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Normal dilations and operator approximations

R. H. BOULDIN and D. D. ROGERS

§ 1. Preliminaries

This paper continues the research presented in [2]; the earlier results are refined and extended in several directions. Consideration is given to best approximation by self-adjoint operators as well as by non-negative operators. A best approximation from the first set is a "self-adjoint approximant" and from the second set is a "positive approximant". For elementary facts about positive approximants the reader is directed to [8] and [5]; for self-adjoint approximants, check [8] and [6]. A general reference for terms not explained is [9].

For a given operator each set of approximants is convex; the main results of this paper identify broad classes of operators for which each of these sets of approximants is infinite-dimensional. (For a discussion of the dimension of a convex set see [16, p. 7].) Moreover, the constructive proofs of these results develop concrete techniques for obtaining approximants for a given operator.

In [8] HALMOS showed that for any (bounded linear) operator $A=B+iC$ ($B=B^*$, $C=C^*$) a positive approximant is $B+(\delta^2-C^2)^{1/2}$ where $\delta=\delta(A)$ is the distance from A to the non-negative operators; this positive approximant, denoted P_0, is referred to as the "Halmos positive approximant". Halmos also showed that B is a self-adjoint approximant for A, or equivalently the distance from A to the self-adjoint operators is $\|C\|$.

The work in this paper exploits a fundamental relation between an operator T and various normal dilations of T. Before establishing this relation we recall the following two lemmas which state some previously known facts in a form appropriate to this work. These facts are proved in [12] and [10], respectively.

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1.1. Lemma. *If N is a normal element of a C^*-algebra and if A is any element, then*

$$\|A-N\| \geqq h(\sigma(A), \sigma(N))$$

where $h(M_1, M_2)=\sup\{\text{dist}(m_1, M_2): m_1 \in M_1\}$.

1.2. Lemma. *For any normal operator N the following formula holds:*

$$\|N-P_\Lambda(N)\| = h(\sigma(N), \Lambda)$$

where $P_\Lambda(N)$ is a best approximation for N from the normal operators with spectrum in the nonempty closed set Λ, denoted $\mathcal{N}(\Lambda)$.

The notation of the preceding lemmas is continued in the next theorem.

1.3. Theorem. *Assume Λ is a closed convex subset of the real line. If T is an operator on H with normal dilation N on $K \supset H$ such that $\sigma(N) \subset \sigma(T)$, then*

$$\|T-P_\Lambda(T)\| = h(\sigma(T), \Lambda) = h(\sigma(N), \Lambda) = \|N-P_\Lambda(N)\|.$$

Furthermore, provided Q is the orthogonal projection of K onto H, $QP_\Lambda(N)Q|H$ is a best approximation for T from $\mathcal{N}(\Lambda)$.

Proof. It follows from the hypothesis and the two preceding lemmas that

$$\|N-P_\Lambda(N)\| = h(\sigma(N), \Lambda) \leqq h(\sigma(T), \Lambda) \leqq \|T-P_\Lambda(T)\|.$$

Because $QLQ|H$ belongs to $\mathcal{N}(\Lambda)$ on H for any $L \in \mathcal{N}(\Lambda)$ on K, the following inequality holds

$$\|T-P_\Lambda(T)\| \leqq \|T-QP_\Lambda(N)Q\| = \|Q)(N-P_\Lambda(N)Q\| \leqq \|N-P_\Lambda(N)\|.$$

The inequalities prove the theorem.

We use Theorem 1.3 in each of the next four sections. In sections § 2 and § 3 it is assumed that T is subnormal with minimal normal extension N, and in sections § 4 and § 5 it is assumed that T is a Toeplitz operator with N the corresponding Laurent operator. It should be noted that other general hypotheses guarantee that $\sigma(N) \subset \sigma(T)$ — for example, if $\sigma(T)$ is a spectral set for T —, then such a normal dilation N exists.

§ 2. Positive approximants of a subnormal operator

In the next theorem and throughout the remainder of this section the symbol T will denote a subnormal operator defined on H and the symbol N will denote a normal operator defined on K that is the minimal normal extension of T. Also, N equals $B+iC$ where B and C are self-adjoint operators.

2.1. Theorem. *For any subnormal operator T one has $\|T-P_\Lambda(T)\|=\|N-P_\Lambda(N)\|$ where N is the minimal normal extension of T and Λ is a nonempty, closed, convex subset of the real line. Moreover, the compression of any $P_\Lambda(N)$ to H is a best approximation for T from $\mathcal{N}(\Lambda)$.*

Proof. Recall $\sigma(T)$ differs from $\sigma(N)$ only by filling in some holes (see [9, Problems 157, 158]). Thus, the above theorem is a special case of Theorem 1.3.

