Acta Sci. Math., 39 (1977), 265—272

Congruence-equalities and Mal'cev conditions in regular equational classes

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FREESE and NATION have shown in [1] that there is no lattice equality holding in all congruence lattices of semilattices. It follows easily that this result remains true if one replaces the variety of semilattices by any variety defined by a set of regular equations. On the other hand not every algebraic lattice is the congruence lattice of a semilattice, see HALL [4] and PAPERT [5]. WILLE has introduced in [9] the notion of a congruence equality using the binary term $\circ$ (relational product) in addition to the binary terms $\vee$ (join) and $\wedge$ (meet). We are going to show in this paper that the result of Freese and Nation is also true for a certain class of congruence-equalities in $\wedge$, $\vee$ and $\circ$, and on the other hand we provide congruence-equalities which are nontrivial and which do hold in semilattices. This also gives us examples of congruence-equalities which do not imply any lattice equation.

Two such congruence equalities are characterized in terms of Mal'cev conditions and it turns out that they are within the class of regular varieties equivalent to the Mal'cev conditions

$$\exists p(p(x,x)=x,\, p(x,y)=p(y,x)), \quad \text{resp.} \quad \exists p(p(x,x,x)=x,\, p(x,y,z)=p(z,x,y)).$$

Finally we characterize the above Mal'cev conditions within the class of all varieties in terms of fixed points of involutions similar to [3]. For basic facts and notations used in this paper see GRÄTZER [2]. For the notion of equivalence see, e. g., TAYLOR [8].

1. Regular varieties

1.1. Definition. (PŁONKA [7]) An equation $p=q$ is called *regular* if the set of variables and constants appearing in p is the same as that in q. A *variety* is *regular* if it can be defined by a set of regular equations.

Received May 26, 1976.

1.2. Example. The variety of *semilattices* is a regular variety. The defining equations are: $x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Next we formulate two basic lemmas. The first can be easily proved, and the second was essentially proved in [10].

1.3. Lemma. *Let $\Delta = (n_i | i \in I)$ be a type with corresponding function symbols f_i, $i \in I$. Let $\mathbf{2} := \{0, 1\}$ be a two-element set and define an algebra $\mathbf{2}_\Delta$ by setting*

$$f_i(x_1, \ldots, x_{n_i}) := \begin{cases} 1 & \text{if} \quad x_1 = x_2 = \ldots = x_{n_i} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If there are 0-ary function symbols define them to be 0. Let $\mathbf{SL}_\Delta$ be the variety generated by $\mathbf{2}_\Delta$. Then,

(i) $\mathbf{SL}_\Delta$ *is equivalent to* $\mathbf{SL}_{(2)}$, *the variety of all semilattices iff $n_i \geqq 2$ for some i, and $n_i \neq 0$ for all i.*

(ii) $\mathbf{SL}_\Delta$ *is equivalent to* $\mathbf{SL}_{(0,2)}$, *the variety of all 0-semilattices, iff $n_i \geqq 2$ for some i and $n_i = 0$ for some i.*

(iii) $\mathbf{SL}_\Delta$ *is equivalent to* $\mathbf{\Omega}_\Delta$, *the variety of pointed sets iff $n_i \leqq 1$ for all i, and $n_i = 0$ for some i.*

(iv) $\mathbf{SL}_\Delta$ *is equivalent to the variety of sets otherwise.*

1.4. Lemma. [10] *Let $\mathfrak{B}$ be a variety of type Δ, containing no nullary operation. Then $\mathfrak{B}$ is regular if and only if $\mathfrak{B}$ contains $\mathbf{SL}_\Delta$ as a subvariety. If Δ contains a 0-ary operation, the only if part is still true.*

2. Congruence equalities

Congruence equalities were introduced by Wille [9].

2.1. Definition. A *congruence equality* is an expression $\alpha = \beta$ where α and β are terms in variables and the binary polynomial symbols $\wedge$, $\vee$ and $\circ$. A congruence-equality $\alpha = \beta$ is said to be *congruence-valid* in an algebra $\mathfrak{A}$ if for any interpretation of the variables occurring in $\alpha = \beta$ by congruences of $\mathfrak{A}$ the equation holds if we interpret $\wedge$ as meet, $\circ$ as relational product and $\vee$ as *relational join,* that means: If σ and τ are binary relations on A, we define: $\sigma \vee \tau := \bigcup_{n \in \mathbf{N}} \{\underbrace{\sigma \circ \tau \circ \sigma \circ \ldots \circ \tau}_{\text{n-times}} | n \in \mathbf{N}\}$.

