Acta Sci. Math., 39 (1977), 205—231

Tensor operations on characteristic functions of C_0 contractions

H. BERCOVICI and D. VOICULESCU

By the results of [14], [15] and [1] every contraction T of class C_0 acting on a separable Hilbert space is quasi-similar to a unique Jordan operator. If T has finite defect indices then its Jordan model also shares this property and B. SZ.-NAGY and C. FOIAŞ proved in [14] that the determinant of the characteristic function of T and of the Jordan model coincide in this case.

Also in the case of finite defect indices, from the work of E. A. NORDGREN and B. MOORE ([10] and [8]; cf. also [16]) it is known that the inner functions appearing in the Jordan model of T can be computed from the minors of the determinant of the characteristic function of T.

It is an immediate problem to find characterizations for the inner functions in the Jordan model of a general C_0 contraction, and to look for special characterizations in the case of weak contractions of class C_0 ([13], chapter VIII) when the characteristic function has a determinant.

Also, the determinant being a representation of the unitary group on a finite-dimensional space, more generally we may perform on the characteristic function of a contraction tensor operations of the type associated to irreducible representations of unitary groups, and ask about the properties of the operators having these functions as characteristic functions.

In the first part of this paper we consider tensor operations corresponding to irreducible representations of unitary groups applied to characteristic functions of operators of class C_0, the main result being that these operations preserve the quasisimilarity of the associated operators, provided the given operators have equal defect indices. This assertion is also adapted for the case of unequal defect indices, using impure characteristic functions.

As a corollary we characterize the inner functions in the Jordan model of a C_0 contraction by means of the smallest scalar inner multiples of the exterior

Received October 10, 1976, in augmented form March 29, 1977.

powers of the characteristic function. We also obtain estimates for the defect operator of a C_0 contraction in terms of the Jordan model.

In the second part of the paper we construct higher order algebraic adjoints of the characteristic function of a weak contraction. This enables us, using the results of the first part, to extend the above mentioned result of E. A. Nordgren and B. Moore to the case of weak contractions of class C_0.

We also prove that the determinant of the characteristic function of such a contraction is an inner function.

Using the results of the first part concerning defect operators, we prove that a C_0 contraction is a weak contraction, if and only if its Jordan model is a weak contraction. This extends a result of L. E. Isaev [5] on dissipative operators, which via Cayley transform (see [13] ch. IX) shows that a C_0 contraction with Jordan model $S(m_a)$, $m_a(\lambda)=\exp\left(-a(1+\lambda)/(1-\lambda)\right)$ $(a>0)$, is a weak contraction.

Part I

§ 1. Notation and preliminaries

1. We shall consider separable (finite or infinite dimensional) Hilbert spaces over the complex field **C**.

We shall denote by $\mathfrak{H}$, $\mathfrak{K}$, ... Hilbert spaces; $\langle .,. \rangle$ will denote the scalar product in any such space. If $\mathfrak{Y}$ is a subspace of $\mathfrak{H}$ we denote by $P_{\mathfrak{Y}}$ the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{Y}$ and by $\mathfrak{Y}^{\perp}$ or $\mathfrak{H}\ominus\mathfrak{Y}$ the orthogonal complement of $\mathfrak{Y}$. $(M)^{-}$ denotes the norm-closure of the subset $M\subset\mathfrak{H}$. If $\{Y_\alpha\}_{\alpha\in A}$ is a family of subsets of $\mathfrak{H}$, $\bigvee_{\alpha\in A} Y_\alpha$ will denote the closed linear span of $\bigcup_{\alpha\in A} Y_\alpha$. $X\vee Y$ will denote the closed linear span of $X\cup Y$.

If $\mathfrak{H}$ and $\mathfrak{K}$ are Hilbert spaces we shall denote by $\mathfrak{H}\otimes\mathfrak{K}$ their tensor product, which is also a Hilbert space. Recall that

(1.1) $$\langle f\otimes g, f'\otimes g'\rangle = \langle f, f'\rangle\langle g, g'\rangle \quad \text{for} \quad f, g\in\mathfrak{H}, f' g'\in\mathfrak{H}'.$$

$\mathfrak{H}^{\otimes n}$ will denote the tensor product $\mathfrak{H}\otimes\mathfrak{H}\otimes\ldots\otimes\mathfrak{H}$ (n times).

We denote by $\mathscr{L}(\mathfrak{H}, \mathfrak{K})$ the linear space of all linear bounded operators $X\colon \mathfrak{H}\to\mathfrak{K}$, $\mathscr{L}(\mathfrak{H})=\mathscr{L}(\mathfrak{H}, \mathfrak{H})$. If S is any subset of $\mathscr{L}(\mathfrak{H})$, $(S)'$ denotes the commutant of S. $\mathscr{U}(\mathfrak{H})$ denotes the group of unitary operators on $\mathfrak{H}$.

If $T\in\mathscr{L}(\mathfrak{H})$, the operator $\Gamma_n(T)\in\mathscr{L}(\mathfrak{H}^{\otimes n})$ is determined by

(1.2) $$\Gamma_n(T)(h_1\otimes h_2\otimes\ldots\otimes h_n) = Th_1\otimes Th_2\otimes\ldots\otimes Th_n, \quad h_j\in\mathfrak{H} \quad (1\leqq j\leqq n).$$

The map Γ_n is multiplicative, commutes with the $*$-operation and restricted to $\mathscr{U}(\mathfrak{H})$ is a unitary representation.

2. Let us recall that H^∞ is the Banach algebra of bounded analytic functions in the unit disc $D=\{z\in\mathbf{C}\,|\,|z|<1\}$. We denote by H_i^∞ the set of inner functions in H^∞, that is $m\in H_i^\infty$ if and only if m has (dt-)almost everywhere radial limits $m(e^{it})$ of modulus one. We shall abuse notation sometimes, writing $m=m'$ for two inner functions such that m/m' is a constant (of modulus one).

If $\{f_\alpha\}_{\alpha\in A}$ is a family of H^∞-functions, not all 0, we denote by $\bigwedge_{\alpha\in A} f_\alpha$ the greatest common inner divisor of the functions f_α.

Consider also the Hardy space H^2 and, for a Hilbert space $\mathfrak{H}$, the vector-valued Hardy space $H^2(\mathfrak{H})$ which can be identified with $\mathfrak{H}\otimes H^2$.

If $T\in\mathscr{L}(\mathfrak{H})$ and $S\in\mathscr{L}(H^2)$ we shall consider $T\otimes S$ as an operator on $H^2(\mathfrak{H})$. For $f\in H^\infty(\mathfrak{H}), g\in H^\infty(\mathfrak{K})$ we shall denote (somewhat ambiguously) by $f\otimes g$ the element of $H^2(\mathfrak{H}\otimes\mathfrak{K})$ defined by

$$(f\otimes g)(z)=f(z)\otimes g(z),\quad z\in D. \tag{1.3}$$

For any two Hilbert spaces $\mathfrak{H}$, $\mathfrak{K}$ the operator-valued Hardy space $H^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$ is the set of all bounded, $\mathscr{L}(\mathfrak{H},\mathfrak{K})$-valued analytic functions in the unit disc.

A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$ is contractive if $\|\Theta(z)\|\leqq 1, z\in D$. Any function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$ may be considered as an element of $\mathscr{L}(H^2(\mathfrak{H}), H^2(\mathfrak{K}))$ that commutes with scalar H^∞-multiplications.

We say that two functions

$$\Theta_i\in H^\infty(\mathscr{L}(\mathfrak{H}_i,\mathfrak{K}_i))\quad (i=1,2)$$

coincide if there are unitary operators $U\colon \mathfrak{H}_1\to\mathfrak{H}_2$, $V\colon\mathfrak{K}_1\to\mathfrak{K}_2$ such that $\Theta_2(\lambda)U=V\Theta_1(\lambda)$ for all $\lambda\in D$.

A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$ is *inner* if it is isometric as an element of $\mathscr{L}(H^2(\mathfrak{H}), H^2(\mathfrak{K}))$. Θ is $*$-*inner* if the function $\Theta^\sim$ defined by

$$\Theta^\sim(z)=\Theta(\bar z)^*,\quad z\in D \tag{1.4}$$

is inner. Θ is *two-sided inner* if it is simultaneously inner and $*$-inner. We denote by $H_i^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$ the set of two-sided inner functions in $H^\infty(\mathscr{L}(\mathfrak{H},\mathfrak{K}))$.

For any $\Theta\in H^\infty(\mathscr{L}(\mathfrak{H}))$ we denote by $\Gamma_n(\Theta)$ the element of $H^\infty(\mathscr{L}(\mathfrak{H}^{\otimes n}))$ defined by

$$(\Gamma_n(\Theta))(z)=\Gamma_n(\Theta(z)),\quad z\in D. \tag{1.5}$$

If $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{H}))$ then $\Gamma_n(\Theta)\in H_i^\infty(\mathscr{L}(\mathfrak{H}^{\otimes n}))$.

3. For any $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{H}))$ we define $S(\Theta)$ as the operator acting on

$$\mathfrak{H}(\Theta)=H^2(\mathfrak{H})\ominus\Theta H^2(\mathfrak{H}) \tag{1.6}$$

and defined by

$$(S(\Theta)^*u)(z)=z^{-1}(u(z)-u(0)),\quad z\in D,\quad u\in\mathfrak{H}(\Theta). \tag{1.7}$$

If Θ is pure then it coincides with the characteristic function of $S(\Theta)$ and in this case dim $\mathfrak{H}$ equals the defect indices of $S(\Theta)$ [13]. Recall that, for a contraction $T\in\mathscr{L}(\mathfrak{K})$, the *defect operators* are $D_T=(I-T^*T)^{1/2}$, $D_{T^*}=(I-TT^*)^{1/2}$ and the *defect indices* $\mathfrak{d}_T$, $\mathfrak{d}_{T^*}$ are the ranks of D_T and D_{T^*}, respectively.

Let μ_T denote the *multiplicity* of T, i.e. the least cardinal of cyclic sets for T.

We shall need the lifting of commutants theorem of [13] in the following form.

If $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{H}))$, $\Theta'\in H_i^\infty(\mathscr{L}(\mathfrak{H}'))$ and $X\in\mathscr{L}(\mathfrak{H}(\Theta), \mathfrak{H}(\Theta'))$ satisfy the relation

$$S(\Theta')X = XS(\Theta)$$

then there is an $A\in H^\infty(\mathscr{L}(\mathfrak{H}, \mathfrak{H}'))$ such that

$$A\Theta H^2(\mathfrak{H}) \subset \Theta' H^2(\mathfrak{H}') \quad \text{and} \tag{1.8}$$

$$Xh = P_{\mathfrak{H}(\Theta')}Ah, \quad h\in\mathfrak{H}(\Theta). \tag{1.9}$$

The operator X is one-to-one if and only if, for $h\in H^2(\mathfrak{H})$,

$$h\in\Theta H^2(\mathfrak{H}) \Leftrightarrow Ah\in\Theta' H^2(\mathfrak{H}'), \tag{1.10}$$

and has dense range if and only if

$$AH^2(\mathfrak{H}) \vee \Theta' H^2(\mathfrak{H}') = H^2(\mathfrak{H}'). \tag{1.11}$$

Let us recall that X is called a *quasi-affinity* if it is one-to-one and has dense range.

The operator $S(\Theta)$ is of class C_0 if and only if Θ has a scalar multiple, that is, if

$$\Theta H^2(\mathfrak{H}) \supset mH^2(\mathfrak{H}) \tag{1.12}$$

for some $m\in H_i^\infty$. The minimal function of $T=S(\Theta)$ is then the greatest common inner divisor m_T of the functions m satisfying (1.12) [13].

A Jordan operator is an operator $S(\Theta)$ determined by a function of the form

$$\Theta = \begin{bmatrix} m_1 & & 0 \\ & m_2 & \\ & & \cdot \\ 0 & & & \cdot \\ & & & & \cdot \end{bmatrix}$$

where $m_j\in H_i^\infty$ and m_{j+1} divides m_j for each j. We shall denote it also by $S(M)$, $M=\{m_j\}_{j=1}^\infty$. By the results of [14], [15], [1] every C_0 contraction acting on a separable Hilbert space is quasisimilar to a unique Jordan model $S(M)$.