A curious consequence of the preceding theorem is that the norm of the imaginary part of the subnormal operator T, denoted $\|\text{im } T\|$, equals the norm of the imaginary part of the minimal normal extension, denoted $\|\text{im } N\|$. The first norm is the distance $\|T-P_{\mathbf{R}}(T)\|$ and the second norm is the distance $\|N-P_{\mathbf{R}}(N)\|$.

Let T be a subnormal operator on H. There is a subspace H_1 which reduces T to a normal operator and is maximal with respect to this property. Moreover, the orthogonal complement of H_1, denoted $H_1^\perp$, includes no subspace M invariant under T such that $T|M$ is normal. (See Proposition 1.1 of [1], for example.) Thus, T is the direct sum of a normal operator and a completely nonnormal operator. Since positive approximants of a normal operator are studied extensively in [3], attention is now concentrated on completely nonnormal subnormal operators.

Let Γ denote that set of z such that the distance from z to $[0, \infty)$ is exactly $\delta(T)$ and $\text{re}\, z$ does not exceed $\|T\|$.

2.2. Lemma. *Let T be a completely nonnormal subnormal operator. Then T has infinitely many distinct approximate eigenvalues, say $\{z_1, z_2, \ldots\}$, such that $\{z_1, z_2, \ldots\}$ does not intersect Γ.*

Proof. If $\sigma(T)$ were contained in Γ, then it would follow that T is normal (see [15, Corollary 2] or [11, Theorem 1]). Thus, $\sigma(T)$ and the topological boundary of $\sigma(T)$, denoted bdry $\sigma(T)$, contains some z_0 such that $z_0 \notin \Gamma$. If z_0 were isolated from its complement in $\sigma(T)$, then it would follow that z_0 is an eigenvalue for T and the corresponding eigenspace reduces T to a normal operator (see [14, Theorem 2 and Lemma 6]). Thus, z_0 must be an accumulation point for $\sigma(T)$ and it follows that bdry $\sigma(T)$ contains an infinite number of points off the contour Γ. Since bdry $\sigma(T)$ consists of approximate eigenvalues, the lemma is proved.

In the next lemma and throughout the remainder of this section the symbol P_0 will denote the Halmos positive approximant of the normal operator $N=B+iC$; thus, P_0 is $B+(\delta^2-C^2)^{1/2}$. It should not be confused with the Halmos positive approximant for the subnormal operator T.

2.3. Lemma. *Let $E(\cdot)$ denote the spectral measure for the normal operator N and let K_0 denote the subspace $(P_0K)^- \cap ((\delta^2-C^2)K)^-$.*

(i) *The subspaces K_0 and $E(\Gamma^c)K$ are equal.*

(ii) *If D is a compact set not intersecting Γ, then $E(D)K$ reduces (δ^2-C^2) and P_0 to invertible operators.*

Proof. The first statement follows from Lemma 2.1 of [4].

It follows from Lemma 1.2 that $|(\operatorname{re} z)_- + i(\operatorname{im} z)| \leqq \delta(N)$ for every z in $\sigma(N)$ where x_- denotes the maximum of $\{-x, 0\}$. Since D and Γ are both compact, there is a positive distance between them. It follows that there is a positive number γ such that $|(\operatorname{re} z)_- + i(\operatorname{im} z)| \leqq \delta - \gamma$ for every z in the intersection of D and $\sigma(N)$. Consequently the sets $\{\delta^2-(\operatorname{im} z)^2 : z \in D \cap \sigma(N)\}$ and $\{\operatorname{re} z + (\delta^2-(\operatorname{im} z)^2)^{1/2} : z \in D \cap \sigma(N)\}$ are bounded away from zero, and these sets are the spectra of $\delta^2 - C^2$ and P_0 restricted to $E(D)K$, respectively.

2.4. Theorem. *Let T be a completely nonnormal subnormal operator defined on H with minimal normal extension N defined on K. Then the real dimension of the convex set $\mathscr{P}(T)$ of positive approximants of T, denoted* $\dim \mathscr{P}(T)$, *is infinite.*

Proof. Let z_1 and z_2 be approximate eigenvalues of T off the contour Γ. Let α_j and β_j be real numbers such that $z_j = \alpha_j + i\beta_j$ and let $\{e_{j1}, e_{j2}, \ldots\}$ be a normalized approximate eigenvector for T corresponding to z_j. Because H is invariant under N, z_j is an approximate eigenvalue for N and $\{e_{j1}, e_{j2}, \ldots\}$ is a corresponding approximate eigenvector for N.