We have to be careful because if γ, θ are congruences then $\gamma \circ \theta$ need not be a congruence. If σ and τ happen to be congruences, then $\sigma \vee \tau$ is the join of σ and τ.

We call a congruence-equality *trivial* if it holds in each partition lattice. We say that $\alpha = \beta$ is *congruence-valid* in a variety $\mathfrak{B}$ if it is congruence-valid for each algebra $\mathfrak{A} \in \mathfrak{B}$.

Now it is obvious what we mean by a *congruence-inequality* $\alpha \leqq \beta$ and in fact we can replace each congruence-equality $\alpha = \beta$ by the congruence-inequalities $\alpha \leqq \beta$ and $\alpha \geqq \beta$. Clearly, a congruence inequality $\alpha \leqq \beta$ which holds in a variety $\mathfrak{B}$ will hold in each variety $\mathfrak{B}'$ which is equivalent to $\mathfrak{B}$ as well.

For the proof of our first theorem we need the following simple lemma:

2.2. Lemma. *Let $\alpha \leqq \beta$ be a nontrivial congruence-inequality. Then there exists a finite set X, such that $\alpha \leqq \beta$ fails to hold in $\pi(X)$, the partition lattice of X.*

Proof. The proof essentially uses the ideas of theorem 6.15 in Wille [9]. $\alpha \leqq \beta$ is nontrivial, thus there exists a set X such that $\alpha \leqq \beta$ does not hold for the partitions of X. Let $x_1, \ldots, x_n$ be the variables occurring in $\alpha \leqq \beta$. Let $\mathbf{i}$ be an interpretation map assigning to x_i, $0 < i \leqq n$, the partition θ_i of X such that for a certain pair (a, b) we have $(a, b) \in \mathbf{i}(\alpha)$ and $(a, b) \notin \mathbf{i}(\beta)$.

Let γ now be an arbitrary expression in $\wedge$, $\vee$ and $\circ$ and the variables amongst $\{x_1, \ldots, x_n\}$. Let x, y be arbitrary elements of X. Define recursively:

1) If γ is a variable,

$$R^{\gamma}_{(x,y)} := \begin{cases} \{x, y\} & \text{if } (x, y) \in \mathbf{i}(\gamma) \\ \emptyset & \text{otherwise.} \end{cases}$$

2) If $\gamma = \sigma \circ \tau$,

$$R^{\gamma}_{(x,y)} := \begin{cases} R^{\sigma}_{(x,z)} \cup R^{\tau}_{(z,y)} & \text{for some } z \text{ with } (x, z) \in \mathbf{i}(\sigma) \text{ and } (z, y) \in \mathbf{i}(\tau) \\ \emptyset & \text{if } (x, y) \notin \mathbf{i}(\gamma). \end{cases}$$

3) If $\gamma = \sigma \vee \tau$,

$$R^{\gamma}_{(x,y)} := \begin{cases} R^{\sigma}_{(x,z_1)} \cup R^{\tau}_{(z_1,z_2)} \cup \ldots \cup R^{\tau}_{(z_n,y)} & \text{for some } z_1, \ldots, z_n \text{ with} \\ & x\,\mathbf{i}(\sigma)\,z_1\,\mathbf{i}(\tau)\,z_2 \ldots z_n\,\mathbf{i}(\tau)\,y \\ \emptyset & \text{if } (x, y) \notin \mathbf{i}(\gamma). \end{cases}$$

4) If $\gamma = \sigma \wedge \tau$,

$$R^{\gamma}_{(x,y)} := \begin{cases} R^{\sigma}_{(x,y)} \cup R^{\tau}_{(x,y)} & \text{if } (x, y) \in \mathbf{i}(\gamma) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $X_0 := R^{\alpha}_{(a,b)}$ is finite and nonempty. Define $\theta_i^0 := \theta_i \cap X_0 \times X_0$ and $\mathbf{i}_0 \colon x_i \to \theta_i^0$, $0 < i \leqq n$. Then clearly by the construction we have $(a, b) \in \mathbf{i}_0(\alpha)$ and $(a, b) \notin \mathbf{i}_0(\beta)$. Thus $\alpha = \beta$ does not hold for the partitions of the finite set X_0.

