

(6) $$\Delta\Theta_2 = \Phi_2 A \quad \text{and} \quad \Delta'\Phi_2 = \Theta_2 B.$$

Thus it follows that $\Phi_2AB=\Delta\Delta'\Phi_2$, and hence,

(7) $$\det A\cdot\det B = \det\Delta\cdot\det\Delta'.$$

Since $\det\Delta\cdot\det\Delta'$ is relatively prime to $\det\Phi$, (7) implies that $\det A$ is relatively prime to $\det\Phi$, hence to $\det\Phi_1$. To prove $\mathfrak{L}'=\overline{X\mathfrak{L}}$ suppose that $f\in\mathfrak{L}'\ominus\overline{X\mathfrak{L}}$. Then, again using (2)-(5), we see that f is orthogonal to $\Delta\Theta_2H_n^2$, and hence to $\Phi_2AH_n^2$, by (6). Moreover, (5) implies $f=\Phi_2 g$ for some $g\in H_n^2\ominus\Phi_1H_n^2$. Then for every $h\notin H_n^2$

$$0 = (f, \Delta\Theta_2 h) = (\Phi_2 g, \Phi_2 Ah) = (g, Ah).$$

Since $\det A$ is relatively prime to $\det\Phi_1$, AH_n^2 and $\Phi_1H_n^2$ span the whole H_n^2. This implies $g=0$, hence $f=0$, proving $\mathfrak{L}'=\overline{X\mathfrak{L}}$. The relation $\mathfrak{L}=\overline{Y\mathfrak{L}'}=\overline{YX\mathfrak{L}}$ is proved in a similar way. This completes the proof.

Corollary 2. *Let Θ, Φ, X and Y be as in Theorem 1, and $\mathfrak{L}$ a hyperinvariant subspace for $S(\Theta)$. If*

$$S(\Theta) = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix} \quad \textit{and} \quad S(\Phi) = \begin{bmatrix} S_1' & * \\ 0 & S_2' \end{bmatrix}$$

are the triangulations corresponding to the decompositions

$$\mathfrak{H}(\Theta) = \mathfrak{L} \oplus \mathfrak{L}^{\perp} \quad \textit{and} \quad \mathfrak{H}(\Phi) = \overline{X\mathfrak{L}} \oplus \overline{X\mathfrak{L}}^{\perp},$$

respectively, then S_1 *and* S_2 *are quasi-similar to* S_1' *and* S_2', *respectively.*

Proof. For the quasi-similarity of S_1 and S_1', use the quasi-affinities $X|\mathfrak{L}$ and $Y|X\mathfrak{L}$. Relation (6) implies that $S(\Theta_2)$ and $S(\Phi_2)$ are quasi-similar. S_2 and S_2' are unitarily equivalent to $S(\Theta_2)$ and $S(\Phi_2)$, respectively (see [5]).

2. A *normal* matrix M over H^∞ is, by definition, of the form

$$M = \mathrm{diag}\,(m_1, m_2, \ldots, m_n),$$

where, for each i, m_i is a scalar inner function and m_{i-1} is a divisor of m_i $(m_0 = 1)$. The operator $S(M)$ induced by a normal matrix M is called a *Jordan operator*. By the Sz.-Nagy—Foiaş theorem [4] every operator $S(\Theta)$ with inner Θ is quasi similar to a Jordan operator. Therefore on the basis of Theorem 1 and Corollary 2 the subsequent discussions will be confined to the case of Jordan operators.