Let D be a compact set not intersecting Γ and containing $\{z_1, z_2\}$ in its interior. Define τ by the equation

$$2\tau = \inf \{\operatorname{re} z + (\delta^2-(\operatorname{im} z)^2)^{1/2}, (\delta^2-(\operatorname{im} z)^2)^{1/2} : z \in D \cap \sigma(N)\}$$

and note that the proof of (ii) of Lemma 2.3 implies that τ is positive. The functional calculus for N readily shows that $\lim \{\|f_{jk} - e_{jk}\| : k = 1, 2, \ldots\}$ is zero, where $f_{jk} = E(D)e_{jk}$ for $j = 1, 2$. It follows that $\lim \{\|(I-Q)f_{jk}\| : k = 1, 2, \ldots\}$ is zero, where Q is the orthogonal projection of K onto H. Replace the original sequences with subsequences if necessary so that $\{f_{1n}, f_{2n}\}$ is linearly independent for each n.

By definition the operator $A(\varrho; n)$ is zero on $(E(D)K)^\perp$, on $K(n) = \operatorname{span} \{f_{1n}, f_{2n}\}$ it is the matrix

$$\begin{pmatrix} \tau & \varrho \\ \varrho & \tau \end{pmatrix}$$

and on $E(D)K \ominus K(n)$ it is τI. It will be shown that $P_0 - A(\varrho; n)$ is a positive approximant of N for ϱ in an interval $(0, \varrho_0)$ for all n sufficiently large. If $(N-P_0)|E(D)K$ is written as a matrix relative to $K(n) \oplus (E(D)K \ominus K(n))$, then the nondiagonal entries converge to zero in operator norm by the choice of z_1 and z_2. Thus, it suffices to show that both $\|(N-P_0+A(\varrho; n))|E(D)K \ominus K(n)\|$ and $\|(N-P_0+A(\varrho; n))|K(n)\|$ are strictly less than ϱ for appropriate ϱ and n. The first inequality follows from (ii) of Lemma 2.3 and the choice of τ, and the second inequality is proved in the next paragraph.

Define $R(n)$ to be $(N-P_0)|K(n)$ minus the diagonal operator with entries $-(\delta^2-\beta_1^2)^{1/2}+i\beta_1$, $-(\delta^2-\beta_2^2)^{1/2}+i\beta_2$, respectively, and note that $(N-P_0+ + A(\varrho; n))|K(n)$ equals

$$\begin{pmatrix}\tau-(\delta^2-\beta_1^2)^{1/2}+i\beta_1 & 0\\ 0 & \tau-(\delta^2-\beta_2^2)^{1/2}+i\beta_2\end{pmatrix}+\begin{pmatrix}0 & \varrho\\ \varrho & 0\end{pmatrix}+R(n).$$

By the choice of τ, the norm of the first operator is strictly less than δ and the norms of the remaining two operators can be made arbitrarily small by the choice of ϱ and n, respectively. Thus, $P_0-A(\varrho; n)$ is a positive approximant of N.

Let m be any positive integer; a distinguished set of m positive approximants for N will be constructed. Let $\{z_1, \dots, z_{m+1}\}$ be a set of $m+1$ distinct approximate eigenvalues of T. For each pair $\{z_1, z_j\}$, the preceding construction results in a positive approximant $P_0-A(\varrho; n; j)$ for $j=2, \dots, m+1$.

By Theorem 2.1, $Q(P_0-A(\varrho; n; j))|H$ is a positive approximant for T, where Q is the orthogonal projection of K onto H. Recall from the second paragraph of this proof that $\lim \{\|f_{jk}-e_{jk}\| : k=1, 2, \dots\}$ is zero for $j=1, \dots, n+1$. It follows that $\{Qf_{j1}, Qf_{j2}, \dots\}$ is an approximate eigenvector for T corresponding to z_j for $j=1, \dots, m+1$. The linear independence of $\{Qf_{1n}, \dots, Qf_{m+1n}\}$ for all n sufficiently large is clear. In order to show that the dimension of $\mathscr{P}(T)$ is at least m it suffices to show the linear independence of