2.3. Theorem. *Let $\alpha \leqq \beta$ a nontrivial congruence-inequality where α is arbitrary and β is of the form $\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_k$ where each σ_i is a term in $\vee$ and $\circ$. Then each regular variety contains a finite algebra where $\alpha \leqq \beta$ is not congruence-valid.*

Proof. If a congruence-inequality holds in a variety $\mathfrak{B}$ then it obviously holds in each subvariety of $\mathfrak{B}$ and in each variety which is equivalent to $\mathfrak{B}$. By lemma 1.5

we need to prove our statement only for $\mathbf{SL}_\Delta$. As the variety of sets and the variety of pointed sets do not fulfil any nontrivial congruence equality we need in view of lemma 1.4 only consider $\mathbf{SL}_{(2)}$ and $\mathbf{SL}_{(0,2)}$, semilattices and 0-semilattices. Let now X be a set, π a partition of X and $\mathbf{FSL}(X)$ (resp. $\mathbf{FSL}_0(X)$) be the free semilattice resp. 0-semilattice generated by X. Let θ_π be the congruence generated by π in $\mathbf{FSL}(X)$ (resp. $\mathbf{FSL}_0(X)$) and let p and q be elements of $\mathbf{FSL}(X)$ (resp. $\mathbf{FSL}_0(X)$). We assume that p and q are in reduced normal form. Then we have $p\theta_\pi q$ if and only if for each variable x in p there is a variable y in q such that $x\pi y$ and vice versa.

By a repeated use of this argument one obtains that for a set $\pi_1, \ldots, \pi_n$ of partitions of X and $x, y \in X$ we have:

$$(*) \qquad x\theta_{\pi_1}\circ \ldots \circ \theta_{\pi_n} y \quad \text{if and only if} \quad x\pi_1 \circ \ldots \circ \pi_n y.$$

Now let $\alpha \leqq \beta$ a congruence-inequality of the form required in our theorem. Then there exists by lemma 2.2 a finite set X and partitions $\pi_1, \ldots, \pi_n$ of X and an interpretation $\mathbf{i}$ assigning the variables $x_1, \ldots, x_n$ of $\alpha \leqq \beta$ to the partitions $\pi_1, \ldots, \pi_n$ such that for some $x, y \in X$ we have $(x, y) \in \mathbf{i}(\alpha)$ and $(x, y) \notin \mathbf{i}(\beta)$.

Take now $\mathbf{FSL}(X)$ resp. $\mathbf{FSL}_0(X)$ and define $\bar{\mathbf{i}}: x_i \mapsto \theta_{\pi_i}$. Of course we still have $(x, y) \in \bar{\mathbf{i}}(\alpha)$, but by $(*)$ we have $(x, y) \notin \bar{\mathbf{i}}(\beta)$. Thus $\alpha \leqq \beta$ does not hold in $\mathbf{FSL}(X)$ nor in $\mathbf{FSL}_0(X)$; and both are finite algebras, which concludes the proof.

2.4. Definition. A variety is *n-permutable* iff the congruence-inequality $\theta_1 \circ \theta_2 \circ \ldots \circ \theta_2 \subseteqq \theta_2 \circ \theta_1 \circ \ldots \circ \theta_1$, with n factors on each side, holds in $\mathfrak{B}$.

2.5. Corollary. *Regular varieties are not n-permutable for any n.*

Now we are going to show that we cannot drop the assumption on the form of β.

3. Mal'cev conditions

For basic facts concerning Mal'cev conditions see e.g. Taylor [8].

3.1. Definition. A strong Mal'cev condition is an expression of second order logic of the form $\exists p_1, \ldots, p_n(\Sigma)$ where Σ is a finite conjunction of equations universally quantified in individual variables, containing the function variables $p_1, \ldots, p_n$. A strong Mal'cev condition $\mathbf{M} := \exists p_1, \ldots, p_n(\Sigma)$ holds in a variety $\mathfrak{B}$ (shortly $\mathfrak{B} \vdash \mathbf{M}$) iff there exist polynomials $p_1, \ldots, p_n$ in the language of $\mathfrak{B}$ such that Σ holds in $\mathfrak{B}$.