4. For a finite group G we shall denote by $C^*(G)$ the C^*-algebra of G [2], and by $\hat{G}$ the set of all (equivalence classes) of irreducible unitary representations of G. The elements of $C^*(G)$ will be written in the form $\sum_{g\in G} c_g\, g$ where $c_g\in\mathbf{C}$, so that for any unitary representation π of G the associated representation of $C^*(G)$ is given by

$$\pi\left(\sum_{g\in G} c_g g\right) = \sum_{g\in G} c_g \pi(g).$$

Let $\mathfrak{S}_n$ be the group of permutations of the set $\{1, 2, ..., n\}$. The group $\mathfrak{S}_{n-1}$ will be identified with the subgroup of $\mathfrak{S}_n$ consisting of those permutations of $\mathfrak{S}_n$ that leave n fixed and $C^*(\mathfrak{S}_{n-1})$ will be considered as a sub-algebra of $C^*(\mathfrak{S}_n)$.

$\hat{\mathfrak{S}}_n$ is known to be in one-to-one correspondence with signatures $\tau=(t_1\geqq ...\geqq t_n)$, t_j non-negative integers, $\sum_{j=1}^{n} t_j=n$, and the corresponding minimal central projections p_τ of $C^*(\mathfrak{S}_n)$ are given by the central Young symmetrizers [18], [6], [9]. It is known [17], Ch. V, § 18, that an irreducible representation of signature $\tau=(t_1\geqq t_2\geqq ...\geqq t_n)$ restricted to $\mathfrak{S}_{n-1}$ contains the irreducible representation of signature $\tau'=(t_1'\geqq t_2'\geqq ...$ $...\geqq t_{n-1}')$ if and only if

$$t_1 \geqq t_1' \geqq t_2 \geqq t_2' \geqq ... \geqq t_{n-1} \geqq t_{n-1}' \geqq t_n \tag{1.13}$$

(this will be written $\tau'\prec\tau$) and that the multiplicity of τ' is one in this case.

Consider now a Hilbert space $\mathfrak{K}$. On $\mathfrak{K}^{\otimes n}$ there is a unitary representation π_n of $\mathfrak{S}_n$ given by

$$\pi_n(\sigma)(k_1\otimes ...\otimes k_n) = k_{\sigma^{-1}(1)}\otimes ...\otimes k_{\sigma^{-1}(n)}, \quad \sigma\in\mathfrak{S}_n. \tag{1.14}$$

By one of the basic results of HERMANN WEYL ([18], [6], see also [11], [7] for the adaptation to the case when dim $\mathfrak{K}$ is infinite) we have

$$(\Gamma_n(\mathscr{U}(\mathfrak{K})))' = (\Gamma_n(\mathscr{L}(\mathfrak{K})))' = \pi_n(C^*(\mathfrak{S}_n)). \tag{1.15}$$

The irreducible representations of $\mathscr{U}(\mathfrak{K})$ which will be considered are also labelled by signatures, so we shall first make a convention. A signature will be a decreasing sequence $\tau=(t_1\geqq t_2\geqq ...)$ of nonnegative integers, of finite or infinite length $l(\tau)$. By $\iota(\tau)$ we shall denote the number of nonzero elements among the t_j's and $|\tau|$ will stand for $\sum_{j=1}^{l(\tau)} t_j$.

Thus for instance the set $\hat{\mathfrak{S}}_n$ is in a one-to-one correspondence with those signatures τ for which $l(\tau)=|\tau|=n$. Two signatures $\tau=(t_1\geqq t_2\geqq ...)$ and $\tau'=$ $=(t_1'\geqq t_2'\geqq ...)$ are *essentially equivalent* if $\iota(\tau)=\iota(\tau')$ and $t_j=t_j'$ for $j=1, 2, ..., \iota(\tau)$.

For a signature τ with $l(\tau)=\dim\mathfrak{K}$, $|\tau|<\infty$, there corresponds an irreducible representation ϱ_τ of $\mathscr{U}(\mathfrak{K})$ on a Hilbert space $\mathfrak{K}^\tau$ (these are the irreducible representations of "positive" signatures; cf. [18], [6] for the case dim $\mathfrak{K}<\infty$ and [11] for the extension to the case dim $\mathfrak{K}=\infty$).

The representation ϱ_τ can be defined as follows: consider $\tilde{\tau}$, the signature of length $|\tau|$ essentially equivalent to τ, and let $q_{\tilde{\tau}}$ be any minimal projection in $C^*(\mathfrak{S}_{|\tau|})$ such that $q_{\tilde{\tau}}\leqq p_{\tilde{\tau}}$. Then ϱ_τ is defined as the restriction of $\Gamma_{|\tau|}$ to $\pi_{|\tau|}(q_{\tilde{\tau}})\mathfrak{K}^{\otimes|\tau|}$. Clearly

ϱ_τ extends to a multiplicative homomorphism of the multiplicative semigroup $\mathscr{L}(\mathfrak{K})$ which is holomorphic. Also clearly the restriction of $\Gamma_{|\tau|}$ to $\pi_{|\tau|}(p_\tau)\mathfrak{K}^{\otimes|\tau|}$ is a finite multiple of ϱ_τ.

Another classical fact we need is that for τ with $l(\tau)=|\tau|=n$ we have $\pi_n(p_\tau)\neq 0$ if and only if $\iota(\tau)\leqq\dim\mathfrak{K}$.

§ 2. Tensor operations on operator-valued functions

Let $\mathfrak{K}$ be a Hilbert space. For any $k\in\mathfrak{K}$ we shall consider the map $T_k\colon \mathfrak{K}^{\otimes n}\to\mathfrak{K}^{\otimes(n+1)}$ defined by

$$T_k(k_1\otimes k_2\otimes\ldots\otimes k_n) = k_1\otimes\ldots\otimes k_n\otimes k. \tag{2.1}$$

Clearly T_k is proportional to an isometry and

$$T_k^*(k_1\otimes\ldots\otimes k_{n+1}) = \langle k_{n+1}, k\rangle k_1\otimes\ldots\otimes k_n. \tag{2.2}$$

Lemma 2.1. *Consider two signatures $\tau'\prec\tau$, $l(\tau')=|\tau'|=n$, $l(\tau)=|\tau|=n+1$ such that $\iota(\tau)\leqq\dim\mathfrak{K}$. Then we have:*

$$\bigvee_{k\in\mathfrak{K}} \pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau)\quad \mathfrak{K}^{\otimes(n+1)} = \pi_n(p_{\tau'})\mathfrak{K}^{\otimes n}. \tag{2.3}$$

Proof. Let us denote by $\mathfrak{F}$ the space on the left hand side of (2.3). Then $\mathfrak{F}$ is $\pi_n(\mathfrak{S}_n)$-invariant and $\Gamma_n(\mathscr{U}(\mathfrak{K}))$-invariant.

Indeed, for $\sigma\in\mathfrak{S}_n$ we have

$$\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau)\pi_{n+1}(\sigma) = \pi_n(\sigma)\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau),$$

since $p_{\tau'}, p_\tau$ commute with $C^*(\mathfrak{S}_n)$ and $T_k\pi_n(\sigma)=\pi_{n+1}(\sigma)T_k$. Also,

$$\mathfrak{K}^{\otimes n}\ominus\mathfrak{F} = \bigcap_{k\in\mathfrak{K}} \operatorname{Ker}[\pi_{n+1}(p_\tau)T_k\pi_n(p_{\tau'})]$$

and for any $U\in\mathscr{U}(\mathfrak{K})$ we have

$$\Gamma_n(U)\operatorname{Ker}[\pi_{n+1}(p_\tau)T_k\pi_n(p_{\tau'})] = \operatorname{Ker}[\pi_{n+1}(p_\tau)T_{Uk}\pi_n(p_{\tau'})]$$

so that $\mathfrak{K}^{\otimes n}\ominus\mathfrak{F}$ is invariant for $\Gamma_n(\mathscr{U}(\mathfrak{K}))$ and hence so is $\mathfrak{F}$.

Therefore $P_{\mathfrak{F}}\in(\pi_n(C^*(\mathfrak{S}_n))\cup\Gamma_n(\mathscr{U}(\mathfrak{K})))'$ and $P_{\mathfrak{F}}\leqq\pi_n(p_{\tau'})$. Hence by Hermann Weyl's theorem and because of the minimality of $p_{\tau'}$ in the center of $C^*(\mathfrak{S}_n)$ either $P_{\mathfrak{F}}=0$ or $P_{\mathfrak{F}}=\pi_n(p_{\tau'})$. So it will be sufficient to prove that $\mathfrak{F}\neq\{0\}$.

Observe that $\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau)=T_k^*\pi_{n+1}(p_{\tau'}p_\tau)$. On the other hand, p_τ is the central support of $p_{\tau'}p_\tau$ in $C^*(\mathfrak{S}_{n+1})$ as explained in the next paragraph. Thus, from $\pi_{n+1}(p_\tau)\neq 0$ we infer $\pi_{n+1}(p_{\tau'}p_\tau)\neq 0$. Now $\bigcap_{k\in K}\operatorname{Ker}T_k^*=\{0\}$ so we can find $k\in\mathfrak{K}$ such that $T_k^*\pi_{n+1}(p_{\tau'}p_\tau)\neq 0$.

If ϱ is an irreducible representation of the finite-dimensional C^*-algebra A, there is a minimal central projection p of A such that $\ker \varrho=(1-p)A$. Let $A_1\subset A_2$ be finite dimensional C^*-algebras with $1_{A_2}\in A_1$, ϱ_i irreducible representations of A_i, and p_i the corresponding minimal central projection of A_i $(i=1,2)$. Then $\varrho_2|A_1$ contains ϱ_1 if and only if $p_1p_2\neq 0$. Indeed, if $\varrho_2|A_1$ contains ϱ_1 we obviously have $\ker(\varrho_2|A_1)\subset\ker\varrho_1$, so that $p_1p_2\neq 0$ (since $p_1\notin\ker\varrho_1$). Conversely, if $p_1p_2\neq 0$ the two-sided ideal $J=\{x\in A_1;\ p_1p_2x=0\}$ of A_1 contains $\ker\varrho_1$ and $p_1\notin J$. Since ϱ_1 is irreducible and A_1 is finite-dimensional, $\ker\varrho_1$ is a maximal ideal of A_1, so that $J=\ker\varrho_1$. It follows that $\ker(\varrho_2|A_1)\subset\ker\varrho_1$ and this in turn implies that $\varrho_2|A_1$ contains ϱ_1.

This completes the proof.

Lemma 2.2. *Consider two signatures* $\tau'\prec\tau$, $l(\tau')=|\tau'|=n$, $l(\tau)=|\tau|=n+1$, *such that* $\iota(\tau)\leqq\dim\mathfrak{K}$ *and let* $\Theta\in H^\infty(\mathscr{L}(\mathfrak{K}))$. *For any* $k\in\mathfrak{K}$ *we have:*

$$\left((\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau))\otimes I_{H^2}\right)\Gamma_{n+1}(\Theta)H^2(\mathfrak{K}^{\otimes(n+1)})\subset \tag{2.4}$$
$$\subset\left((\pi_n(p_{\tau'})\otimes I_{H^2})\Gamma_n(\Theta)H^2(\mathfrak{K}^{\otimes n})\right)^-.$$

Proof. Clearly both terms of (2.4) are invariant with respect to multiplication operators by scalar H^∞-functions. Hence it is easily seen that it will be enough to prove that a function of the form

$$z\to\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau)\left(\Theta(z)k_1\otimes\ldots\otimes\Theta(z)k_{n+1}\right)$$

is in

$$\left(\pi_n(p_{\tau'})\otimes I_{H^2}\right)\Gamma_n(\Theta)H^2(\mathfrak{K}^{\otimes n}).$$

Writing $p_\tau=\sum_{\sigma\in\mathfrak{S}_{n+1}}c_\sigma\sigma$ the assertion becomes obvious from the following computation:

$$\pi_n(p_{\tau'})T_k^*\pi_{n+1}(p_\tau)\left(\Theta(z)k_1\otimes\ldots\otimes\Theta(z)k_{n+1}\right)=$$
$$=\pi_n(p_{\tau'})T_k^*\sum_{\sigma\in\mathfrak{S}_{n+1}}c_\sigma\left(\Theta(z)k_{\sigma^{-1}(1)}\otimes\ldots\otimes\Theta(z)k_{\sigma^{-1}(n+1)}\right)=$$
$$=\sum_{\sigma\in\mathfrak{S}_{n+1}}c_\sigma\langle\Theta(z)k_{\sigma^{-1}(n+1)},k\rangle\pi_n(p_{\tau'})\Gamma_n(\Theta(z))\left(k_{\sigma^{-1}(1)}\otimes\ldots\otimes k_{\sigma^{-1}(n)}\right).$$

Let us now consider $\Theta\in H^\infty(\mathscr{L}(\mathfrak{K}))$ and let τ be a signature with $|\tau|<\infty$ and $l(\tau)=\dim\mathfrak{K}$. Consider also $\tilde{\tau}$, the signature of length $|\tau|$ essentially equivalent to τ. We define an inner function $d^\tau(\Theta)$ by

$$d^\tau(\Theta)=\wedge\{m\in H_i^\infty\,|\,mH^2(\mathfrak{K}^\tau)\subset(\varrho_\tau(\Theta)H^2(\mathfrak{K}^\tau))^-\} \tag{2.5}$$

(by convention we put $\wedge\varnothing=0$, $\varnothing$-the empty set).