$$\{QA(\varrho;\ n;\ 2)|H, \dots, QA(\varrho;\ n;\ m+1)|H\};$$

thus, consider the matrix of $QA(\varrho; n; j)|H$ compressed to span $\{Qf_{1n}, \dots, Qf_{m+1n}\}$ relative to $\{Qf_{1n}, \dots, Qf_{m+1n}\}$. Make an appropriate choice for ϱ, and note that it is determined by $z_1, z_2, \dots, z_{m+1}, \tau$. Because each entry in the matrix for $QA(\varrho; n; j)|H$ relative to $\{Qf_{1n}, \dots, Qf_{m+1n}\}$ converges to the corresponding entry in the matrix of the compression of $A(\varrho; n; j)$ relative to $\{f_{1n}, \dots, f_{m+1n}\}$ as $n\to\infty$, it is not difficult to show that the first set of matrices are linearly independent for appropriately large n.

In fact, choose n so large that each entry in the matrix of the compression of $QA(\varrho; n; j)|H$ differs from the corresponding entry of the matrix of $A(\varrho; n; j)$ by less than ϱ/m. Denote those matrices by $M_1, \dots, M_m$ and assume that $c_1, \dots, c_m$ are real constants such that

$$0 = c_1 M_1 + \dots + c_m M_m.$$

By considering each entry in the first row, one obtains m equations of similar form, and each equation implies an inequation of the form

$$|c_j|\varrho < (\varrho/m)\sum_{k=1}^{m}|c_k|.$$

Adding up these inequalities results in a contradiction, which proves the theorem.

Recall the standard decomposition of a subnormal operator that was discussed prior to Lemma 2.2. If T is the orthogonal direct sum $T_1 \oplus T_2$, then it is clear that $\delta(T)$ is the maximum of $\{\delta(T_1), \delta(T_2)\}$. Consequently, unless $\delta(T_1)$ equals $\delta(T_2)$ there is much arbitrariness in the approximation of T. For example, if $\delta(T_2)$ exceeds $\delta(T_1)$, then $(P_1 - A)\oplus P_2$ is a positive approximant for T, where P_2 is any positive approximant for T_2, P_1 is any positive approximant for T_1 and A is any nonnegative operator dominated by P_1 and having norm dominated by $\delta(T_2)-\delta(T_1)$.

It should be noted that the construction carried out for the minimal normal extension N in the proof of Theorem 2.4 proves the following corollary.

2.5. Corollary. *If the spectrum of the normal operator N has an accumulation point not on the contour Γ consisting of all z with distance to* $[0, \infty)$ *exactly equal to $\delta(N)$, then the dimension of $\mathscr{P}(N)$ is infinite.*

A convex set, for example a disc in the plane, may have uncountably many extreme points, and the only implication about the dimension of the convex set is that it exceeds one. On the other hand, conclusions about the dimension of a convex set have immediate nontrivial implications about the number of extreme points.

2.6. Corollary. *If T is a completely nonnormal subnormal operator, then $\mathscr{P}(T)$ has an infinite number of extreme points.*

A consequence of some results of T. Sekiguchi in [13] is that $\mathscr{P}(T)$ has uncountably many extreme points.

§ 3. Self-adjoint approximants of a subnormal operator

Recall that $E(\cdot)$ denotes the spectral measure of the normal operator $N=B+iC$ with $B=B^*$, $C=C^*$, defined on the Hilbert space K.

3.1. Lemma. *If D is a compact set not intersecting the set $\Sigma=\{z\colon (\operatorname{im} z)^2=\|C\|^2,\ |z|\le\|N\|\}$, then $E(D)K$ reduces $(\|C\|^2-C^2)^{1/2}$ to an invertible operator.*

Proof. Clearly $|\operatorname{im} z|$ does not exceed $\|C\|$ for any z in $\sigma(N)$. Since D and Σ are both compact, there is a positive distance between them. It follows that there is a positive number ν such that $|\operatorname{im} z|\le\|C\|-\gamma$ for every z in the intersection of D and $\sigma(N)$. Consequently the set $\{(\|C\|^2-(\operatorname{im} z)^2)^{1/2}\colon z\in D\cap\sigma(N)\}$ is bounded away from zero, and this set is the spectrum of $(\|C\|^2-C^2)^{1/2}$ restricted to $E(D)K$.

Since self-adjoint approximants of a normal operator are studied in [6] and [10], attention is now concentrated on completely nonnormal subnormal operators.

