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On the spectrum of contractions of class $C_{\cdot 1}$

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1. In this paper we shall consider (bounded) operators in complex separable Hilbert spaces. We shall use notations from [8], and **Z** will denote the integers, **N** the natural numbers, **C** the field of complex numbers. We denote the open unit disc by D, the unit circle by C, and the annulus $\{\lambda \in \mathbf{C}: 1/2 \leqq |\lambda| \leqq 1\}$ by K. For a contraction $T \in \mathcal{L}(\mathfrak{H})$ we denote by $\sigma(T)$ its spectrum, by $\sigma_p(T)$ its point spectrum, $D_T = (I - T^*T)^{1/2}$ denotes the defect operator, $\mathfrak{D}_T = \overline{D_T \mathfrak{H}}$ the defect space, and $\mathfrak{d}_T = \dim \mathfrak{D}_T$ the defect number of T.

B. SZ.-NAGY and C. FOIAŞ call the contraction T of *class* $C_{\cdot 1}$ if $T^{*n}x \not\to 0$ for all $x \in \mathfrak{H}$, $x \neq 0$ (see [8], Ch. II. Section 4). In [8] Ch. VII, 6.3, or [8], Th. 2* they prove that if $T \in C_{\cdot 1}$ and δ_{T^*} is finite then $\sigma(T) = \bar{D}$ or $\sigma(T) \subseteqq C$. Moreover, in the first case, $\sigma_p(T) \supseteqq D$ and $T \notin C_{11}$, while in the second, $T \in C_{11}$. In the case $\mathfrak{d}_{T^*} = \infty$ it is posible that $T \in C_{11}$ and $\sigma(T) = \bar{D}$ (see [8], Ch. VI, Section 4).

This raises the following questions:

a) If $T \in C_{01}$, does it follow that $\sigma(T) \cap D \neq \emptyset$?

b) If $T \in C_{01}$ and $\sigma(T) \cap D \neq \emptyset$, then does it follow that $\sigma_p(T) \neq \emptyset$?

c) If $T \in C_{\cdot 1}$, does it follow that $\sigma(T) = \bar{D}$ or $\sigma(T) \subseteqq C$?

d) If $T \in C_{\cdot 1}$ and $\sigma_p(T) \cap D \neq \emptyset$, does it follow that $\sigma_p(T) \supset D$?

e) If $T \in C_{\cdot 1}$ and $1 \notin \sigma(T)$, does it follow that $\sigma(T) \subset C$?

GILFEATHER [2] gave a negative answer to a) and b). Using weighted shifts he proved that

a) there is an operator $T \in C_{01}$ with $\sigma(T) = C$, and

b) there is an operator $T \in C_{01}$ with $\sigma(T) = \bar{D}$ and $\sigma_p(T) = \emptyset$.

The aim of this note is to give a negative answer to c) and d).

2. Theorem 1. *There exists $T \in C_{01}$ with $\sigma(T) = K$.*

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Proof. Let $\mathfrak{H}$ be a space with orthonormal basis $\{\varphi_n\}_{n\in\mathbf{Z}}$ and let T be the weighted shift in $\mathfrak{H}$ defined by

$$T\varphi_n = w_n \varphi_{n+1} \quad (n\in\mathbf{Z}) \quad \text{where} \quad w_n = 1 \quad \text{for} \quad n \leqq 0 \quad \text{and } 1/2 \text{ for} \quad n > 0.$$

T is a contraction ($\|T\| = \sup |w_n| = 1$, see [4]). It is of class C_{01} since $\prod_{n>0} w_n$ diverges and $\prod_{n<0} w_n$ converges (see [2]). The spectrum of T is K, since

$$1/2 = \lim_{n\to\infty} \inf_{k\in\mathbf{Z}} \{(w_k w_{k+1} \dots w_{k+n-1})^{1/n}\}, \quad 1 = \lim_{n\to\infty} \sup_{k\in\mathbf{Z}} \{(w_k w_{k+1} \dots w_{k+n-1})^{1/n}\}$$

(see [3], [6] and [4]).

We shall see now that the alternative of problem c) does not occur even if $T\in C_{11}$.

Theorem 2. *There exists an operator $T\in C_{11}$ with $\sigma(T)=K$.*

Proof. Let $\mathfrak{H}$ be a space with orthonormal basis $\{\varphi_{ij}\}_{(i,j)\in\mathbf{N}\times\mathbf{Z}}$ and let $T\in L(\mathfrak{H})$ be defined by

$$T\varphi_{ij} = w_{ij}\varphi_{i,j+1} \quad (i\in\mathbf{N}, j\in\mathbf{Z}),$$

where

$$w_{ij} = 1 \quad \text{for} \quad j\notin[0,i] \quad \text{and } 1/2 \text{ for} \quad j\in[0,i].$$

One can verify that $T\in C_{11}$ and $0\notin\sigma(T)$. Taking $h_n = \sum_{k=0}^{n-1} n^{-1/2}\varphi_{nk}\in\mathfrak{H}$ we have $\|h_n\|=1$ and $\|\mathrm{Th}_n - (1/2)h_n\| = \|(1/2)n^{-1/2}\varphi_{nn} - (1/2)n^{-1/2}\varphi_{n0}\| = (2n)^{-1/2}\to 0$; hence $1/2\in\sigma(T)$. We have $\sigma(T)\neq\bar{D}$ and $\sigma(T)\not\subset C$. Consider the unitary operators $S_t\in\mathscr{L}(\mathfrak{H})$ defined by

$$S_t\varphi_{mn} = \exp(-\mathrm{int})\varphi_{mn}.$$

We have $S_t^{-1}TS_t = \exp(\mathrm{it})\, T$ from which we deduce the circular symmetry of $\sigma(T)$. Moreover, by condition $T\in C_{11}$, the spectrum of T has no components far from C (since then there would exist a non-trivial subspace $\mathfrak{H}_0$ of $\mathfrak{H}$, with $TH_0\subset\mathfrak{H}_0$ and $\sigma(T|\mathfrak{H}_0)\subset D$, so that $T^n h_0\to 0$ for $h_0\in\mathfrak{H}_0$). Since $\|T^{-1}\|=2$ and since, by [6], $\sigma(T)$ is an annulus, it follows that $\sigma(T)=K$.