Remark that in case Θ is an inner function, $\varrho_\tau(\Theta)$ is still an inner function and $d^\tau(\Theta)$ is the minimal function of $S(\varrho_\tau(\Theta))$ in case $\varrho_\tau(\Theta)$ has a scalar multiple and zero otherwise. In case τ is of the form $(1,1,\ldots,1,0,\ldots)$ with j nonzero terms, that is, ϱ_τ is the representation in antisymmetric tensors of degree j, we shall use the notation $d_j(\Theta)$ for $d^\tau(\Theta)$.

Since the restriction of $\Gamma_{|\tau|}$ to $\pi_{|\tau|}(p_{\tilde{\tau}})\mathfrak{K}^{\otimes|\tau|}$ is a multiple of ϱ_τ, we have

$$(2.6)\qquad d^\tau(\Theta) = \wedge\{m\in H_i^\infty | mH^2(\pi_{|\tau|}(p_{\tilde{\tau}})\mathfrak{K}^{\otimes|\tau|}) \subset \subset (\Gamma_{|\tau|}(\Theta)H^2(\pi_{|\tau|}(p_{\tilde{\tau}})\mathfrak{K}^{\otimes|\tau|}))^-\}.$$

For the next lemma let τ', τ be signatures with $|\tau'|=n$, $|\tau|=n+1$ (n finite), $l(\tau')=l(\tau)=\dim\mathfrak{K}$ and such that denoting by $\tilde{\tau}'$ and $\tilde{\tau}$ the signatures of length n, $n+1$, essentially equivalent to τ', τ, respectively, we have

Lemma 2.3. *For Θ in $H^\infty(\mathscr{L}(\mathfrak{K}))$ and τ', τ as above, $d^{\tau'}(\Theta)$ divides $d^\tau(\Theta)$.*

Proof. Consider $m\in H_i^\infty$ such that

$$mH^2(\pi_{n+1}(p_{\tilde{\tau}})\mathfrak{K}^{\otimes(n+1)}) \subset (\Gamma_{n+1}(\Theta)H^2(\pi_{n+1}(p_{\tilde{\tau}})\mathfrak{K}^{\otimes(n+1)}))^-.$$

It follows from Lemma 2.2. that

$$m\Big(\bigvee_{k\in\mathfrak{K}}((\pi_n(p_{\tilde{\tau}'})T_k^*\pi_{n+1}(p_{\tilde{\tau}}))\otimes I_{H^2})H^2(\mathfrak{K}^{n+1})\Big)^- \subset$$

$$\subset ((\pi_n(p_{\tilde{\tau}'})\otimes I_{H^2})\Gamma_n(\Theta)H^2(\mathfrak{K}^{\otimes n}))^- = (\Gamma_n(\Theta)H^2(\pi_n(p_{\tilde{\tau}'})\mathfrak{K}^{\otimes n}))^-$$

and hence by Lemma 2.1

$$mH^2(\pi_n(p_{\tilde{\tau}'})\mathfrak{K}^{\otimes n}) \subset (\Gamma_n(\Theta)H^2(\pi_n(p_{\tilde{\tau}'})\mathfrak{K}^{\otimes n}))^-$$

so that by (2.6) $d^{\tau'}$ divides m. Q.E.D.

Let us also record the following simple fact for further use.

Remark 2.4. Let $\mathfrak{X}_i, \mathfrak{Y}_i$ $(i=1,2)$ be Hilbert spaces, $A_i\in H^\infty(\mathscr{L}(\mathfrak{X}_i, \mathfrak{Y}_i))$, $B\in H^\infty(\mathscr{L}(\mathfrak{X}_1\otimes\mathfrak{X}_2, \mathfrak{Y}_1\otimes\mathfrak{Y}_2))$, $B(z)=A_1(z)\otimes A_2(z)$ $(z\in D)$ and suppose $f_i\in(A_iH^2(\mathfrak{X}_i))^-\cap \cap H^\infty(\mathfrak{Y}_i)$. Then we have $f_1\otimes f_2\in(BH^2(\mathfrak{X}_1\otimes\mathfrak{X}_2))^-$. Indeed, consider $h_i^{(n)}\in H^\infty(\mathfrak{X}_i)$ such that

$$\lim_{n\to\infty}\|A_ih_i^{(n)}-f_i\| = 0 \quad \text{in} \quad H^2(\mathfrak{Y}_i).$$

Then in

$$H^2(\mathfrak{Y}_1\otimes\mathfrak{Y}_2)$$

we have

$$\lim_{n\to\infty}\|B(h_1^{(n)}\otimes h_2^{(m)})-f_1\otimes A_2h_2^{(m)}\| = 0$$

and

$$\lim_{m\to\infty}\|f_1\otimes f_2-f_1\otimes A_2h_2^{(m)}\| = 0$$

which is the desired result.

For the following theorem consider $\Theta \in H^\infty(\mathscr{L}(\mathfrak{K}))$, $\Theta' \in H^\infty(\mathscr{L}(\mathfrak{K}'))$ and suppose there are $A \in H^\infty(\mathscr{L}(\mathfrak{K}, \mathfrak{K}'))$, $B \in H^\infty(\mathscr{L}(\mathfrak{K}', \mathfrak{K}))$ such that the following set of relations holds

$$(2.7)\qquad \begin{cases} A\Theta H^2(\mathfrak{K}) \subset (\Theta' H^2(\mathfrak{K}'))^-, \\ B\Theta' H^2(\mathfrak{K}') \subset (\Theta H^2(\mathfrak{K}))^-, \\ BAH^2(\mathfrak{K}) \vee \Theta H^2(\mathfrak{K}) = H^2(\mathfrak{K}). \end{cases}$$

Theorem 2.5. *Let Θ, Θ', A, B be as before and suppose* (2.7) *holds. Let further τ, τ' be essentially equivalent signatures with $l(\tau) = \dim \mathfrak{K}$, $l(\tau') = \dim \mathfrak{K}'$, $|\tau| < \infty$ and $\iota(\tau) = \iota(\tau') \leqq \min(\dim \mathfrak{K}, \dim \mathfrak{K}')$. Then $d^\tau(\Theta)$ divides $d^{\tau'}(\Theta')$.*

Proof. If $d^{\tau'}(\Theta') = 0$, the assertion of the theorem is obvious, so assume $d^{\tau'}(\Theta') = m \in H_i^\infty$. Let $\tilde{\tau}$ denote the signature of length $n = |\tau|$ that is essentially equivalent to τ.

Consider $f_1, f_2, \dots, f_n \in H^\infty(\mathfrak{K})$, $g_1, g_2, \dots, g_n \in H^\infty(\mathfrak{K})$ and

$$(2.8)\qquad s = (\pi_n(p_{\tilde{\tau}}) \otimes I_{H^2})\big((BAf_1 + \Theta g_1) \otimes \dots \otimes (BAf_n + \Theta g_n)\big).$$

Using (2.7) it is easily seen that the elements s form a total subset of $H^2(\pi_n(p_{\tilde{\tau}})\mathfrak{K}^{\otimes n})$, so that it will be sufficient to prove that

$$(2.9)\qquad ms \in \big(\Gamma_n(\Theta) H^2(\pi_n(p_{\tilde{\tau}})\mathfrak{K}^{\otimes n})\big)^-.$$

Now, s is a finite sum of elements of the form

$$(2.10)\qquad r = \big((\pi_n(p_{\tilde{\tau}})\pi_n(\sigma)) \otimes I_{H^2}\big)(BAf_1' \otimes \dots \otimes BAf_j' \otimes \Theta g_1' \otimes \dots \otimes \Theta g_{n-j}')$$

where $0 \leqq j \leqq n$, $\sigma \in \mathfrak{S}_n$ and f_i', g_i' are some of the f' and g. Thus to prove (2.9) it will be enough to show that

$$(2.11)\qquad mr \in \big(\Gamma_n(\Theta) H^2(\pi_n(p_{\tilde{\tau}})\mathfrak{K}^{\otimes n})\big)^-.$$

Because $\sum_{\gamma \in \hat{\mathfrak{S}}_j} p_\gamma = 1$ and $\mathfrak{S}_j$ is considered as a subgroup of $\mathfrak{S}_n$ $(j \leqq n)$, we have $\sum_{\gamma \in \hat{\mathfrak{S}}_j} p_{\tilde{\tau}} p_\gamma = p_{\tilde{\tau}}$ and $p_{\tilde{\tau}} p_\gamma \neq 0$ if and only if the restriction of the representation of signature $\tilde{\tau}$ to $\mathfrak{S}_j$ contains the representation of signature γ. So, $p_{\tilde{\tau}} p_\gamma \neq 0$ if and only if there are $\gamma_k \in \hat{\mathfrak{S}}_k$ $(j < k < n)$ such that

$$(2.12)\qquad \gamma \prec \gamma_{j+1} \prec \dots \prec \gamma_{n-1} \prec \tilde{\tau}.$$

Hence denoting by $\check{\gamma}$ the signature of length $\dim \mathfrak{K}'$ that is essentially equivalent to γ, using Lemma 2.3 several times we conclude that $d^{\check{\gamma}}(\Theta')$ divides $d^{\tau'}(\Theta') = m$.

Now we have:

$$mr =$$

$$= (\pi_n(p_{\tilde{\tau}})\pi_n(\sigma) \otimes I_{H^2}) \sum_{\substack{\gamma \in \hat{\mathfrak{S}}_j \\ p_{\tilde{\tau}} p_\gamma \neq 0}} \big(m\,(\pi_j(p_\gamma) \otimes I_{H^2})\,(BAf_1' \otimes \dots \otimes BAf_j')\big) \otimes (\Theta g_1' \otimes \dots \otimes \Theta g_{n-j}').$$

To end the proof it will be sufficient to show that

$$m(\pi_j(p_\gamma)\otimes I_{H^2})(BAf_1'\otimes\ldots\otimes BAf_j') \quad \text{is in} \quad (\Gamma_j(\Theta)H^2(\mathfrak{K}^{\otimes j}))^-,$$

because then using Remark 2.4 we will have that mr is in

$$((\pi_n(p_{\bar{\gamma}})\pi_n(\sigma)\otimes I_{H^2})\Gamma_n(\Theta)H^2(\mathfrak{K}^{\otimes n}))^- = (\Gamma_n(\Theta)H^2(\pi_n(p_{\bar{\gamma}})\mathfrak{K}^{\otimes n}))^-$$

which is the desired result.

Now further $m(\pi_j(p_\gamma)\otimes I_{H^2})(Af_1'\otimes\ldots\otimes Af_j')$ is in $d^{\bar{\gamma}}$ $(\Theta')H^2(\pi_j(p_\gamma)\mathfrak{K}'^{\otimes j})$, since $d^{\bar{\gamma}}(\Theta')$ divides m, and hence is in $(\Gamma_j(\Theta')H^2(\pi_j(p_\gamma)\mathfrak{K}'^{\otimes j}))^-\subset(\Gamma_j(\Theta')H^2(\mathfrak{K}'^{\otimes j}))^-$. Thus it will be sufficient to prove that

$$(\Gamma_j(B)\Gamma_j(\Theta')H^2(\mathfrak{K}'^{\otimes j}))^- \subset (\Gamma_j(\Theta)H^2(\mathfrak{K}^{\otimes j}))^-$$

in order that

$$m(\pi_j(p_\gamma)\otimes I_{H^2})(BAf_1'\otimes\ldots\otimes BAf_j')\in(\Gamma_j(\Theta)H^2(\mathfrak{K}^{\otimes j}))^-.$$

To this end remark that the elements of the form $B\Theta'h_1\otimes\ldots\otimes B\Theta'h_j$ with $h_i\in H^\infty(\mathfrak{K}')$ are total in $(\Gamma_j(B)\Gamma_j(\Theta')H^2(\mathfrak{K}'^{\otimes j}))^-$ and

$$B\Theta'h_1\otimes\ldots\otimes B\Theta'h_j\in(\Gamma_j(\Theta)H^2(\mathfrak{K}^{\otimes j}))^-$$

because fo (2.7) and Remark 2.4. Q.E.D.

§ 3. Applications to quasi-similar C_0 operators

The following Proposition is an easy application of Theorem 2.5.