3.2. Theorem. *Let T be a completely nonnormal subnormal operator defined on H with minimal normal extension N defined on K. Then the real dimension of the convex set $\mathcal{S}(T)$ of self-adjoint approximants of T, denoted* $\dim \mathcal{S}(T)$, *is infinite.*

Proof. This proof uses the same techniques as the proof of Theorem 2.4 with a few modifications which will be indicated. Choose $\{z_1, z_2\}$ as in the earlier proof and let D be a compact set not intersecting Σ and containing $\{z_1, z_2\}$ in its interior. Define τ by the equation

$$2\tau = \inf\{(\|C\|^2-(\operatorname{im} z)^2)^{1/2} : z\in D\cap \sigma(N)\}$$

and note that the proof of Lemma 3.1 implies that τ is positive. Proceed with the construction in the proof of Theorem 2.4.

It will be shown that $B_0-A(\varrho;n)$ is a self-adjoint approximant of N for ϱ in an interval $(0, \varrho_0)$ for all n sufficiently large where henceforth, B_0 denotes $B+ +(\|C\|^2-C^2)^{1/2}$. (Recall that Theorem 1 of [6] shows that B_0 dominates every self-adjoint approximant of N and Proposition 2 of [6] shows that B_0 is a self-adjoint approximant.) As in the earlier proof, it suffices to show that $\|(N-B_0+A(\varrho;n))| |E(D)K\ominus K(n)\|$ and $\|(N-B_0+A(\varrho;n))|K(n)\|$ are strictly less than $\|C\|$ for appropriate ϱ and n. The first inequality follows from the choice of τ and Lemma 3.1, and the second inequality in the next paragraph.

Define $R(n)$ to be $(N-B_0)|K(n)$ minus the diagonal operator with entries $-(\|C\|^2-\beta_1^2)^{1/2}+i\beta_1$, $-(\|C\|^2-\beta_2^2)^{1/2}+i\beta_2$ respectively. The remainder of the proof of this theorem proceeds by conspicuous analogy to the proof of Theorem 2.4. The resulting self-adjoint approximants of T are $Q(B_0-A(\varrho;n;j))|H$.

Recall the discussion immediately subsequent to Theorem 2.4. Analogously, unless $\|\operatorname{im} T_1\|$ equals $\|\operatorname{im} T_2\|$ there is much arbitrariness in the self-adjoint approximation of $T=T_1\oplus T_2$. For example, if $\|\operatorname{im} T_1\|$ exceeds $\|\operatorname{im} T_2\|$ and R_j is a self-adjoint approximant for T_j, with $j=1, 2$, then $R_1\oplus(R_2-A)$ is a self-adjoint approximant for T provided A is any self-adjoint operator whose norm is dominated by $\|\operatorname{im} T_1\|-\|\operatorname{im} T_2\|$.

The discussion prior to Corollary 2.5 and the discussion prior to Corollary 2.6 indicate the methods used to prove the next two results.

3.3. Corollary. *If the spectrum of the normal operator $N=B+iC$ has an accumulation point not contained in the set Σ consisting of all z with distance to $(-\infty, \infty)$ equal to $\|C\|$, then the dimension of $\mathcal{S}(N)$ is infinite.*

3.4. Corollary. *If T is a completely nonnormal subnormal operator, then $\mathcal{S}(T)$ has an infinite number of extreme points.*

§ 4. Positive approximants of Toeplitz operators

Notation. For $\delta=\delta(T)$, let $\Gamma=\{z$ in $\mathbf{C}$: dist $(z,[0,\infty))=\delta\}$. Let μ denote Lebesgue measure on the unit circle Δ, normalized so that $\mu(\Delta)=1$. For $p=2$ or $p=\infty$, we denote by $L^p(\Delta)$ the usual Lebesgue spaces. If φ is in $L^\infty(\Delta)$, then the definitions of the Laurent operator L_φ and Toeplitz operator T_φ are as in [9]. By [9, Problem 196] $\sigma(L_\varphi)\subset\sigma(T_\varphi)$.

From Theorem 1.3 it follows that $\delta(T_\varphi)=\delta(L_\varphi)=h$ (ess range φ, $[0,\infty)$) for all φ in $L^\infty(\Delta)$. We examine next the dimension of the convex set $\mathscr{P}(L_\varphi)$ $[\mathscr{P}(T_\varphi)]$ of positive approximants of Laurent [Toeplitz] operators.