3.2. Definition. An *involution* is an automorphism of order two.

3.3. Theorem. *For an arbitrary variety $\mathfrak{B}$ the following are equivalent:*

(i) *The strong Mal'cev condition $\exists p(p(x, x) = x \wedge p(x, y) = p(y, x))$ holds in $\mathfrak{B}$.*

(ii) *If φ is an involution of an algebra $\mathfrak{A} \in \mathfrak{B}$ then for each $x \in \mathfrak{A}$ there exists a fixed point y of φ such that $(x, \varphi x) \in \theta$ implies $(x, y) \in \theta$ for arbitrary congruences θ of $\mathfrak{A}$.*

A similar theorem with automorphisms of order n holds for the Mal'cev condition $\exists p(p(x, \ldots, x)=x \wedge p(x_1, \ldots, x_n)=p(x_2, \ldots, x_n, x_1))$.

Proof. (i)→(ii): Assume (i) and let φ be an involution of $\mathfrak{A}\in\mathfrak{B}$. Take $x\in\mathfrak{A}$. Then define $y:=p(x, \varphi x)$. We have: $\varphi(y)=\varphi(p(x, \varphi x))=p(\varphi(x), \varphi^2(x))= =p(\varphi(x), x)=p(x, \varphi(x))=y$. Thus y is a fixed point of φ. Assume $(x, \varphi x)\in\theta$. Then $x=p(x, x)\theta p(x, \varphi x)=y$. Thus $(x, y)\in\theta$.

(ii)→(i) Let $\mathbf{F}_{\mathfrak{B}}(x, y)$ be the free algebra in $\mathfrak{B}$ generated by the two distinct elements x and y. Then the map $\varphi: x\to y, y\to x$ extends uniquely to a homomorphism φ of $\mathbf{F}_{\mathfrak{B}}(x, y)$ which is moreover an involution. For x we then have an element $z\in\mathbf{F}_{\mathfrak{B}}(x, y)$ which is a fixed point of φ. Here $z=p(x, y)$ for some polynomial p and $\varphi z=z$, thus $\varphi p(x, y)=p(x, y)$. As $\varphi p(x, y)=\varphi p(x, \varphi x)=p(\varphi x, \varphi^2 x)=p(y, x)$ we conclude $p(x, y)=p(y, x)$. Now $(x, \varphi x)\in\theta_{(x,y)}$, the smallest congruence which collapses x and y. By (ii) we have: $(x, z)\in\theta_{(x,y)}$ which means $(x, p(x, y))\in\theta_{(x,y)}$ and thus $p(x, x)=x$. Hence, $p(x, y)=p(y, x)$ and $p(x, x)=x$ holds in the variety $\mathfrak{B}$. Wille [9] and Pixley [6] have shown that in a variety each congruence-inequality in $\wedge$, $\vee$ and $\circ$ is equivalent to a countable conjunction of countable disjunctions of strong Mal'cev conditions.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{g}$ be the following congruence inequalities:

$$\mathbf{e}_1: \theta_0\wedge(\theta_1\circ\theta_2)\wedge(\theta_3\circ\theta_4) \leqq \theta_1\circ\{(\theta_2\circ\theta_3)\wedge\{[(\theta_1\circ\theta_3)\wedge(\theta_2\circ\theta_4)]\circ\theta_0\}\}\circ\theta_4,$$

$$\mathbf{e}_2: (\theta_1\circ\theta_2)\wedge(\theta_3\circ\theta_4) \leqq \theta_1\circ\{(\theta_2\circ\theta_3)\wedge\{[(\theta_1\circ\theta_3)\wedge(\theta_2\circ\theta_4)]\circ[(\theta_1\circ\theta_2)\wedge(\theta_3\circ\theta_4)]\}\}\circ\theta_4.$$

($\mathbf{e}_2$ is obtained by replacing θ_0 in $\mathbf{e}_1$ by $(\theta_1\circ\theta_2)\wedge(\theta_3\circ\theta_4)$.

$$\mathbf{g}: \theta_0\wedge\{\theta_1\circ[\theta_2\wedge(\theta_3\circ\theta_4)]\}\wedge\{[\theta_5\wedge(\theta_6\circ\theta_7)]\circ\theta_8\} \subseteqq$$
$$\subseteqq \theta_1\circ\theta_6\circ\{(\theta_0\circ\theta_3\circ\theta_7)\wedge\{\theta_5\circ\theta_2\circ[(\theta_6\circ\theta_1\circ\theta_3)\wedge(\theta_7\circ\theta_8\circ\theta_4)]\}\}\circ\theta_4\circ\theta_8.$$

Then we have the following theorems:

3.4. Theorem. *For a regular variety the following are equivalent:*

(i) $\mathbf{e}_1$ *is congruence-valid in* $\mathfrak{B}$.