Proposition 3.1. *Let $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{K}))$, $\Theta'\in H_i^\infty(\mathscr{L}(\mathfrak{K}'))$ and let τ,τ' be essentially equivalent signatures with $l(\tau)=\dim\mathfrak{K}$, $l(\tau')=\dim\mathfrak{K}'$ and $\iota(\tau)=\iota(\tau')\leqq\leqq\min(\dim\mathfrak{K},\dim\mathfrak{K}')$. If $S(\Theta)$ and $S(\Theta')$ are quasi-similar, we have*

$$d^\tau(\Theta) = d^{\tau'}(\Theta'). \tag{3.1}$$

Proof. Let X and Y be two quasi-affinities such that $S(\Theta')X=XS(\Theta)$ and $S(\Theta)Y=YS(\Theta')$. From the lifting theorem (see (1.8—11)) it follows that we can find $A\in H^\infty(\mathscr{L}(\mathfrak{K},\mathfrak{K}'))$ and $B\in H^\infty(\mathscr{L}(\mathfrak{K}',\mathfrak{K}))$ such that

$$X = P_{\mathfrak{H}(\Theta')}A|\mathfrak{H}(\Theta),\quad Y = P_{\mathfrak{H}(\Theta)}B|\mathfrak{H}(\Theta'), \tag{3.2}$$

$$A\Theta H^2(\mathfrak{K})\subset\Theta'H^2(\mathfrak{K}'),\quad B\Theta'H^2(\mathfrak{K}')\subset\Theta H^2(\mathfrak{K}) \tag{3.3}$$

and

$$ABH^2(\mathfrak{K}')\vee\Theta'H^2(\mathfrak{K}') = H^2(\mathfrak{K}'),\quad BAH^2(\mathfrak{K})\vee\Theta H^2(\mathfrak{K}) = H^2(\mathfrak{K}) \tag{3.4}$$

so that the assumptions of Theorem 2.5 are satisfied. It follows that $d^\tau(\Theta)$ divides $d^{\tau'}(\Theta')$ and $d^{\tau'}(\Theta')$ divides $d^\tau(\Theta)$ and this proves (3.1). Q.E.D.

Let T be any operator unitarily equivalent to some $S(\Theta)$ with a pure $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{K}))$. It is easy to see that the functions $d^\tau(\Theta)$ and $d_j(\Theta)$ depend only on T and not on the particular function Θ, so we shall denote them by $d^\tau(T)$ and $d_j(T)$, respectively.

Corollary 3.2. *If T and T' are two quasisimilar C_0 operators and $\mathfrak{d}_T = \mathfrak{d}_{T'}$, then $d^\tau(T) = d^\tau(T')$ for each τ with $l(\tau) = \mathfrak{d}_T$.*

Proof. T and T' are unitarily equivalent to $S(\Theta)$ and $S(\Theta')$, respectively, where $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{K}))$, $\Theta' \in H_i^\infty(\mathscr{L}(\mathfrak{K}'))$ with $\dim \mathfrak{K} = \dim \mathfrak{K}' = \mathfrak{d}_T$. The corollary obviously follows from Proposition 3.1. Q.E.D.

Consider now a C_0 operator T with Jordan model $S = S(m_1) \oplus S(m_2) \oplus \ldots$. If $\mathfrak{d}_S < \mathfrak{d}_T$ we shall put $m_j = 1$ for $\mathfrak{d}_S < j \leq \mathfrak{d}_T$. So we have

$$S = \bigoplus_{j=1}^{\mathfrak{d}_T} S(m_j). \tag{3.5}$$

Corollary 3.3. *For any C_0 operator T and any signature $\tau = (t_1 \geqq t_2 \geqq \ldots)$, $|\tau| < \infty$, $l(\tau) = \mathfrak{d}_T$, we have*

$$d^\tau(T) = m_1^{t_1} m_2^{t_2}, \ldots, m_n^{t_n}, \quad n = \iota(\tau). \tag{3.6}$$

Proof. We have only to apply Proposition 3.1 to Θ coinciding with the characteristic function of T and to

$$\Theta' = \operatorname{diag}(m_1, m_2, \ldots) \in H_i^\infty(\mathscr{L}(\mathfrak{K}')) \quad \text{with} \quad \dim \mathfrak{K}' = \mathfrak{d}_T.$$

Since $\tau = (t_1 \geqq t_2 \geqq \ldots)$ represents the highest weight to the representation ϱ_τ (see [18], [6] to the finite-dimensional and [1] for the infinite-dimensional case) it is immediate that:

$$d^\tau(\Theta') = m_1^{t_1}, \ldots, m_n^{t_n}.$$

Q.E.D.

Corollary 3.4. *For any C_0 operator T, the functions m_j appearing in the Jordan model can be computed as*

$$m_j = d_j(T)/d_{j-1}(T), \quad 1 \leqq j \leqq \mathfrak{d}_T \quad \textit{where} \quad d_0(T) = 1. \tag{3.7}$$

Proof. The preceding Corollary gives for $\tau_j = (1, \ldots, 1, 0, \ldots)$ (with j nonzero terms)

$$d_j(T) = d^{\tau_j}(T) = m_1 \ldots m_j, \quad j \leqq \mathfrak{d}_T$$

so relation (3.7) becomes obvious. Q.E.D.

Since the quasisimilarity class of a C_0 operator is determined by the Jordan model, Corollary 3.4 shows that a C_0 operator T is determined up to quasisimilarity by the least inner multiples of the exterior powers of any function coinciding with the characteristic function of T. This enables us to prove the following theorem.

Theorem 3.5. *Let* $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{K}))$, $\Theta'\in H_i^\infty(\mathscr{L}(\mathfrak{K}'))$ *be such that* $d_1(\Theta)\neq 0$ *and* $\dim\mathfrak{K}=\dim\mathfrak{K}'$. *If* $S(\Theta)$ *and* $S(\Theta')$ *are quasisimilar then* $S(\varrho_\tau(\Theta))$ *and* $S(\varrho_\tau(\Theta'))$ *are quasisimilar for each signature* τ *such that* $l(\tau)=\dim\mathfrak{K}$, $|\tau|<\infty$.

Proof. By Corollary 3.4 we have only to show that $d_j(\varrho_\tau(\Theta))=d_j(\varrho_\tau(\Theta'))$ for each $j\leq\dim\mathfrak{K}^\tau$. Let $\tau_j=(1,1,\dots,1,0,\dots)$ (with j nonzero terms), $l(\tau_j)=\dim\mathfrak{K}^\tau$.

The representation $\varrho_{\tau_j}\circ\varrho_\tau$ of $\mathscr{U}(\mathfrak{K})$ is a subrepresentation of the representation of $\mathscr{U}(\mathfrak{K})$ on $\mathfrak{K}^{\otimes j|\tau|}$ and hence a finite direct sum of representations $\varrho_{\tau'}$, with $l(\tau')=$ $=\dim\mathfrak{K}$, $|\tau'|<\infty$:

$$\varrho_{\tau_j}\circ\varrho_\tau=\bigoplus_{\tau'}\varrho_{\tau'}. \tag{3.8}$$

From (3.8) it follows then that

$$\varrho_{\tau_j}(\varrho_\tau(\Theta))=\bigoplus_{\tau'}\varrho_{\tau'}(\Theta),\quad \varrho_{\tau_j}(\varrho_\tau(\Theta'))=\bigoplus_{\tau'}\varrho_{\tau'}(\Theta')$$

and hence $d_j(\varrho_\tau(\Theta))$ is the least inner common multiple of the $d^{\tau'}(\Theta)$ and $d_j(\varrho_\tau(\Theta'))$ the least inner common multiple of the $d^{\tau'}(\Theta')$. Since $d^{\tau'}(\Theta)=d^{\tau'}(\Theta')$ by Proposition 3.1, we infer that $d_j(\varrho_\tau(\Theta))=d_j(\varrho_\tau(\Theta'))$. Q.E.D.

§ 4. Defect operators of C_0 contractions

For an operator $A\in\mathscr{L}(\mathfrak{K})$ and a closed subspace $\mathfrak{M}\subset\mathfrak{K}$ we consider

$$\gamma[A,\mathfrak{M}]=\inf_{\substack{k\in\mathfrak{M}\\ \|k\|=1}}\|Ak\|,\quad \gamma_j(A)=\sup_{\operatorname{codim}\mathfrak{M}=j-1}\gamma[A,\mathfrak{M}].$$

As is known from the minimax principle, $\gamma_j(A)$ $(1\leq j\leq\dim\mathfrak{K})$ are eigenvalues of $(A^*A)^{1/2}$ in increasing order. In case $\dim\mathfrak{K}<\infty$ all eigenvalues of $(A^*A)^{1/2}$ repeated according to their multiplicity appear in the sequence of the $\gamma_j(A)$. In case $\dim\mathfrak{K}=\infty$, $\gamma_1(A)$ is the least eigenvalue of $(A^*A)^{1/2}$, discrete eigenvalues smaller than the least essential eigenvalue appear in increasing order repeated according to their multiplicity and the sequence becomes stationary if the least essential eigenvalue of $(A^*A)^{1/2}$ is reached.

For the next two lemmas, τ_j denotes the signature

$$\tau_j=(1,\dots,1,0\dots),\quad l(\tau_j)=\dim\mathfrak{K},\quad \iota(\tau_j)=j.$$

Lemma 4.1. *Let* $A\in\mathscr{L}(\mathfrak{K})$ *and* τ_j *be as above. Then we have:*

$$\gamma_1(\varrho_{\tau_j}(A))=\gamma_1(A)\gamma_2(A)\dots\gamma_j(A). \tag{4.1}$$

Proof. Remark first that applying ϱ_{τ_j} to the polar decomposition of A we get the polar decomposition of $\varrho_{\tau_j}(A)$, so we can suppose A is positive. Moreover, in view of the minimax definition of γ_j, we have $|\gamma_j(A)-\gamma_j(B)|\leqq\|A-B\|$, and thus by continuity it will be sufficient to consider the case when $A\geqq 0$ has finite spectrum.

In this case, ϱ_{τ_j} being the representation in antisymmetric tensors of degree j, $\varrho_{\tau_j}(A)$ has finite spectrum, the eigenvalues being products $\lambda_1\ldots\lambda_j$ of eigenvalues of A, a given eigenvalue appearing in such a product at most a number of times equal to its multiplicity. Clearly $\gamma_1(A)\ldots\gamma_j(A)$ is then the least eigenvalue of $\varrho_{\tau_j}(A)$.

Q.E.D.

Lemma 4.2. *Let T be a C_0 operator, let $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{K}))$ coincide with the characteristic function of T and let $\{m_j\}_{j=1}^{\mathfrak{d}_T}$ be inner functions for the Jordan model of T with $m_j\equiv 1$ for $\mu_T<j\leqq\mathfrak{d}_T$. Then we have*

$$\gamma_1(\Theta(\lambda))\ldots\gamma_j(\Theta(\lambda))\geqq|m_1(\lambda)\ldots m_j(\lambda)| \tag{4.2}$$

where $1\leqq j\leqq\mathfrak{d}_T$ and $\lambda\in D$.

Proof. In view of Corollary 3.3, $m_1\ldots m_j$ is the least inner multiple of $\varrho_{\tau_j}(\Theta)\in H_i^\infty(\mathscr{L}(\mathfrak{K}^{\tau_j}))$. Hence there is a contractive function $\Omega\in H^\infty(\mathscr{L}(\mathfrak{K}^{\tau_j}))$ such that

$$\Omega(\lambda)\varrho_{\tau_j}(\Theta(\lambda)) = m_1(\lambda)\ldots m_j(\lambda)I_{\mathfrak{K}^{\tau_j}}.$$

Since $\|\Omega(\lambda)\|\leqq 1$ this clearly implies

$$\gamma_1(\varrho_{\tau_j}(\Theta(\lambda)))\geqq|m_1(\lambda)\ldots m_j(\lambda)|$$

and by Lemma 4.1

$$\gamma_1(\varrho_{\tau_j}(\Theta(\lambda)))\leqq\gamma_1(\Theta(\lambda))\ldots\gamma_j(\Theta(\lambda)),$$

which gives the desired inequality. Q.E.D.