Theorem 3. *Let* M *be a normal matrix over* H^∞. *A subspace* $\mathfrak{L}$ *of* $\mathfrak{H}(M)$ *is hyperinvariant for* $S(M)$ *if and only if there are normal matrices*

$$\Theta = \mathrm{diag}\,(u_1, \ldots, u_n) \quad \textit{and} \quad \Phi = \mathrm{diag}\,(v_1, \ldots, v_n)$$

satisfying

$$M = \Theta\Phi \quad \textit{and} \quad \mathfrak{L} = \Theta(H_n^2 \ominus \Phi H_n^2). \tag{8}$$

Proof. By the lifting theorem ([5] p. 258) for every operator X on $\mathfrak{H}(M)$, commuting with $S(M)$, there is a matrix Δ over H^∞ satisfying

$$Xh = P_M \Delta h \quad (h \in \mathfrak{H}(M)) \quad \text{and} \quad \Delta M H_n^2 \subseteqq M H_n^2. \tag{9}$$

The latter condition is equivalent to the existence of a matrix Λ over H^∞ satisfying

$$\Delta M = M\Lambda. \tag{10}$$

Conversely every matrix Δ over H^∞ that is accompanied with a matrix Λ satisfying (10) induces an operator X on $\mathfrak{H}(M)$, commuting with $S(M)$, by the first part of (9).

Suppose that $\mathfrak{L}$ is of the form (8). To prove the hyperinvariance of $\mathfrak{L}$ for $S(M)$, it suffices to show the invariance of $\mathfrak{L}$ for the operator X defined by (9). The existence of Λ satisfying (10) implies that if $i > j$ then the inner function $m_j^{-1} m_i$ is a divisor of the $\Delta_{i,j}$ that is the (i,j)-th entry of Δ. Since Θ and Φ are normal matrices with $M = \Theta\Phi$, for $i > j$ the inner function $u_j^{-1} u_i$ is a divisor of $m_j^{-1} m_i$, hence a divisor of $\Delta_{i,j}$. This gaurantees the existence of a matrix Λ' over H^∞ satisfying

$$\Delta\Theta = \Theta\Lambda', \tag{11}$$

and consequently the invariance of $\mathfrak{L}$ for X.

Suppose conversely that $\mathfrak{L}$ is hyperinvariant for $S(M)$. Let P_i be the orthogonal projection from $\mathfrak{H}(M)$ onto the i-th component space. Since P_i commutes with $S(M)$, the hyperinvariance of $\mathfrak{L}$ implies that

$$\mathfrak{L} = P_1\mathfrak{L} \oplus \ldots \oplus P_n\mathfrak{L}$$

and each $P_i\mathfrak{L}$ is an invariant subspace for $S(m_i)$. By the Beurling theorem there are inner divisors u_i and v_i of m_i satisfying

$$m_i = u_i v_i \quad \text{and} \quad P_i\mathfrak{L} = u_i(H^2 \ominus v_i H^2). \tag{12}$$

Set $\Theta = \mathrm{diag}\,(u_1, \ldots, u_n)$ and $\Phi = \mathrm{diag}\,(v_1, \ldots, v_n)$, then Θ and Φ satisfy (8). It remains to prove the normality of Θ and Φ. To this end, take the matrix Δ over H^∞ whose (i, j)-th entry $\Delta_{i,j}$ is defined by

$$\Delta_{ij} = 1 \quad (i \leqq j) \quad \text{and} \quad \Delta_{ij} = m_j^{-1} m_i \quad (i > j).$$

Clearly there exists a matrix Λ over H^∞ satisfying (10). The hyperinvariance of $\mathfrak{L}$ implies the existence of a matrix Λ' satisfying (11). This means if $i<j$ then u_i is a divisor of u_j and $u_i^{-1}u_j$ is a divisor of $m_i^{-1}m_j$. The former condition guarantees the normality of Θ while the latter does the normality of Φ. This completes the proof.