4.1. Theorem. *Let φ be in $L^\infty(\Delta)$.*

(i) *If $\mu(\varphi^{-1}(\Gamma))<1$, then both $\mathscr{P}(L_\varphi)$ and $\mathscr{P}(T_\varphi)$ are infinite-dimensional.*

(ii) *If $\mu(\varphi^{-1}(\Gamma))=1$, then L_φ has a unique positive approximant; T_φ has a unique positive approximant if and only if* im φ *is constant.*

Proof. (i) Notice $\delta(T)>0$ in this case. Thus Γ is nondegenerate and the spectra of L_φ and T_φ lie inside Γ. Define the sets $F_k=\{\zeta:\text{dist}(\zeta,[0,\infty))\leqq\delta(1-1/k)\}$. Because $\mu(\varphi^{-1}(\Gamma))<1$, there exists $k\geqq 1$ such that $\mu(\varphi^{-1}(F_k))>0$. Fix such a k and write $\varphi^{-1}(F_k)=\bigcup_{j=1}^{\infty}S_j$, where $\{S_j\}$ is a pairwise disjoint collection of measurable sets, each having non-zero measure. Define non-negative functions $p(j)$ in $L^\infty(\Delta)$ by

$$p(j)(z)=\begin{cases}(\text{re}\,\varphi(z))_+ & z \text{ in } S_j\\ \text{re}\,\varphi(z)+(\delta^2-(\text{im}\,\varphi(z))^2)^{1/2} & \text{otherwise}\end{cases}$$

for $j=1,2,\ldots$. It is straightforward to verify that $\|L_{p(j)}-L_\varphi\|=\delta$. Hence each $L_{p(j)}$ is a positive approximant of L_φ, and $T_{p(j)}$ is a positive approximant of T_φ.

We next show that both $\mathscr{P}(L_\varphi)$ and $\mathscr{P}(T_\varphi)$ are infinite-dimensional by proving that

$$\{L_{p(j)}-L_{\text{re}\,\varphi+(\delta^2-(\text{im}\,\varphi)^2)^{1/2}}:j=1,2,\ldots\}$$

is linearly independent; this also shows that

$$\{T_{p(j)}-T_{\text{re}\,\varphi+(\delta^2-(\text{im}\,\varphi)^2)^{1/2}}:j=1,2,\ldots\}$$

is linearly independent since [9, Problem 196] Toeplitz operators and Laurent operators defined by the same function in $L^\infty(\Delta)$ have the same norm.

If $c_1,\ldots,c_n$ are real numbers such that $\sum_{j=1}^{n}c_j\big(L_{p(j)}-L_{\text{re}\,\varphi+(\delta^2-(\text{im}\,\varphi)^2)^{1/2}}\big)=0$, then choose r with $1\leqq r\leqq n$ and apply this linear combination to the characteristic function of S_r, which is in $L^2(\Delta)$. This clearly yields a function that is zero off S_r, and for z in S_r it is

$$c_r\big(-(\text{re}\,\varphi(z))_- -(\delta^2-(\text{im}\,\varphi(z))^2)^{1/2}\big)=0.$$

For $\varphi(z)$ in F_k, however, $-(\operatorname{re}\varphi(z))_- - (\delta^2 - (\operatorname{im}\varphi(z))^2)^{1/2}$ is bounded below in absolute value by a strictly positive constant that depends only on k (which is fixed). Because $\mu(S_r) > 0$, this proves $c_r = 0$.

(ii) If $\mu(\varphi^{-1}(\Gamma)) = 1$, then the essential range of φ is included in Γ. Hence [5, Theorem 5.6] L_φ has a unique positive approximant.

If $\operatorname{im}\varphi$ is constant, then T_φ is normal with spectrum included in Γ, and so it also has [5, Theorem 5.6] a unique positive approximant.

If $\operatorname{im}\varphi$ is not a constant, then the Halmos positive approximant $T_{\operatorname{re}\varphi} + (\delta^2 - (T_{\operatorname{im}\varphi})^2)^{1/2}$ and $T_{\operatorname{re}\varphi + (\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}}$ are two distinct positive approximants of T_φ. Proof: that both are positive approximants is straightforward to verify. If they were equal, then it would follow that $\delta^2 = (T_{\operatorname{im}\varphi})^2 + (T_{(\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}})^2$. To show this is impossible, let $e_k(z) = z^k$, $k = 0, 1, 2, \ldots$ be the usual orthonormal basis of $H^2(\Delta)$; with respect to this basis Toeplitz operators have matrices that are constant along each diagonal [9, Problem 194]. Hence there exists $k > 0$ such that $\langle T_{\operatorname{im}\varphi} e_k, e_0 \rangle \neq 0$ because $T_{\operatorname{im}\varphi}$ is self-adjoint and not a scalar. Notice that for a self-adjoint Toeplitz operator the fact that the entries in its corresponding matrix are constant along diagonals implies that the sum of the squares of each entry in a given column is exactly one term plus the same sum for the adjacent column on the left. Thus,