Proposition 4.3. *Let T be a C_0 operator acting on $\mathfrak{H}$ and $\{m_j\}_{j=1}^\infty$ inner functions for the Jordan model of T, with $m_j\equiv 1$ in case $\mu_T<j$.*

a) *If* $\sum_{j=1}^\infty(1-|m_j(0)|)<\infty$, *then* $\operatorname{tr}(I-T^*T)<\infty$.

b) *If* $\lim_{j\to\infty}|m_j(0)|=1$, *then $I-T^*T$ is compact.*

Proof. a) The assumptions are that the Jordan model $S=S(m_1)\oplus S(m_2)\oplus\ldots$ is a weak contraction ([13] ch. VIII) since $\operatorname{tr}(I-S^*S)=\sum_{j=1}^\infty(1-|m_j(0)|^2)\leqq\leqq 2\sum_{j=1}^\infty(1-|m_j(0)|)<\infty$. As usual for weak contractions there will be no loss of generality to assume that $m_j(0)\neq 0$ (one uses a conformal automorphism of the unit disc as in [13] ch. VIII). Thus the infinite product $\prod_{j=1}^\infty|m_j(0)|$ converges to some $c>0$. Hence by Lemma 4.2 for Θ the characteristic function of T, we infer that

$$\prod_{1\leqq j\leqq\mathfrak{d}_T}\gamma_j(\Theta(0))>0.$$

Since in case $\mathfrak{d}_T=\infty$ this implies $\lim_{j\to\infty} \gamma_j(\Theta(0))=1$, it follows that

$$\operatorname{tr}(I_{\mathfrak{D}_T}-\Theta(0)^*\Theta(0)) = \sum_{1\leq j\leq \mathfrak{d}_T} (1-\gamma_j(\Theta(0))^2)$$

and

$$\sum_{1\leq j\leq \mathfrak{d}_T} (1-\gamma_j(\Theta(0))^2) < \infty$$

since

$$\prod_{1\leq j\leq \mathfrak{d}_T} \gamma_j(\Theta(0)) > 0. \quad \text{But} \quad I_{\mathfrak{D}_T}-\Theta(0)^*\Theta(0) = D_T^2|\mathfrak{D}_T,$$

so that $\operatorname{tr}(I-T^*T)<\infty$.

b) The proof is quite similar to that of a), so we can be brief in details. Again we may suppose T is invertible and hence $m_j(0)\neq 0$. Then $\lim_{j\to\infty} |m_j(0)|=1$ gives

$$\lim_{j\to\infty} |m_1(0)\dots m_j(0)|^{1/j} = 1.$$

Using Lemma 4.2 this implies

$$\lim_{j\to\infty} (\gamma_1(\Theta(0))\dots\gamma_j(\Theta(0)))^{1/j} = 1$$

so that $\lim_{j\to\infty} \gamma_j(\Theta(0))=1$ which gives that $I-T^*T$ is compact.

Q.E.D.

Remark 4.4. As we shall see in § 8 the converse of 4.3 a) is also true. For 4.3 b) the converse is in general false. An example can be constructed as follows.

Let μ be a finite non-negative measure on $[0, 2\pi]$, singular with respect to Lebesgue measure and without atoms. Consider the inner functions

$$m_{j,n}(\lambda) = \exp\left[-\int_{2\pi(j-1)/n}^{2\pi j/n} \frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\mu(t)\right], \quad 1\leq j\leq n$$

and the operators

$$T=\bigoplus_{n=1}^{\infty}\left(\bigoplus_{j=1}^{n} S(m_{j,n})\right), \quad S=S(m_{1,1})\oplus S(m_{1,1})\oplus\dots.$$

Then S is the Jordan model of T, $I-T^*T$ is compact and $|m_{1,1}(0)|$, $|m_{1,1}(0)|$, … tends to $|m_{1,1}(0)|<1$.

Proposition 4.5. *Let T be a C_0 operator, let $\{m_j\}_{j=1}^{\infty}$ be inner functions for the Jordan model of T ($m_j\equiv 1$ in case $\mu_T<j$) and let $\Theta\in H_i^\infty(\mathcal{L}(\mathfrak{K}))$ coincide with the characteristic function of T. Suppose moreover $m_1(0)\neq 0$ and $n\in\mathbf{N}$ is such that $|m_n(0)|<\lim_{j\to\infty}|m_j(0)|$. Then the following conditions are equivalent:*

(i) $|m_1(0)\dots m_n(0)|=\gamma_1(\Theta(0))\dots\gamma_n(\Theta(0))$,

(ii) *T is unitarily equivalent to $T_1\oplus T_2$, where $\mathfrak{d}_{T_1}=n$ and T_1, T_2 are quasisimilar to $S(m_1)\oplus\dots\oplus S(m_n)$ and respectively to $S(m_{n+1})\oplus S(m_{n+2})\oplus\dots$.*

Proof. (i)⇒(ii). The condition $0 \neq |m_n(0)| < \lim_{j\to\infty} |m_j(0)|$ implies that $\gamma_n(\Theta(0))$ is less than the least essential eigenvalue of $(\Theta(0)^*\Theta(0))^{1/2}$, for otherwise we would have $\gamma_n(\Theta(0)) = \gamma_{n+1}(\Theta(0)) = \dots$ which in view of Lemma 4.2 would imply $\lim_{j\to\infty} |m_j(0)| \leqq \gamma_n(\Theta(0))$ and hence $|m_n(0)| < \gamma_n(\Theta(0))$ which when combined with (i) would give $|m_1(0) \dots m_{n-1}(0)| > \gamma_1(\Theta(0)) \dots \gamma_{n-1}(\Theta(0))$, contradicting Lemma 4.2. Thus replacing Θ by some equivalent inner operator-valued function in $H_i^\infty(\mathscr{L}(\mathfrak{K}))$ we may assume there is an orthonormal set $\{e_1, \dots, e_n\}$ in $\mathfrak{K}$ such that $\Theta(0)e_j = \gamma_j(\Theta(0))e_j$ for $1 \leqq j \leqq n$. Consider $f = \pi_n(p_{\tau_n})(e_1 \oplus \dots \oplus e_n)$. Then

$$\varrho_{\tau_n}(\Theta(0))f = \gamma_1(\Theta(0)) \dots \gamma_n(\Theta(0))f$$

and since $p_{\tau_n} = (n!)^{-1} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma$ ($\varepsilon(\sigma)$ is the sign of the permutation σ), we have $f \neq 0$.

But $\Omega\varrho_{\tau_n}(\Theta) = m_1 \dots m_n I_{\mathfrak{K}^{\tau_n}}$ for some contractive Ω, and we infer $\|\Omega(0)f\| = \|f\|$ so that $\Omega(\lambda)f = \mu f$ for some constant μ, $|\mu| = 1$. This in turn implies $\varrho_{\tau_n}(\Theta(\lambda))f = \mu^{-1}m_1(\lambda) \dots m_n(\lambda)f$ for all $\lambda \in D$. In view of the known properties of p_{τ_n} this last equality implies that $\mathfrak{B} = \mathbf{C}e_1 + \dots + \mathbf{C}e_n$ is invariant for $\Theta(\lambda)$ for all $\lambda \in D$. Since Θ is two-sided inner we infer that $\mathfrak{B}$ is a reducing subspace for $\Theta(\lambda)$, $\lambda \in D$. Hence $\Theta = \Theta_1 \oplus \Theta_2$ where $\Theta_1 = \Theta|\mathfrak{B}$, $\Theta_2 = \Theta|\mathfrak{K} \ominus \mathfrak{B}$.

Thus we define $T_i = S(\Theta_i)$ for $i = 1, 2$ and clearly T is unitarily equivalent to $T_1 \oplus T_2$ and $\mathfrak{d}_{T_1} = n$. Remark also that $\varrho_{\tau_n}(\Theta_1)$ coincides with $m_1 \dots m_n$. Let $S(m_1') \oplus \dots \oplus S(m_n')$ and $S(m_1'') \oplus S(m_2'') \oplus \dots$ be the Jordan models of T_1 and T_2 (we do not exclude the possibility that some m_j' or m_j'' be 1). Then we have:

$$m_1 \dots m_n = m_1' \dots m_n' = \bigvee_{k=0}^{n} m_1' \dots m_k' m_1'' \dots m_{n-k}'' \tag{4.3}$$

(use for instance Proposition 3.1 with $\tau = \tau_n$). From 4.3 we infer that $m_1' \dots m_{n-1}' m_1''$ divides $m_1' \dots m_n'$ and hence m_1'' divides m_n'. Thus $S(m_1') \oplus \dots \oplus S(m_n') \oplus S(m_1'') \oplus \oplus S(m_2'') \oplus \dots$ is the Jordan model of $T_1 \oplus T_2$ and hence $m_j' = m_j$, $m_k'' = m_{n+k}$ ($1 \leqq j \leqq n$, $k = 1, 2, \dots$). This ends the proof of (i)⇒(ii).

(ii)⇒(i). Let Θ_1, Θ_2 coincide with the characteristic functions of T_1, T_2. Then $\varrho_{\tau_n}(\Theta_1)$ coincides with $m_1 \dots m_n$ so that $\gamma_1(\Theta_1(0)) \dots \gamma_n(\Theta_1(0)) = \gamma_1(\tau_n(\Theta_1(0))) = |m_1(0) \dots m_n(0)|$ (use Proposition 3.1 for instance and then Lemma 4.1). Now clearly $\gamma_j(\Theta_1(0)) \geqq \gamma_j(\Theta(0))$ and hence $\gamma_1(\Theta(0)) \dots \gamma_n(\Theta(0)) \leqq |m_1(0) \dots m_n(0)|$ which in view of Lemma 4.2 gives $\gamma_1(\Theta(0)) \dots \gamma_n(\Theta(0)) = |m_1(0) \dots m_n(0)|$.

Q.E.D

Remark 4.6. If T is a contraction and Θ is its characteristic function then $\gamma_j(\Theta(0)) = \gamma_j(T)$. Thus, let T be a C_0 contraction with Jordan model $S(m_1) \oplus \oplus S(m_2) \oplus \dots$ such that $m_1(0) \neq 0$ and $\lim_{j\to\infty} |m_j(0)| = 1$. Proposition 4.5 shows that the Jordan model of T can be characterized within the class $\mathscr{T}$ of contractions, which are quasisimilar with T by its extremal properties. Indeed, define $\mathcal{T}_n$ recurrently, by $\mathcal{T}_0=\mathcal{T}$ and $\mathcal{T}_{n+1}=\{T'\in\mathcal{T}_n \mid \gamma_{n+1}(T')=\inf_{S\in\mathcal{T}_n}\gamma_{n+1}(S)\}$. Then the only member up to unitary equivalence of $\bigcap_{n=0}^{\infty}\mathcal{T}_n$ is the Jordan model of T.

Part II

§ 5. Preliminaries

1. We begin with a short review of the properties of infinite determinants (see [4], ch. IV, § 1), in order to discuss (in the next section) minors of such determinants.

Let $\mathfrak{K}$ be a complex separable Hilbert space and $\mathscr{C}_1(\mathfrak{K})$ the ideal of nuclear operators, endowed with the trace-norm

(5.1) $$\|X\|_1 = \operatorname{tr}|X|,\ |X| = (X^*X)^{1/2} \quad (X\in\mathscr{C}_1(\mathfrak{K})).$$

Consider $X\in I+\mathscr{C}_1(\mathfrak{K})$ and let $\{\lambda_j(X)\}_{j=1}^{\infty}$ be the eigenvalues of X (repeated according to their multiplicities). We have

$$\sum_{j=1}^{\infty}|1-\lambda_j(X)| \leqq \operatorname{tr}|I-X| < \infty$$

and it follows that the infinite product defining the determinant

(5.2) $$\det(X) = \prod_{j=1}^{\infty}\lambda_j(X)$$

converges absolutely. Moreover, $\det(I+Y)$ as a function of $Y\in\mathscr{C}_1(\mathfrak{K})$ is analytic (in particular continuous on the Banach space $\mathscr{C}_1(\mathfrak{K})$). This follows from [4], Ch. IV, Corollary 1.1 and property 8° on p. 207, combined with Proposition 2 on p. 11 of [3].

Also for $\{e_j\}_{j=1}^{\infty}$ an orthonormal basis of $\mathfrak{K}$ and $X\in I+\mathscr{C}_1(\mathfrak{K})$, we have

(5.3) $$\det(X) = \lim_{N\to\infty}\det[\langle Xe_i, e_j\rangle]_{1\leqq i,\, j\leqq N}$$

(cf. [4], property 2° on p. 203).

Furthermore, for $X, X'\in I+\mathscr{C}_1(\mathfrak{K})$ we have (cf. the proof of property 7° on p. 206 of [4]):

(5.4) $$\det(XX') = \det(X)\det(X').$$

In view of (5.2) the following assertions are easily seen to be true: a) if $X\in I+\mathscr{C}_1(\mathfrak{K})$ is unitary then $|\det(X)|=1$; b) if $X\in I+\mathscr{C}_1(\mathfrak{K})$ is a contraction then

$|\det(X)|\leqq 1$; c) $X\in I+\mathscr{C}_1(\mathfrak{K})$ is invertible if and only if $\det(X)\neq 0$; d) the determinant is invariant under similarities.