Corollary 4. *Let M be a normal matrix over H^∞, and $\mathfrak{L}_1$, $\mathfrak{L}_2$ subspaces of $\mathfrak{H}(M)$ hyperinvariant for $S(M)$. If $S(M)|\mathfrak{L}_1$ is quasi-similar to $S(M)|\mathfrak{L}_2$ then $\mathfrak{L}_1 = \mathfrak{L}_2$.*

Proof. Take the normal matrices Θ_i and Φ_i $(i=1, 2)$ satisfying (8) with Θ_i, Φ_i and $\mathfrak{L}_i$ in place of Θ, Φ and $\mathfrak{L}$, respectively. Since $S(\Phi_i)$ is unitarily equivalent to $S(M)|\mathfrak{L}_i$, it follows that $\Phi_1 = \Phi_2$. This implies that $\Theta_1 = \Theta_2$ and $\mathfrak{L}_1 = \mathfrak{L}_2$. This completes the proof.

Recall that the *minimal function* m_S of an operator S of class $C_0(N)$ is defined as the greatest common inner divisor of all inner functions m for which $m(S) = 0$. If $S(M)$ with normal matrix $M = \mathrm{diag}\,(m_1, \ldots, m_n)$ is the Jordan model of S then the minimal function m_S coincides with m_n. The minimal function is preserved under quasi-similarity.

Corollary 5. *Let M be a normal matrix over H^∞. If $\mathfrak{L}$ is a subspace of $\mathfrak{H}(M)$ hyperinvariant for $S \equiv S(M)$, then*

$$m_S = m_{S_1} \cdot m_{S_2}, \tag{13}$$

where the operators S_1 and S_2 are defined by the triangulation $S = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ corresponding to the decomposition $\mathfrak{H}(M) = \mathfrak{L} \oplus \mathfrak{L}^\perp$

Proof. Take normal matrices Θ and Φ over H^∞ satisfying (8). Since $S(\Phi)$ and $S(\Theta)$ are unitarily equivalent to S_1 and S_2, respectively, it follows that $m_{S_1}=v_n$ and $m_{S_2}=u_n$, which implies (13).

Remark. In the above situation $m_S=m_{S_1}\cdot m_{S_2}$ for an arbitrary invariant subspace $\mathfrak{L}$ if and only if $M=\operatorname{diag}(1, \dots, 1, m_n)$.

3. When m is a scalar inner function, for the operator $S(m)$ the invariance of a subspace is equivalent to its hyperinvariance. The lattice $\mathcal{I}_m$ of all (hyper-) invariant subspaces is totally ordered if and only if m is of the form

$$(14)\qquad \left(\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}\right)^n \quad (|\alpha|<1,\ n \text{ a positive integer})$$

or of the form

$$(15)\qquad e_s(\lambda)\equiv \exp\left(s\frac{\lambda+\alpha}{\lambda-\alpha}\right) \quad (|\alpha|=1;\ s>0),$$

according as $\dim\mathfrak{H}(m)=n$ or $\dim\mathfrak{H}(m)=\infty$ (cf. [5] p. 136). This can be generalized to the case of inner matrices.

Theorem 6. *Let M be a normal matrix over H^∞ and* $\dim\mathfrak{H}(M)=\infty$. *The lattice $\mathcal{I}_M$ of hyperinvariant subspaces for $S(M)$ is totally ordered if and only if m_n is of the form* (15) *and each m_i coincides with either* 1 *or m_n.*

Proof. By Theorem 3 the total orderedness of the lattice $\mathcal{I}_M$ is equivalent to the condition that if normal matrices Θ_i $(i=1,2)$ are (left) divisors of M such that $\Theta_1^{-1}M$ and $\Theta_2^{-1}M$ are normal too, then one of Θ_1 and Θ_2 is a (left) divisor of the other. Suppose that $\mathcal{I}_M$ is totally ordered. Take arbitrary inner divisors u and v of m_n, and set $u_i=u\wedge m_i$ and $v_i=v\wedge m_i$ ($a\wedge b$ denotes the greatest common inner divisor of a and b). Then the normal matrices Θ_1 and Θ_2 defined by

$$\Theta_1=\operatorname{diag}(u_1, u_2, \dots, u_{n-1}, u) \quad \text{and} \quad \Theta_2=\operatorname{diag}(v_1, v_2, \dots, v_{n-1}, v),$$