(ii) $\mathbf{e}_2$ *is congruence-valid in* $\mathfrak{B}$.

(iii) *The strong Mal'cev condition* $\exists p(p(x, x)=x\wedge p(x, y)=p(y, x))$ *holds in* $\mathfrak{B}$.

3.5. Theorem. *For a regular variety t.f.a.e.:*

(i) $\mathbf{g}$ *is congruence-valid in* $\mathfrak{B}$.

(ii) *The strong Mal'cev condition* $\exists p(p(x, x, x)=x, p(x, y, z)=p(y, z, x))$ *holds in* $\mathfrak{B}$.

We prove only the first theorem, the proof of the second is essentially the same but needs a little bit more of computation.

Proof. (iii)→(i): Assume in $\mathfrak{B}$ there exists an idempotent and commutative binary polynomial p. Take $(x, y)\in\theta_0\wedge(\theta_1\circ\theta_2)\wedge(\theta_3\circ\theta_4)$. Then there exist a and b

such that $x\theta_0 y$, $x\theta_1 a\theta_2 y$, $x\theta_3 b\theta_4 y$. Using p we get:

$$x = p(x,x)\theta_1 p(a,x)\theta_2 p(y,x)\theta_3 p(y,b)\theta_4 p(y,y) = y \quad \text{and}$$

$$p_1(a,x)\,[(\theta_1\circ\theta_3)\wedge(\theta_2\circ\theta_4)]\,p(b,x)\theta_0 p(b,y)$$

As $p(y,b)=p(b,y)$ we get:

$$(x,y)\in\theta_1\circ\{(\theta_2\circ\theta_3)\wedge\{[(\theta_1\circ\theta_3)\wedge(\theta_2\circ\theta_4)]\circ\theta_0\}\}\circ\theta_4.$$

(i)→(ii) is trivial. Only in the next step will we use regularity.

(ii)→(iii): First we use Wille's algorithm to write down the Mal'cev condition for $\mathbf{e}_2$. We get, that in the class of all varieties $\mathbf{e}_2$ is equivalent to the following strong Mal'cev condition: $\exists p_1, p_2, \ldots, p_8$ with

$$x = p_1(x,x,v,y), \tag{1}$$
$$p_1(x,y,v,y) = p_2(x,y,v,y) \tag{2}$$
$$p_2(x,u,x,y) = p_3(x,u,x,y), \tag{3}$$
$$p_3(x,u,y,y) = y, \tag{4}$$
$$p_1(x,x,v,y) = p_5(x,x,v,y), \tag{5}$$
$$p_5(x,u,x,y) = p_4(x,u,x,y), \tag{6}$$
$$p_4(x,x,v,y) = p_7(x,x,v,y), \tag{7}$$
$$p_7(x,y,v,y) = p_3(x,y,v,y), \tag{8}$$
$$p_1(x,y,v,y) = p_6(x,y,v,y), \tag{9}$$
$$p_6(x,u,y,y) = p_4(x,u,y,y), \tag{10}$$
$$p_4(x,u,x,y) = p_8(x,u,x,y), \tag{11}$$
$$p_8(x,u,y,y) = p_3(x,u,y,y). \tag{12}$$

Now if this Mal'cev condition holds in a regular variety, each of its equations must be regular. We can thus conclude: From (1) it follows that p_1 depends only on the first two places, therefore in (2) p_2 can depend at most on the first, second and fourth place. From (4) it follows that p_3 depends at most on the last two places thus p_2 depends at most on the first, third and fourth place. Together with the above then p_2 depends at most on the first and fourth place. Thus we can replace (1) to (4) in a regular variety by

$$x = p_1(x,x), \tag{1'}$$
$$p_1(x,y) = p_2(x,y), \tag{2'}$$
$$p_2(x,y) = p_3(x,y), \tag{3'}$$
$$p_3(y,y) = y. \tag{4'}$$

Carrying these cancellations out in (5) up to (12) we finally obtain: $\exists p_1, \ldots, p_8$ with (1′) to (4′) and