$$\delta^2 = \langle T^2_{\operatorname{im}\varphi} e_k, e_k \rangle + \langle T^2_{(\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}} e_k, e_k \rangle = \|T_{\operatorname{im}\varphi} e_k\|^2 + \|T_{(\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}} e_k\|^2$$

$$\geqq \|T_{\operatorname{im}\varphi} e_0\|^2 + |\langle T_{\operatorname{im}\varphi} e_k, e_0 \rangle|^2 + \|T_{(\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}} e_0\|^2 >$$

$$> \langle T^2_{\operatorname{im}\varphi} e_0, e_0 \rangle + \langle T^2_{(\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}} e_0, e_0 \rangle = \delta,$$

a contradiction.

This proves Theorem 4.1.

The previous theorem shows that the Halmos approximant of T_φ is distinct from the compression to $H^2(\Delta)$ of the Halmos approximant of L_φ (if $\operatorname{im}\varphi$ is not a scalar). The former, of course, always dominates the latter [5, Theorem 4.2]. The next result gives one more comparison of these two operators.

4.2. Theorem. *If* $\operatorname{im}\varphi$ *is continuous, then* $T_{\operatorname{re}\varphi} + (\delta^2 - (T_{\operatorname{im}\varphi})^2)^{1/2}$ *is a compact perturbation of* $T_{\operatorname{re}\varphi + (\delta^2 - (\operatorname{im}\varphi)^2)^{1/2}}$.

Proof. Let π denote the canonical homomorphism [7, p. 127] onto the Calkin algebra. By [7, Proposition 7.22], if ξ is a continuous function on Δ, then $(T_\xi)^2 - T_{\xi^2}$ is compact, *i.e.* $\pi((T_\xi)^2) = \pi(T_{\xi^2})$. If ξ is also non-negative, then $\pi(T_\xi) = \pi(T_{\xi^{1/2}})^2$, and hence $\pi(T_{\xi^{1/2}}) = (\pi(T_\xi))^{1/2}$. Thus, since $\pi(p^{1/2}) = \pi(p)^{1/2}$ for all $p \geqq 0$, it follows that

$$\pi((\delta^2 - (T_{\operatorname{im}\varphi})^2)^{1/2}) = (\pi(\delta^2 - (T_{\operatorname{im}\varphi})^2))^{1/2} = (\pi(\delta^2) - \pi(T_{(\operatorname{im}\varphi)^2}))^{1/2} =$$

$$= (\pi(T_{\delta^2 - (\operatorname{im}\varphi)^2}))^{1/2} = \pi(T_{(\delta^2 - (\operatorname{im}\varphi))^{1/2}}). \qquad \text{Q.E.D.}$$

§ 5. Self-adjoint approximants of Toeplitz operators

The results of the previous section on positive approximation have analogues for self-adjoint approximation. Of course, both the distance from L_φ to the self-adjoint operators on $L^2(\Delta)$ and the distance from T_φ to the self-adjoint operators on $H^2(\Delta)$ are $\|\mathrm{im}\,\varphi\|_\infty$ [8] for any φ in $L^\infty(\Delta)$. We now examine the dimension of the convex set $\mathscr{S}(L_\varphi)$ $[\mathscr{S}(T_\varphi)]$ of self-adjoint approximants of Laurent [Toeplitz] operators. We use from § 3 the definition $\Sigma=\{z\colon (\mathrm{im}\,z)^2=\|\mathrm{im}\,\varphi\|_\infty^2,\ |z|\leqq\|\varphi\|_\infty\}$.

