Distributive lattices whose prime ideals are principal

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A well known theorem of I. S. COHEN [1] states that if R is a commutative ring with 1, and every prime ideal in R is principal, then every ideal in R is principal. In this note, the analogue of this theorem is proved for distributive lattices with 1.

Theorem. *Let L be a distributive lattice with* 1. *If every prime ideal in L is principal, then every ideal is principal.*

Proof. Suppose the theorem is false. Let $\mathscr{C}$ denote the non-empty collection of non-principal ideals of L. It is clear that $\mathscr{C}$ is closed under the formation of unions of chains in $\mathscr{C}$. So, by Zorn's lemma we get a maximal element M in $\mathscr{C}$ which is not principal. Since L is principal, $L \neq M$. So, there exist elements $a, b \notin M$ such that $a \wedge b \in M$. Now as M is a maximal element in $\mathscr{C}$, $M \vee (a]$ and $M \vee (b]$ are principal ideals, and, by distributivity $M = (M \vee (a]) \wedge (M \vee (b])$ contradicting that M is not principal. Hence the result.

Obviously, the condition of distributivity cannot be dropped from the Theorem as stated. However, G. Grätzer pointed out that the proof of our theorem carries over to meet-irreducible ideals of general lattices. In a distributive lattice, prime ideals are exactly the meet-irreducible ideals. So we have the following result.

Theorem. *Let L be a lattice with* 1. *If in L every meet-irreducible ideal is principal, then every ideal in L is principal.*

Reference

[1] I. S. COHEN, Rings with restricted minimum condition, *Duke Math. J.*, **17** (1950), 27—42.

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