are (left) divisors of M and $\Theta_i^{-1}M$ $(i=1,2)$ is a normal matrix over H^∞. The divisibility of Θ_2 by Θ_1 or Θ_1 by Θ_2 implies that one of u and v is a divisor of the other. The arbitrariness of u and v implies that m_n is of the form (15) because $\dim\mathfrak{H}(M)=\infty$ implies $\dim\mathfrak{H}(m_n)=\infty$. There exists an m_i such that $m_{i-1}^{-1}m_i=e_S$ $(1\leqq i\leqq n)$. In fact if any $m_{i-1}^{-1}m_i$ is not equal to e_S, then there exists i and j such that $1\leqq i<j\leqq n$, $m_{i-1}^{-1}m_i=e_a$ $(s>a>0)$ $m_{j-1}^{-1}m_j=e_b$ $(s>b>0)$ and $a+b\leqq s$. Now set c and d so that $0<c\leqq a$, $0<d\leqq b$ and $c<d$. Consider the normal matrices Ω_1 and Ω_2 defined by

$$\Omega_1=\operatorname{diag}(1, \dots, 1, \overset{(i)}{e_c}, \dots, e_c) \quad \text{and} \quad \Omega_2=\operatorname{diag}(1, \dots, 1, \overset{(j)}{e_d}, \dots, e_d).$$

Clearly Ω_i is a (left) divisor of M and $\Omega_i^{-1}M$ is a normal matrix. By Theorem 3, the subspace $\Omega_1 H_n^2 \ominus H_n^2 M$ and $\Omega_2 H_n^2 \ominus H_n^2 M$ are hyperinvariant for $S(M)$, but any one of them is not included in the other, a contradiction. Consequently

$$M = \operatorname{diag}(1, \ldots, 1, e_s, \ldots, e_s).$$

The "only if" part follows from the next lemma.

Lemma 7. *Let the operator V on $\mathfrak{H}_0$ be unicellular, i.e. let the lattice of all invariant subspaces for V be totally ordered. Then for any finite direct* sum $T = V \oplus \ldots \oplus V$, *acting on $\mathfrak{H} = \mathfrak{H}_0 \oplus \ldots \oplus \mathfrak{H}_0$, the lattice of all hyperinvariant subspaces for T is totally ordered.*

Proof. Let P_i be a projection from $\mathfrak{H}$ to the i-th component space. For any subspace $\mathfrak{L}$ of $\mathfrak{H}$ hyperinvariant for T, as in the proof of Theorem 3, we have $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots \oplus \mathfrak{L}_n$, where $\mathfrak{L}_i = P_i \mathfrak{L}$. The operator D_2, which causes interhchange of the first component with the second one for each vector, commutes with T, hence $D_2 \mathfrak{L} \subseteqq \mathfrak{L}$. This implies that $\mathfrak{L}_2 \subseteqq \mathfrak{L}_1, \mathfrak{L}_1 \subseteqq \mathfrak{L}_2$ and hence $\mathfrak{L}_1 = \mathfrak{L}_2$. Similarly we have $\mathfrak{L}_1 = \mathfrak{L}_i$. Thus for arbitrary hyperinvariant subspaces $\mathfrak{L}$ and $\mathfrak{L}'$ for T we have $\mathfrak{L} = \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_1$ and $\mathfrak{L}' = \mathfrak{L}_1' \oplus \ldots \oplus \mathfrak{L}_1'$. Since $\mathfrak{L}_1$ and $\mathfrak{L}_1'$ are invariant for V, it follows that $\mathfrak{L}_1 \subseteqq \mathfrak{L}_1'$ or $\mathfrak{L}_1' \subseteqq \mathfrak{L}_1$. Thus we have $\mathfrak{L} \subseteqq \mathfrak{L}'$ or $\mathfrak{L}' \subseteqq \mathfrak{L}$. This completes the proof of the lemma.

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