5.1. Theorem. *Let φ be in $L^\infty(\Delta)$.*

(i) *If $\mu(\varphi^{-1}(\Sigma))<1$, then both $\mathscr{S}(L_\varphi)$ and $\mathscr{S}(T_\varphi)$ are infinite-dimensional.*

(ii) *If $\mu(\varphi^{-1}(\Sigma))=1$, then L_φ has a unique self-adjoint approximant; the Toeplitz operator T_φ has a unique self-adjoint approximant if and only if* $\mathrm{im}\,\varphi$ *is constant.*

Proof. (i) Notice that in this case $\mathrm{im}\,\varphi$ is not identically zero. For $k=1, 2, 3, \ldots$ define $F_k=\{\zeta\colon |\mathrm{im}\,\zeta|\leqq(1-1/k)^{1/2}\|\mathrm{im}\,\varphi\|_\infty\}$. Because $\mu\varphi^{-1}(F_k)<1$, there exists $k\geqq 1$ such that $\mu(\varphi^{-1}(F_k))>0$. Fix such a k and write $\varphi^{-1}(F_k)=\bigcup_{j=1}^{\infty} S_j$ where $\{S_j\}$ is a pairwise disjoint collection of measurable sets of non-zero measure. Define the real-valued functions $s(j)$ in $L^\infty(\Delta)$ by

$$s(j)(z)=\begin{cases}\mathrm{re}\,\varphi(z) & z \text{ in } S_j\\ \mathrm{re}\,\varphi(z)+(\|\mathrm{im}\,\varphi\|^2-(\mathrm{im}\,\varphi(z))^2)^{1/2} & \text{otherwise.}\end{cases}$$

It is again straightforward to verify that $\|L_{s(j)}-L_\varphi\|=\|\mathrm{im}\,\varphi\|_\infty$. Hence each $L_{s(j)}$ is a self-adjoint approximant of L_φ and each $T_{s(j)}$ is a self-adjoint approximant of T_φ

We prove that both $\mathscr{S}(L_\varphi)$ and $\mathscr{S}(T_\varphi)$ are infinite-dimensional by proving that

$$\{L_{s(j)}-L_{\mathrm{re}\,\varphi+(\|\mathrm{im}\,\varphi\|^2-(\mathrm{im}\,\varphi)^2)^{1/2}}\colon\quad j=1,2,\ldots\}$$

is linearly independent, which also proves that the corresponding Toeplitz operators are linearly independent. If $\sum_{j=1}^{n} c_j(L_{s(j)}-L_{\mathrm{re}\,\varphi+(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi)^2)^{1/2}})=0$, then choose r with $1\leqq r\leqq n$ and apply this linear combination to the characteristic function of S_r, which is in $L^2(\Delta)$. This yields a function that is zero off S_r and for z in S_r it is $-c_r(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi(z))^2)^{1/2}$. For $\varphi(z)$ in F_k, however, $(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi)z))^2)$ is bounded below by $\frac{1}{k}\|\mathrm{im}\,\varphi\|_\infty^2$, which is independent of r. Because $\mu(S_r)>0$, this proves $c_r=0$.

(ii) If $\mu(\varphi^{-1}(\Sigma))=1$, then the essential range of φ is included in Σ. Hence [6] L_φ has a unique self-adjoint approximant.

If $\mathrm{im}\,\varphi$ is constant, then T_φ is normal with spectrum included in Σ, so it also has a unique self-adjoint approximant. If $\mathrm{im}\,\varphi$ is not a constant, then $T_{\mathrm{re}\,\varphi}+$

$+(\|\mathrm{im}\,\varphi\|_\infty^2-(T_{\mathrm{im}\,\varphi})^2)^{1/2}$ and $T_{\mathrm{re}\,\varphi+(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi)^2)^{1/2}}$ are two distinct self-adjoint approximants of T_φ. The proofs of these last two assertions are entirely analogous to those given in Theorem 4.1 and are hence omitted. This completes the proof of Theorem 5.1.

It follows from [6, Theorem 1] that

$$T_{\mathrm{re}\,\varphi}+(\|\mathrm{im}\,\varphi\|_\infty^2-(T_{\mathrm{im}\,\varphi})^2)^{1/2} \geqq T_{\mathrm{re}\,\varphi+(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi)^2)^{1/2}}.$$

The following comparison of these two operators can be proved as in Theorem 4.2.

5.2. Theorem. *If* $\mathrm{im}\,\varphi$ *is continuous, then* $T_{\mathrm{re}\,\varphi}+(\|\mathrm{im}\,\varphi\|_\infty^2-(T_{\mathrm{im}\,\varphi})^2)^{1/2}$ *is a compact perturbation of* $T_{\mathrm{re}\,\varphi+(\|\mathrm{im}\,\varphi\|_\infty^2-(\mathrm{im}\,\varphi)^2)^{1/2}}$.

Remark. It is not known whether $\mathscr{S}(T_\varphi)$ and $\mathscr{P}(T_\varphi)$ must be either zero-dimensional or infinite-dimensional for each φ in $L^\infty(\Delta)$.

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