2. For any Hilbert space $\mathfrak{K}$ we shall indicate by "$\rightharpoonup$" the weak convergence in $\mathfrak{K}$ and in $\mathscr{L}(\mathfrak{K})$. In order to avoid antilinear mappings we shall consider the dual space $\mathfrak{K}^d$. If $T\in\mathscr{L}(\mathfrak{K})$, the dual operator is denoted by T^d $\big(T^d\in\mathscr{L}(\mathfrak{K}^d)\big)$. $(\mathfrak{K}^d)^d$ can be identified in the usual way with $\mathfrak{K}$.

3. For any Hilbert space $\mathfrak{K}$ and $n\geqq 0$ we shall denote by $\mathfrak{K}^{\wedge n}$ the n-th exterior power of $\mathfrak{K}$. For $n=0$ this is just the complex field $\mathbf{C}$ and in general $\mathfrak{K}^{\wedge n}$ coincides with $\mathfrak{K}^{\tau_n}$ for $\tau_n=(1, 1, 1, \dots, 1, 0, \dots)$, $\ell(\tau_n)=\dim\mathfrak{K}$, $\iota(\tau_n)=n$ (cf. § 1.4). $\mathfrak{K}^{\wedge n}$ is generated by vectors of the form

$$(5.5)\qquad k_1\wedge k_2\wedge\dots\wedge k_n=(n!)^{-1/2}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)k_{\sigma(1)}\otimes\dots\otimes k_{\sigma(n)},\quad k_j\in\mathfrak{K}\quad(1\leqq j\leqq n),$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ.

The factor $(n!)^{-1/2}$ has been chosen so that $\|e_1\wedge\dots\wedge e_n\|=1$ for any orthonormal system $\{e_1, e_2, \dots, e_n\}$.

For n, m two positive integers there is a bilinear map

$$\wedge:\mathfrak{K}^{\wedge n}\times\mathfrak{K}^{\wedge m}\to\mathfrak{K}^{\wedge(m+n)}$$

such that $(k_1\wedge k_2\wedge\dots\wedge k_n)\wedge(k_{n+1}\wedge\dots\wedge k_{n+m})=k_1\wedge\dots\wedge k_{n+m}$. For each $A\in\mathscr{L}(\mathfrak{K})$ we shall denote $\varrho_{\tau_n}(A)$ as $A^{\wedge n}$, so that

$$(5.6)\qquad A^{\wedge n}(k_1\wedge\dots\wedge k_n)=Ak_1\wedge\dots\wedge Ak_n.$$

Let $\mathfrak{K}$ now be a Hilbert space of finite dimension n. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathfrak{K}$, we can define a bilinear form

$$B:\mathfrak{K}^{\wedge k}\times\mathfrak{K}^{\wedge(n-k)}\to\mathbf{C}$$

by the formula

$$(5.7)\qquad B(h, g)=\langle h\wedge g, e_1\wedge\dots\wedge e_n\rangle.$$

Choosing in $\mathfrak{K}^{\wedge j}$ the usual orthonormal basis

$$\{e_{i_1}\wedge\dots\wedge e_{i_j}\mid 1\leqq i_1<i_2<\dots<i_j\leqq n\}$$

it is easy to see that the mapping

$$(5.8)\qquad C:\mathfrak{K}^{\wedge(n-k)}\to(\mathfrak{K}^{\wedge k})^d$$

given by $C(g)(h)=B(h, g)$ for $g\in\mathfrak{K}^{\wedge(n-k)}$, $h\in\mathfrak{K}^{\wedge k}$ is a linear isometry. If $A\in\mathscr{L}(\mathfrak{K})$ we have

$$(5.9)\qquad B(A^{\wedge k}h, A^{\wedge(n-k)}g)=\det(A)\,B(h, g)$$

because $A^{\wedge n} = \det(A) I_{\mathfrak{K}^{\wedge n}}$. Let us define

$$F = CA^{\wedge(n-k)}C^{-1} \in \mathcal{L}((\mathfrak{K}^{\wedge k})^d) \tag{5.10}$$

and

$$A^{Adk} = F^d \in \mathcal{L}(\mathfrak{K}^{\wedge k}). \tag{5.11}$$

We have $B(A^{Adk}h, g) = C(g)(A^{Adk}h) = (F(C(g)))(h) = (C(A^{\wedge(n-k)}g))(h) = B(h, A^{\wedge(n-k)}g)$ and since C is isometric,

$$\|A^{\wedge(n-k)}\| = \|F\| = \|A^{Adk}\|. \tag{5.12}$$

Also, as B is nondegenerate we have

$$A^{Adk}A^{\wedge k} = \det(A) I_{\mathfrak{K}^{\wedge k}}. \tag{5.13}$$

It is obvious by the definition of A^{Adk} that

$$(A_1A_2)^{Adk} = A_2^{Adk}A_1^{Adk}, \; A_1, A_2 \in \mathcal{L}(\mathfrak{K}), \tag{5.14}$$

and it can be shown that

$$(A^*)^{Adk} = (A^{Adk})^*. \tag{5.15}$$

Moreover, for invertible A we infer from (5.13) that

$$A^{\wedge k}A^{Adk} = \det(A) I_{\mathfrak{K}^{\wedge k}} \tag{5.16}$$

and by continuity it follows that (5.16) always holds.

For $\{f_1, f_2, \dots, f_k\}$ an orthonormal system in $\mathfrak{K}$ we shall show that

$$\langle A^{Adk}(f_1 \wedge \dots \wedge f_k), f_1 \wedge \dots \wedge f_k \rangle = \det(P + (I-P)A(I-P)) \tag{5.17}$$

where P denotes the orthogonal projection onto the linear span of $\{f_1, f_2, \dots, f_k\}$. Completing the system $\{f_1, \dots, f_k\}$ to an orthonormal basis $\{f_1, \dots, f_n\}$, we have

$$\begin{aligned}
&\langle A^{Adk}(f_1 \wedge \dots \wedge f_k), f_1 \wedge \dots \wedge f_k \rangle = \\
&= \langle (A^{Adk}(f_1 \wedge \dots \wedge f_k)) \wedge f_{k+1} \wedge \dots \wedge f_n, f_1 \wedge \dots \wedge f_n \rangle = \\
&= B(A^{Adk}(f_1 \wedge \dots \wedge f_k), f_{k+1} \wedge \dots \wedge f_n) \cdot \langle f_1 \wedge \dots \wedge f_n, e_1 \wedge \dots \wedge e_n \rangle^{-1} = \\
&= B(f_1 \wedge \dots \wedge f_k, A^{\wedge(n-k)}(f_{k+1} \wedge \dots \wedge f_n)) \cdot \langle f_1 \wedge \dots \wedge f_n, e_1 \wedge \dots \wedge e_n \rangle^{-1} = \\
&= \langle f_1 \wedge \dots \wedge f_k \wedge A^{\wedge(n-k)}(f_{k+1} \wedge \dots \wedge f_n), f_1 \wedge \dots \wedge f_n \rangle = \\
&= \langle (P + A(I-P))^{\wedge n}(f_1 \wedge \dots \wedge f_n), f_1 \wedge \dots \wedge f_n \rangle = \\
&= \det(P + A(I-P)) = \det(P + (I-P)A(I-P)).
\end{aligned}$$

Formulas (5.14), (5.16), (5.17) show that A^{Adk} does not depend on the particular choice of the orthonormal basis $\{e_1, \dots, e_n\}$.

Let us now suppose that A is a positive operator with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_n$ and the corresponding eigenvectors $f_1, f_2, \dots, f_n$. Then $A^{\wedge(n-k)}$ is positive with eigenvalues

$$\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_{n-k}} \quad (1 \leqq i_1 < i_2 < \dots < i_{n-k} \leqq n).$$

It follows that

$$\|A^{Adk}\| = \|A^{\wedge(n-k)}\| = \lambda_1\dots\lambda_{n-k} \leqq (1+|\lambda_1-1|)(1+|\lambda_2-1|)\dots(1+|\lambda_{n-k}-1|) \leqq$$
$$\leqq \exp(|\lambda_1-1|)\exp(|\lambda_2-1|)\dots\exp(|\lambda_{n-k}-1|) \leqq \exp(\mathrm{Tr}|A-I|).$$

Now for any $T\in\mathscr{L}(\mathfrak{K})$, we have

$$\|T^*T-I\|_1 \leqq (1+\|T-I\|_1)^2-1 \quad \text{and} \quad \|(T^*T)^{1/2}-I\|_1 \leqq \|T^*T-I\|_1$$

as can be seen by comparing the eigenvalues of these operators. Therefore,

$$\|(T^*T)^{1/2}-I\|_1 \leqq (1+\|T-I\|_1)^2-1.$$

In particular, for the polar decomposition $T=UA$ of T $\left(A=|T|=(T^*T)^{1/2}\right)$ it follows that:

(5.18) $$\|T^{Adk}\| = \|A^{Adk}U^{Adk}\| \leqq \|A^{Adk}\| \leqq \exp(\mathrm{Tr}|A-I|) \leqq$$
$$\leqq \exp\left((1+\|T-I\|_1)^2-1\right).$$

§ 6. Infinite dimensional adjoints and minors

Let us now consider $\mathfrak{K}$ an infinite dimensional Hilbert space and $A\in\mathscr{L}(\mathfrak{K})$ so that $\mathrm{rank}\,(I-A)<\infty$. From the preceding considerations we easily infer the existence of an operator $A^{Adk}\in\mathscr{L}(\mathfrak{K}^{\wedge k})$ satisfying

(6.1) $$A^{Adk}A^{\wedge k} = A^{\wedge k}A^{Adk} = \det(A)I_{\mathfrak{K}^{\wedge k}};$$

(6.2) $$\langle A^{Adk}(f_1\wedge\dots\wedge f_k), f_1\wedge\dots\wedge f_k\rangle = \det\left(P+(I-P)A(I-P)\right),$$

for P the orthogonal projection onto the linear span of the orthonormal system $\{f_1, \dots, f_k\}$;

(6.3) $$\|A^{Adk}\| \leqq \exp\left((1+\|A-I\|_1)^2-1\right).$$

Also for $A_1, A_2\in\mathscr{L}(\mathfrak{K})$ with rank $(I-A_j)<\infty$, $j=1, 2$, we have

(6.4) $$(A_1A_2)^{Adk} = A_2^{Adk}A_1^{Adk}.$$

Let $A\in\mathscr{L}(\mathfrak{K})$ now be such that $I-A\in\mathscr{C}_1(\mathfrak{K})$ and let A_n be such that rank $(I-A_n)<\infty$ and $\lim\limits_{n\to\infty}\|A-A_n\|_1=0$.

Using the fact that the function

$$\mathscr{C}_1(\mathfrak{K})\ni X \to \det(I+X)$$

is continuous, it follows from (6.2—3) that the sequence A_n^{Adk} converges weakly. The limit, which will be denoted by A^{Adk}, satisfies (6.2—3). Because $A_n^{\wedge k}$ converges to $A^{\wedge k}$ in norm and $\det(A_n)\to\det(A)$ we also obtain property (6.1) for A^{Adk}. Using now (6.2—3) it follows that:

(6.5) $$A, A_n\in I+\mathscr{C}_1(\mathfrak{K}) \quad\text{and}\quad \|A_n-A\|_1\to 0 \quad\text{imply}\quad A_n^{Adk} \rightharpoonup A^{Adk}.$$

Property (6.4) for $A_1, A_2\in I+\mathscr{C}_1(\mathfrak{K})$ follows from (6.1), provided A_1, A_2 are invertible, and can be extended using (6.5) to the case when only A_1 is invertible. Using (6.5) once again it follows that (6.4) holds in the general case.

We have shown in § 5.1 that the function $Y\to\det(I+Y)$ is analytic on the Banach space $\mathscr{C}_1(\mathfrak{K})$. Using (6.2—3) we infer that for $\xi,\eta\in\mathfrak{K}^{\wedge k}$ the mapping

$$\mathbf{C}\ni\lambda\to\langle (I+X+\lambda Y)^{Adk}\xi,\eta\rangle$$

is analytic when $X, Y\in\mathscr{C}_1(\mathfrak{K})$.

From this fact and from (6.3), using [3], Proposition 2 it follows that

$$\mathscr{C}_1(\mathfrak{K})\ni X\to\langle (I+X)^{Adk}\xi,\eta\rangle$$

for $\xi,\eta\in\mathfrak{K}^{\wedge k}$ is analytic.