3. In this section we shall give a class of contractions for which the alternative of c) is true. We shall use the functional model introduced by Sz.-Nagy and Foiaş (see [8], Ch. V and VI). For a contraction $T\in\mathscr{L}(\mathfrak{H})$ we have:

$$\sigma_p(T)\cap D = \{\lambda\in D\colon \Theta_T(\lambda) \quad \text{is not injective}\},$$

$$\sigma(T)\cap D = \{\lambda\in D\colon \Theta_T(\lambda) \quad \text{is not invertible}\},$$

where $\Theta_T(\lambda)\colon \mathfrak{D}_T\to\mathfrak{D}_{T^*}$ is the characteristic function

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T \quad (\lambda\in D).$$

Since T maps $\mathfrak{H}\ominus\mathfrak{D}_T$ unitarily on $\mathfrak{H}\ominus\mathfrak{D}_{T^*}$ we can here replace $\Theta_T(\lambda)$ by

$$T(\lambda) = -T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T.$$

Suppose that $\sigma_p(T^*)\cap D=\emptyset$ and that $D_{T^*}(I-\lambda T^*)^{-1}D_T$ is compact for each $\lambda\in D$. If $\lambda_0\in D\setminus\sigma(T)$, then $T(\lambda_0)$ is invertible, hence it is Fredholm of index 0 (that is $T(\lambda_0)\mathfrak{H}$ is closed and $\dim\operatorname{Ker} T(\lambda_0)=\dim\operatorname{Ker} T^*(\lambda_0)<\infty$). For $\lambda\in D$ we have

$$T(\lambda) = T(\lambda_0)+[T(\lambda)-T(\lambda_0)].$$

Since $T(\lambda)-T(\lambda_0)$ is compact, we deduce that $T(\lambda)$ is Fredholm of index 0 (see [1] or [5]). But $\operatorname{Ker} T^*(\lambda)=\{0\}$ since $\sigma_p(T^*)\cap D=\emptyset$ hence $T(\lambda)$ is invertible. We have proved the following

Proposition. *If $T\in\mathscr{L}(\mathfrak{H})$ is a contraction with $\sigma_p(T^*)=\emptyset$ and if $\Theta_T(\lambda)-\Theta_T(0)$ is compact for each $\lambda\in D$ then $\sigma(T)=\bar{D}$ or $\sigma(T)\subset C$.*

Remark. The hypothesis of this proposition is fulfilled in particular if $T\in C_{\cdot 1}$ with D_T or D_{T^*} compact.

We shall see that even under the hypothesis of the proposition, problem d) has a negative answer.

Theorem 3. *There exists an operator $T\in C_{\cdot 1}$ with $\Theta_T(\lambda)-\Theta_T(0)$ compact and $\sigma_p(T)=\{0\}$.*

Proof. Let $\mathfrak{E}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n\ge 0}$, $\mathfrak{E}_1$ the subspace of $\mathfrak{E}$ generated by $\{e_n\}_{n\ge 1}$, and let $S\in\mathscr{L}(\mathfrak{E})$ be the operator defined by

$$e_0\mapsto 0,\quad e_n\mapsto (1/n)\,e_{n-1}\quad (n>0).$$

Let $F\in\mathscr{L}(\mathfrak{E})$ be the compact operator defined by

$$Fe_0=f=\sum_{k=1}^{\infty}\frac{1}{k+1}e_k,\quad F\mathfrak{E}_1=\{0\}.$$

We have $\overline{S\mathfrak{E}}=\overline{S\mathfrak{E}_1}=\mathfrak{E}$ and $f\notin S\mathfrak{E}$. Consider the analytic contractive function $\{\mathfrak{E},\mathfrak{E},\Theta(\lambda)\}$ defined by

$$\Theta(\lambda)=(\|S\|+\|F\|)^{-1}(S+\lambda F).$$

As $F|\mathfrak{E}_1=0$, we have

$$\overline{\Theta H^2(\mathfrak{E})}\supset\overline{\Theta H^2(\mathfrak{E}_1)}=\overline{SH^2(\mathfrak{E}_1)}=H^2(\mathfrak{E}),$$

that is, $\Theta(\lambda)$ is an outer function. If $\lambda\in D\setminus\{0\}$ and $\Theta(\lambda)x=0$, then $Sx=-\lambda Fx$, hence $Sx=0$ and $Fx=0$. But from the first equality it follows that $x=\alpha e_0$, and from the second that $\alpha=0$, hence $\Theta(\lambda)$ is injective for each $\lambda\in D\setminus\{0\}$. Constructing the contraction $\mathbf{T}$ (see [8], Ch. VI. 3) we obtain a contraction of class $C_{\cdot 1}$ with $\sigma_p(\mathbf{T})=\{0\}$ and $\Theta_{\mathbf{T}}(\lambda)-\Theta_{\mathbf{T}}(0)$ compact.

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