(5′) $$p_1(x, x) = p_5(x),$$
(6′) $$p_5(x) = p_4(x, x),$$
(7′) $$p_4(x, v) = p_7(x, v),$$
(8′) $$p_7(y, v) = p_3(v, y),$$
(9′) $$p_1(x, y) = p_6(x, y),$$
(10′) $$p_6(x, y) = p_4(x, y),$$
(11′) $$p_4(x, x) = p_8(x),$$
(12′) $$p_8(y) = p_3(y, y).$$

Now let us have a look at p_1. By (1′) we get $p_1(x, x)=x$ and we obtain

$$p_1(x, y) = p_2(x, y) = p_3(x, y) = p_7(y, x) = p_4(y, x) = p_6(y, x) = p_1(y, x).$$

Thus we have: $\exists p$ with $p(x, x)=x \wedge p(x, y)=p(y, x)$.

This finishes the proof.

4. Applications

We consider the equational classes of groupoids defined by subsets of the following set $\mathbf{\Sigma}$ of regular equations

$$\mathbf{\Sigma} := \{x(yz) = (xy)z, xy = yx, xx = x\},$$

and define

$\mathfrak{B}_1 = \mathbf{Mod}\,(x(yz)=(xy)z)$	semigroups,
$\mathfrak{B}_2 = \mathbf{Mod}\,(xy=yx)$	commutative groupoids,
$\mathfrak{B}_3 = \mathbf{Mod}\,(xx=x)$	idempotent groupoids,
$\mathfrak{B}_4 = \mathbf{Mod}\,(x(yz)=(xy)z,\ xy=yx)$	commutative semigroups,
$\mathfrak{B}_5 = \mathbf{Mod}\,(x(yz)=(xy)z,\ xx=x)$	idempotent semigroups,
$\mathfrak{B}_6 = \mathbf{Mod}\,(xy=yx,\ xx=x)$	commutative, idempotent groupoids,
$\mathfrak{B}_7 = \mathbf{Mod}\,(x(yz)=(xy)z,\ xy=yx,\ xx=x)$	semilattices.

As projections: $\pi_i^n(x_1, \ldots, x_n) := x_i$ are idempotent and associative we have that the variety of sets is contained up to polynomial equivalence as a subvariety in $\mathfrak{B}_1$, $\mathfrak{B}_3$, $\mathfrak{B}_5$. Furthermore, the variety of pointed sets is up to equivalence contained in $\mathfrak{B}_2$ and in $\mathfrak{B}_4$, so $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$, $\mathfrak{B}_4$ and $\mathfrak{B}_5$ do not fulfil any nontrivial congruence inequalities.

We are going to show now that we can separate the remaining varieties by congruence inequalities.

4.1. Theorem. *The congruence inequalities* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *are nontrivial and hold in commutative, idempotent groupoids. The congruence inequality* $\mathbf{g}$ *holds in semilattices but not in commutative idempotent groupoids.*

Proof. The first part of the theorem is a direct consequence of theorem 3.2. Theorem 3.3. implies that **g** holds in semilattices. Assume **g** holds in commutative idempotent groupoids.

In [3] we characterized the strong Mal'cev condition $\exists p(p(x, y, z)=p(y, z, x))$ and it was shown that it is equivalent to the statement that every automorphism φ of order 3 has a fixed point.

So in order to show that **g** does not hold for all commutative idempotent groupoids we only have to find a commutative idempotent groupoid $\mathcal{G}$ and an automorphism $\varphi: G \to G$ of order 3 which has no fixed point.

Take $\mathcal{G}=(\{0, 1, 2\}, \cdot)$ with $\cdot$ defined as $x \cdot y :=2x+2y \pmod 3$. Take the map $\varphi: G \to G$ with $\varphi(x) :=x+1 \pmod 3$. φ is an automorphism of order 3 but φ has no fixed point. This finishes the proof. Notice that **g** happens to hold in $\mathcal{G}$ because $\mathcal{G}$ is simple.

4.2. Corollary. *The congruence inequalities* $\mathbf{e}_1, \mathbf{e}_2$ *and* **g** *do not imply any lattice inequality.*

Proof. Freese and Nation have shown that there is no lattice inequality holding for the congruence lattices of semilattices, but $\mathbf{e}_1, \mathbf{e}_2$ and **g** are congruence-valid in semilattices.

The author wishes to acknowledge with many thanks the helpful discussions with Professor R. Padmanabhan, who also provided the commutative idempotent groupoid used in the proof of theorem 4.1.

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