This again implies the following stronger fact: *the mapping*

(6.6) $$\mathscr{C}_1(\mathfrak{K})\ni X\to (I+X)^{Adk}\in\mathscr{L}(\mathfrak{K}^{\wedge k})$$

is analytic (in particular continuous with respect to the norm topologies).

Let us also remark that for any contraction $A\in I+\mathscr{C}_1(\mathfrak{K})$ the adjoints A^{Adk} are contractions. This is obvious if $\dim\mathfrak{K}=n<\infty$ (since in this case $\|A^{Adk}\|=\|A^{\wedge(n-k)}\|$) and follows in the general case by a simple limit argument.

We are now going to define the minors of an infinite determinant. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two closed subspaces of $\mathfrak{K}$, $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ the corresponding projections, and suppose there is a unitary operator $U\in I+\mathscr{C}_1(\mathfrak{K})$ such that $U\mathfrak{M}=\mathfrak{N}$. Then for $A\in I+\mathscr{C}_1(\mathfrak{K})$ the minor of $\det(A)$ corresponding to the triple $(\mathfrak{M},\mathfrak{N},U)$ is

(6.7) $$\det(UP_{\mathfrak{M}}A|\mathfrak{N}).$$

The definition makes sense because it is easily seen that $UP_{\mathfrak{M}}A|\mathfrak{N}\in I_{\mathfrak{N}}+\mathscr{C}_1(\mathfrak{N})$. In case $\mathfrak{N}$ (and hence $\mathfrak{M}$ also) is of finite codimension in $\mathfrak{K}$, we shall say that $\det(UP_{\mathfrak{M}}A|\mathfrak{N})$ is a minor of corank $\dim\mathfrak{M}^{\perp}$.

Let $\det(UP_{\mathfrak{M}}A|\mathfrak{N})$ be a minor of corank k of A. Then, by (6.2)

(6.8) $$\det(UP_{\mathfrak{M}}A|\mathfrak{N})=\det(P_{\mathfrak{N}}UAP_{\mathfrak{N}}+(I-P_{\mathfrak{N}}))=$$
$$=\langle (UA)^{Adk}(e_1\wedge\ldots\wedge e_k), e_1\wedge\ldots\wedge e_k\rangle$$

for $\{e_1,\ldots,e_k\}$ an orthonormal basis of $\mathfrak{K}\ominus\mathfrak{N}$. Thus the minors of corank k of A coincide with some matrix elements of $(UA)^{Adk}=A^{Adk}U^{Adk}$.

§ 7. Determinants of contractive analytic functions

Let $\Theta \in H^\infty(\mathscr{L}(\mathfrak{K}))$ be a contractive function (here $\mathfrak{K}$ denotes as usual a separable Hilbert space). Let us suppose that $I-\Theta(\lambda)$ is nuclear for $\lambda \in D$ and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathfrak{K}$. The functions

$$d_n(\lambda) = \det\left[\langle \Theta(\lambda) e_i, e_j \rangle\right]_{1 \leq i, j \leq n} = \det\left(P_n \Theta(\lambda) P_n + (1-P_n)\right)$$

(here P_n denotes the orthogonal projection onto the linear span of $\{e_1, \ldots, e_n\}$) are analytic,

$$(7.1) \qquad |d_n(\lambda)| \leqq 1,$$

and

$$(7.2) \qquad \lim_{n\to\infty} d_n(\lambda) = \det\left(\Theta(\lambda)\right).$$

From (7.1) and (7.2) we infer, by the Vitali—Montel theorem, that $\det(\Theta(\lambda))$ is an analytic function. A similar argument shows that the functions $\lambda \to (\Theta(\lambda))^{Adk}$ are analytic and contractive (cf. § 6) and that

$$(7.3) \qquad \Theta^{\wedge k}\Theta^{Adk} = \Theta^{Adk}\Theta^{\wedge k} = \det(\Theta) I_{\mathfrak{K}^{\wedge k}}$$

In particular, if $\Theta(\lambda)$ is invertible for some $\lambda \in D$, it follows that Θ has a scalar multiple (cf. [13], ch. V, § 6).

In case $\mathfrak{M}, \mathfrak{N}$ are subspaces of $\mathfrak{K}$ of finite codimension and $U \in I + \mathscr{C}_1(\mathfrak{K})$ is a unitary operator such that $U\mathfrak{M} = \mathfrak{N}$, the function $\lambda \to \det\left(UP_{\mathfrak{M}}\Theta(\lambda)|\mathfrak{N}\right)$ is analytic and of modulus $\leqq 1$. We call such a function a minor of Θ of corank $\dim \mathfrak{M}^\perp$.

Let us denote by $\delta_r(\Theta)$ the greatest common inner divisor of the minors of corank r of Θ $(r=0, 1, 2, \ldots)$. For $r=0$, $\delta_0(\Theta)$ coincides with the inner factor of $\det(\Theta(\lambda))$. From (6.8) it follows that $\delta_r(\Theta)$ coincides with the greatest common inner divisor of the matrix elements of Θ^{Adr}.

Lemma 7.1. *$\delta_{r+1}(\Theta)$ divides $\delta_r(\Theta)$ for each r.*

Proof. We have to prove that $\delta_{r+1}(\Theta)$ divides each minor of corank r of Θ. Clearly it suffices to prove that $\delta_1(\Theta)$ divides $\det(\Theta)$ or, equivalently,

$$\det(\Theta) H^2(\mathfrak{K}) \subset \delta_1(\Theta) H^2(\mathfrak{K}).$$

But this easily follows from the relation $\Theta\Theta^{Ad1} = \det(\Theta) I_{\mathfrak{K}}$. Indeed, $\Theta^{Ad1} H^2(\mathfrak{K}) \subset \subset \delta_1(\Theta) H^2(\mathfrak{K})$ and, since Θ is analytic,

$$\det(\Theta) H^2(\mathfrak{K}) = \Theta\Theta^{Ad1} H^2(\mathfrak{K}) \subset \Theta\delta_1(\Theta) H^2(\mathfrak{K}) \subset \delta_1(\Theta) H^2(\mathfrak{K}).$$

Lemma 7.2. *The greatest common inner divisor of the functions $\delta_j(\Theta)$ $(j=1, 2, \ldots)$ is* 1.

Proof. Let us denote by m the greatest common inner divisor of the family $\{\delta_j(\Theta)\}_0^\infty$ and let $\{e_j\}_{j=1}^\infty$, be an orthonormal basis of $\mathfrak{K}$. Since $\Theta^{Ad\,r}H^2(\mathfrak{K}^{\wedge r})\subset \subset mH^2(\mathfrak{K}^{\wedge r})$ for each r, we have

$$|m(0)| \geqq |\langle \Theta(0)^{Ad\,r}(e_1\wedge e_2 \wedge \ldots \wedge e_r), e_1\wedge e_2\wedge \ldots \wedge e_r\rangle| = \\ = |\det((I-P_r)\Theta(0)(I-P_r)+P_r)|,$$

where P_r denotes as usual the orthogonal projection onto the linear span of $\{e_1, \ldots, e_r\}$. We infer

$$|m(0)| \geqq \limsup_{r\to\infty} |\det((I-P_r)\Theta(0)(I-P_r)+P_r)| = 1$$

and the lemma follows.

Let us also note the relations

(7.4) $$\delta_j(\Theta^\sim) = \delta_j(\Theta)^\sim \quad (j=1,2,\ldots)$$

which hold for each function Θ of the type considered in this section.

§ 8. Weak contractions

Let us recall that a contraction T acting on a Hilbert space $\mathfrak{H}$ is a weak contraction if its spectrum does not cover the unit disk D and $I-T^*T$ is a nuclear operator. T is a weak contraction if and only if T^* is a weak contraction.

If a weak contraction T is of class C_{00} (that is $T^n\to 0$ and $T^{*n}\to 0$ strongly as $n\to\infty$), then T is of class C_0 and acts on a necessarily separable space. The proof of this fact goes as follows (cf. [13], Ch. VIII, § 1).

If we put

(8.1) $$T_\lambda = (T-\lambda I)(I-\bar{\lambda}T)^{-1}, \quad \lambda\in D$$

we have

(8.2) $$I-T_\lambda^*T_\lambda = X_\lambda^*(I-T^*T)X_\lambda, \; X_\lambda = (1-|\lambda|^2)^{1/2}(I-\bar{\lambda}T)^{-1}.$$

So T is a weak contraction if and only if T_λ is a weak contraction. Moreover, we have $(T_\lambda)_{-\lambda}=T$. Therefore we may suppose without loss of generality that T is invertible. Let $\{\mu_j\}_1^n$ $(n\leqq\aleph_0)$ be the eigenvalues of $(I-T^*T)|\mathfrak{D}_T$, $\mathfrak{D}_T=((I-T^*T)H)^-$ (multiple eigenvalues repeated according to their multiplicities). We have $\mu_j\neq 1$ because $\ker T=\{0\}$.

Let $\{\varphi_j\}_1^n$ be an orthonormal basis of $\mathfrak{D}_T$ such that $(I-T^*T)\varphi_j=\mu_j\varphi_j$. It is easy to verify that the system $\{\psi_j\}_1^n$, where $\psi_j=(1-\mu_j)^{-1/2}T\varphi_j$, is an orthonormal basis of $\mathfrak{D}_{T^*}$ and that we have also $T^*\psi_j=(1-\mu_j)^{1/2}\varphi_j$.

Let us denote by U the unitary operator determined by

(8.3) $$U\colon \mathfrak{D}_T\to\mathfrak{D}_{T^*}, U\varphi_j=-\psi_j.$$

Then the operator $(U+T)\mathfrak{D}_T$ is nuclear. Indeed,

$$(U+T)h = \sum_{j=1}^{n} \big((1-\mu_j)^{1/2}-1\big)(h, \varphi_j)\psi_j, \quad h \in \mathfrak{D}_T$$

and from the relations

$$\lim_{\mu\to 0} \mu^{-1}\big(1-(1-\mu)^{1/2}\big) = 1/2, \quad \sum_{j=1}^{n} \mu_j < \infty$$

we infer

$$\sum_{j=1}^{n} \big(1-(1-\mu_j)^{1/2}\big) < \infty.$$

Furthermore, if $\Theta_T \in H_i^\infty(\mathscr{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*}))$ is the characteristic function of T, $U-\Theta_T(\lambda)$ is nuclear for $\lambda \in D$. Indeed,

$$U-\Theta_T(\lambda) = (U+T)|\mathfrak{D}_T - \lambda D_{T^*}(I-\lambda T^*)^{-1} D_T|\mathfrak{D}_T \quad (D_T = (I-T^*T)^{1/2})$$

and since D_T and D_{T^*} are Hilbert—Schmidt operators because T is a weak contraction, $\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T$ is nuclear. Thus the function $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{D}_T))$ defined by $\Theta(\lambda) = U^*\Theta_T(\lambda)$ coincides with Θ_T and $I-\Theta(\lambda)$ is nuclear for $\lambda \in D$.

Let us put

$$d_T(\lambda) = \det(\Theta(\lambda)), \quad \delta_j(T) = \delta_j(\Theta), \quad (j=0, 1, 2, \ldots). \tag{8.4}$$

We have $d_T(0) = \prod_{j=1}^{n} (1-\mu_j)^{1/2} \neq 0$ and from (7.3) (with $k=1$) it follows that d_T is a scalar multiple of Θ. As in [13], Theorem VI. 5.2 we obtain

Lemma 8.1. *Each weak contraction T of class C_{00} is a C_0 contraction and its minimal function coincides with $\delta_0(T)/\delta_1(T)$.*

Let us remark that we have a converse: suppose $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{K}))$ is such that $\Theta(\lambda) \in I + \mathscr{C}_1(\mathfrak{K})$, $\lambda \in D$, and $\det(\Theta) \not\equiv 0$. Since $\det(\Theta)$ is then a scalar multiple of Θ (by (7.3) with $k=1$), it follows that Θ coincides with the characteristic function of an operator T of class C_0 and from [13], Ch. IV § 1 it follows that $\operatorname{tr}(I-T^*T) = \operatorname{tr}\big(I-\Theta(0)^*\Theta(0)\big) < \infty$ so that T is a weak contraction. Let us also note that the relations

$$d_{T^*} = d_T^{\sim}, \quad \delta_j(T^*) = \delta_j(T)^{\sim} \quad (j = 0, 1, \ldots) \tag{8.5}$$

hold for each weak contraction T.

Proposition 8.2. *Let T be a weak C_0 contraction acting on the Hilbert space $\mathfrak{H}$ and let $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be the triangularization associated with the T-invariant subspace $\mathfrak{H}_1$. Then T_1 and T_2 are weak C_0 contractions and we have*

$$d_T = d_{T_1} d_{T_2}, \quad \delta_0(T) = \delta_0(T_1)\delta_0(T_2).$$

Proof. We may suppose without loss of generality that T is invertible, thus $m_T(0)\neq 0$. By [13], Proposition III. 6.1, T_1 and T_2 are C_0 operators and m_{T_1}, m_{T_2} are divisors of m. It follows that $m_{T_1}(0)\neq 0$, $m_{T_2}(0)\neq 0$ so that T_1 and T_2 are invertible. Moreover, we have

$$I_{\mathfrak{H}_1}-T_1^*T_1 = P_{\mathfrak{H}_1}(I-T^*T)|\mathfrak{H}_1, \quad I_{\mathfrak{H}_2}-T_2T_2^* = P_{\mathfrak{H}_2}(I-TT^*)|\mathfrak{H}_2,$$

thus T_1 and T_2 are weak contractions.

By [13] Theorem VII.1.1 and Proposition VII.2.1, we can associate with the invariant subspace $\mathfrak{H}_1$ a regular factorization

$$\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda) \tag{8.6}$$

such that the characteristic functions $\Theta_{T_1}(\lambda)$, $\Theta_{T_2}(\lambda)$ coincide with the pure parts of $\Theta_1(\lambda)$, $\Theta_2(\lambda)$, respectively. Then we have

$$\Theta_j(\lambda) = U_j'\begin{bmatrix}\Theta_{T_j}(\lambda) & 0\\ 0 & I_j\end{bmatrix}U_j'' \tag{8.7}$$

where U_j', U_j'' are unitary operators and I_j denotes the identity operator on some Hilbert space ($j=1, 2$). Now, from the consideration preceding Lemma 8.1, it follows that $I-U_j^{0*}\Theta_{T_j}(\lambda)$ is nuclear and $d_{T_j}(\lambda)=\det(U_j^{0*}\Theta_{T_j}(\lambda))$ for some unitary operators U_j^0 ($j=1, 2$). With the notation

$$U_j = U_j'\begin{bmatrix}U_j^0 & 0\\ 0 & I_j\end{bmatrix}U_j''$$

we see that $I-U_j^*\Theta_j(\lambda)$ is nuclear and

$$d_{T_j}(\lambda) = \det(U_j^*\Theta_j(\lambda)). \tag{8.8}$$

Using (8.6) and (8.7) we obtain

$$U^*\Theta_T(\lambda) = U^*U_2U_1[U_1^*(U_2^*\Theta(\lambda))U_1](U_1^*\Theta_1(\lambda))\cdot \tag{8.9}$$

From this relation it follows that $I_{\mathfrak{D}_T}-U^*U_2U_1$ is a nuclear operator such that $\det(U^*U_1U_2)$ exists. Using (8.8-9) and (5.4) we then obtain

$$\begin{aligned} d_T(\lambda) &= \det(U^*U_2U_1)\det(U_1^*(U_2^*\Theta_2(\lambda))U_1)\det(U_1^*\Theta_1(\lambda)) = \\ &= \det(U^*U_2U_1)\det(U_2^*\Theta_2(\lambda))\det(U_1^*\Theta_1(\lambda)) = \\ &= \det(U^*U_2U_1)d_{T_2}(\lambda)d_{T_1}(\lambda). \end{aligned}$$

The relation $\delta_0(T)=\delta_0(T_1)\delta_0(T_2)$ follows by taking the inner factors in the last obtained relations. The proposition is proved.

Remark 8.3. This proposition is a generalization of [13], Lemma IX. 3.1.

Lemma 8.4. *A Jordan operator $S(M)$, $M=\{m_j\}_1^\infty$, is a weak contraction if and only if $\sum_{j=1}^{\infty}(1-|m_j(0)|)<\infty$. In this case we have $d_{S(M)}=\delta_0(S(M))=\prod_{j=1}^{\infty} m_j$, where $\prod_{j=1}^{\infty} m_j$ means the limit of some converging subsequence of $\{m_1m_2\dots m_n\}_{n=1}^\infty$.*

Proof. For any inner function $m\in H^\infty$ we have

$$(I_{\mathfrak{H}(m)}-S(m)S(m)^*)h = P_{\mathfrak{H}(m)}(I-UU^*)h = (h, c_0)P_{\mathfrak{H}(m)}c_0 =$$
$$= (h, c_0)(1-\overline{m(0)}m)$$

($h\in\mathfrak{H}(m)$), where U denotes the unilateral shift on H^2 and c_0 is the constant functions $c_0\equiv 1$. Thus $I-S(m)S(m)^*$ is of rank one and has norm $(1-\overline{m(0)}m, c_0)=1-|m(0)|^2$. It follows that the trace norm of $I-S(M)S(M)^*$ equals $\sum_{j=1}^{\infty}(1-|m_j(0)|^2)$. We have only to remark that

$$1-|m_j(0)|\leqq 1-|m_j(0)|^2\leqq 2(1-|m_j(0)|).$$

The equality $d_{S(M)}=\prod_{j=1}^{\infty} m_j$ obviously follows from the special form of the characteristic function of $S(M)$. So it remains only to prove that $\prod_{j=1}^{\infty} m_j$ is an inner function. To see this, let us remark that $\prod_{j=1}^{\infty} m_j$ and $\prod_{j=n}^{\infty} m_j$ have the same outer factor, such that this outer factor must be 1 because $\left|\prod_{j=n}^{\infty} m_j(\lambda)\right|\to 1$ for each $\lambda\in D$. The lemma is proved.

From now on T will denote a weak C_0 contraction acting on $\mathfrak{H}$, $\Theta\in H_i^\infty(\mathscr{L}(\mathfrak{K}))$ will denote a function coinciding with the characteristic function of T and $\Theta(\lambda)\in I+\mathscr{C}_1(\mathfrak{K})$, $\lambda\in D$. We shall also denote by $S(M)$, $M=\{m_j\}_1^\infty$, the Jordan model of T. From the relation

$$\Theta^{\wedge r}\Theta^{Adr} = \Theta^{Adr}\Theta^{\wedge r} = d_T\cdot I_{\mathfrak{K}^{\wedge r}}, \quad \text{see (7.3)},$$

we infer, because $\Theta^{\wedge r}$ is two-sided inner, that $\delta_0(T)/\delta_r(T)$ is the least inner scalar multiple of $\Theta^{\wedge r}$. Thus we have

$$d_r(T) = \delta_0(T)/\delta_r(T). \tag{8.10}$$

Theorem 8.5. *A C_0 contraction T is a weak contraction if and only if its Jordan model $S(M)$, $M=\{m_j\}_1^\infty$, is a weak contraction.*

Proof. That T is a weak contraction if $S(M)$ is so follows from Proposition 4.3, via Lemma 8.4. So let us assume that T is a weak contraction. Then, by Corollary 3.3.

and relation (8.10) it follows that $m_1 m_2 \ldots m_r$ divides $\delta_0(T)$ for each r. If we suppose T is invertible, we have $\delta_0(T)(0) \neq 0$ and from the inequality

$$|m_1(0) \ldots m_r(0)| \geqq |\delta_0(T)(0)|$$

it follows that the infinite product $\prod_{j=1}^{\infty} |m_j(0)|$ converges. Therefore $\sum_{j=1}^{\infty} (1-|m_j(0)|) < \infty$ and our theorem follows by Lemma 8.4.

Proposition 8.6. *For each weak C_0 contraction T, the determinant function d_T is an inner function.*

Proof. Let us write the inner-outer decomposition of d_T

(8.11) $$d_T = d_i d_o.$$

Because d_i is a scalar multiple of $\Theta^{\wedge k}$, there exists a contractive function $\Omega^{(k)} \in H^\infty(\mathscr{L}(\mathfrak{R}^{\wedge k}))$ such that

(8.12) $$\Omega^{(k)} \Theta^{\wedge k} = \Theta^{\wedge k} \Omega^{(k)} = d_i I_{\mathfrak{R}^{\wedge k}}.$$

Then, by (7.3) and (8.12) we have

$$\Theta^k (d_o \Omega^{(k)} - \Theta^{Adk}) = 0$$

so that (Θ^k being inner)

(8.13) $$\Theta^{Adk} = d_o \Omega^{(k)}.$$

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathfrak{R}$ and denote by P_n the orthogonal projection onto the linear span of $\{e_1, \ldots, e_n\}$. By (8.13) we have

$$\langle \Theta^{Adk}(e_1 \wedge \ldots \wedge e_k), e_1 \wedge \ldots \wedge e_k \rangle = d_o \langle \Omega^{(k)}(e_1 \wedge \ldots \wedge e_k), e_1 \wedge \ldots \wedge e_k \rangle$$

and therefore

$$|d_o(0)| \geqq \limsup_{k\to\infty} |\langle \Theta(0)^{Adk}(e_1 \wedge \ldots \wedge e_k), e_1 \wedge \ldots \wedge e_k \rangle| =$$

$$= \limsup_{k\to\infty} |\det((I-P_k)\Theta(0)(I-P_k)+P_k)| = 1.$$

It follows that $|d_o(0)|=1$ so that $|d_o| \equiv 1$. The proposition follows.

We are now able to prove that the determinant function of a weak C_0 contraction is a quasi-similarity invariant.

Theorem 8.7. *For each C_0 contraction T with Jordan model $S(M)$, $M=\{m_j\}_1^{\infty}$, we have*

(8.14) $$m_j = \delta_{j-1}(T)/\delta_j(T);$$

(8.15) $$d_T = d_{S(M)} = \prod_{j=1}^{\infty} m_j.$$

Proof. From (8.10) it follows that $\delta_{j-1}(T)/\delta_j(T)=d_j(T)/d_{j-1}(T)$ so the relation (8.14) obviously follows from Corollary 3.4.

For the second relation let us write (8.10) under the form

$$d_T \doteq \delta_0(T) = m_1 m_2 \dots m_n \cdot \delta_n(T) \tag{8.16}$$

(cf. Corollary 3.4). From Lemma 7.2 and Lemma 1 of [12] it follows that d_T coincides with the least common inner multiple of the family $\{m_1 m_2 \dots m_n\}_{n=1}^{\infty}$, which coincides with $d_{S(M)}$.

The theorem follows.

References

[1] H. Bercovici, C. Foiaş, B. Sz.-Nagy, Compléments à l'étude des opérateurs de classe C_0. III, *Acta Sci. Math.*, **37** (1975), 313—322.

[2] J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars (Paris, 1969).

[3] A. Douady, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, *Ann. Inst. Fourier*, **16: 1** (1966), 1—98.

[4] И. Ц. Гохберг, М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве*, Наука (Москва, 1965).

[5] Л. Е. Исаев, *Докл. АН СССР*, **178** (1968) 783—785.

[6] Д. П. Желобенко, *Компактные группы Ли и их представления*, Наука (Москва, 1970).

[7] А. А. Кириллов, Представления безконечномерной унитарной группы, *Докл. Акад Наук*, **212** (1973), 288—290.

[8] B. Moore III, E. A. Nordgren, On quasisimilarity and quasiequivalence, *Acta Sci. Math.*, **34** (1973), 311—316.

[9] М. А. Наймарк, *Теория представлений групп*, Наука (Москва, 1976).

[10] E. A. Nordgren, On quasiequivalence of matrices over H^{∞}, *Acta Sci. Math.*, **34** (1973), 301—310.

[11] I. E. Segal, The structure of a class of representations of the unitary group on a Hilbert space, *Proc. Amer. Math. Soc.*, **8** (1957), 197—203.

[12] B. Sz.-Nagy, Diagonalization of matrices over H^{∞}, *Acta Sci. Math.*, **38** (1976), 223—238.

[13] B. Sz.-Nagy, C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland /Akadémiai Kiadó (Amsterdam/ Budapest, 1970).

[14] B. Sz.-Nagy, C. Foiaş, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[15] B. Sz.-Nagy, C. Foiaş, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296.

[16] J. Szűcs, Diagonalization theorems for matrices over certain domains, *Acta Sci. Math.*, **36** (1974), 193—201.

[17] H. Weyl, *Gruppentheorie und Quantenmechanik*, 2nd ed. (Leipzig, 1931).

[18] H. Weyl, *The classical groups* (Princeton, 1939).

[19] P. Y. Wu, Jordan model for weak contractions, *Acta Sci. Math.*, to appear.

H. BERCOVICI
BD. BĂLCESCU 32
BUCUREŞTI, ROMANIA

D. VOICULESCU
INCREST
CALEA VICTORIEI 114
SECTOR 1
BUCUREŞTI 22, ROMANIA