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Similarity and interpolation between projectors

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1. Introduction

If two bounded linear projectors E_0, E_1 on a Banach space X are "close" in some sense, are they similar? Can we connect them by a projector-valued continuous path $t \to E_t (0 \leqq t \leqq 1)$? If $\|E_1 - E_0\| < 1$ for some operator norm then [6, I. 4.6, with the footnote and Problem 4.13] gives a positive answer to both questions. For pairs of orthoprojectors on a Hilbert space, the papers [1] and [2, Sec. 3] give a complete set of unitary invariants, with an extensive bibliography on the history of the subject, and the latter work (formula 1.18) expresses a particular unitary U (called direct rotation) such that $E_1 = UE_0U^{-1}$ in the form

$$U = \exp(J\theta), \quad \theta \geqq 0, \; J \text{ normal}, \quad J^3 = -J,$$

which offers a path $E_t = \exp(tJ\theta)E_0 \exp(-tJ\theta)$.

We shall give a similar expression for Banach space projectors in (12), except that we do not try to separate J from θ in $J\theta = -iW$, essentially because square roots of arbitrary operators are generally unavailable.

Differentiable paths between finite decompositions of identity into projectors and their relation to similarity are studied in [6, II. 4.5].

We base our exposition on the concept of an approximate projector (Sec. 2) from which we derive an expression of the bisector $E_{1/2}$ of E_0 and E_1. Bisections are then repeated, leading to $E_{1/4}$, $E_{3/4}$ etc., until analytic operational calculus is applicable to help extend the domain of E_t to all $t \in [0, 1]$. Throughout, we use that holomorphic branch of the natural logarithm whose value at 1 is 0. Accordingly, we define $z^{1/2} = \exp(1/2 \ln z)$ etc.

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2. Approximate projectors and involutions

We call $P \in B(X)$ an *approximate projector* if $\|P-P^2\|<1/4$, and $Q \in B(X)$ an *approximate involution* if $\|1-Q^2\|<1$. The correspondence between approximate projectors and involutions is obviously $Q=2P-1$.

The spectrum $\sigma(P)$ of an approximate projector lies, by the Spectral Mapping Theorem, inside the Bernoulli lemniscate L: $|z(1-z)|=1/4$. Let us compute the spectral projector E of P corresponding to the part of $\sigma(P)$ inside the right loop $L_+: |z(1-z)|=1/4$, $\operatorname{Re}(z) \geqq 1/2$. We have (orienting L_+ correctly)

$$E = (2\pi i)^{-1} \int_{L_+} (z-P)^{-1} dz.$$

Denote $P-P^2=R$ and substitute $z-z^2=v$, hence $z=\frac{1}{2}(1+\sqrt{1-4v})$, $dz = -(1-4v)^{-1/2} dv$, and we can verify

$$(z-P)^{-1} = (1-z-P)(v-R)^{-1}.$$

The path L_+ transforms into the positively oriented circle $C: |v|=1/4$. By virtue of the compactness of $\sigma(P)$, we can deform homotopically C into a smaller circle C_1 to avoid the singular point $v=1/4$. Altogether,

$$E = (2\pi i)^{-1} \int_{C_1} [\tfrac{1}{2}+\tfrac{1}{2}(2P-1)(1-4v)^{-1/2}](v-R)^{-1} dv,$$

i.e.

(1) $$E = \tfrac{1}{2}+\tfrac{1}{2}(2P-1)(1-4R)^{-1/2} = P+\tfrac{1}{2}(2P-1)[(1-4R)^{-1/2}-1].$$

Note that $(1-4R)^{-1/2}$ can be obtained by evaluating the MacLaurin series for $(1-4v)^{-1/2}$ at R in place of v.

For the involution $T=2E-1$, (1) simplifies into

(2) $$T = Q(1-(1-Q^2))^{-1/2} \quad \text{where} \quad Q = 2P-1.$$

Since the power series above has positive coefficients, we have the following estimate where $r=\|P-P^2\|$:

(3) $$\|E-P\| \leqq \tfrac{1}{2}\|2P-1\|[(1-4r)^{-1/2}-1].$$

Intuitively, the closer P is to being a projector, the closer it is to E.

Taking first two terms in the power series, we derive an iterative scheme for computing E, namely:

(4) Set $P_0=P$, and given P_k, compute $R_k=P_k-P_k^2$. Stop if $\|E-P_k\|$ from (3) is satisfactorily small. Else, compute $P_{k+1}=P_k+(2P_k-1)R_k \equiv P_k^2+2P_kR_k$ and return to testing P_{k+1}.

We can prove that $R_{k+1}=R_k^2(3+4R_k)$, hence from $\|R_k\|<1/4$ there follows $4\|R_{k+1}\|\leqq(4\|R_k\|)^2$ (quadratic convergence), and that (1) applied to P_k yields the same E.

3. The bisector of two close projectors

For the sake of exposition, we shall assume that E_0 and E_1 are two projectors satisfying $\|E_1-E_0\|<1$. Their mean $P=\frac{1}{2}(E_0+E_1)$ need not be a projector but it is an approximate projector in the sense of the previous section. Indeed, we can verify directly that $P-P^2=\frac{1}{4}(E_1-E_0)^2$, hence $\|P-P^2\|<\frac{1}{4}$, and using (1), we can define

$$E_{1/2}=\tfrac{1}{2}(E_0+E_1)+\tfrac{1}{2}(E_0+E_1-1)\left[(1-(E_1-E_0)^2)^{-1/2}-1\right] \tag{5}$$

and call $E_{1/2}$ the *bisector* of E_0 and E_1. For the associated involution $T_{1/2}=2E_{1/2}-1$, we obtain

$$T_{1/2}=(E_0+E_1-1)[1-(E_1-E_0)^2]^{-1/2}=\tfrac{1}{2}(T_0+T_1)[1-\tfrac{1}{4}(T_1-T_0)^2]^{-1/2}. \tag{6}$$

4. The trigonometry of projectors

With a pair of projectors E_0, E_1 and their involutions $T_i=2E_i-1$ $(i=0, 1)$, we associate the following operators (compare [1]):

(7) $S_1=(E_1-E_0)^2$, the *separation* of (E_0, E_1);
$C_1=(E_0+E_1-1)^2$, the *closeness* of (E_0, E_1);
$V_1=T_0T_1$;

and we can verify the following properties:

(8) (i) $C_1=\frac{1}{4}(T_0+T_1)^2=\frac{1}{4}(2+V_1+V_1^{-1})$; $S_1=\frac{1}{4}(T_1-T_0)^2=\frac{1}{4}(2-V_1-V_1^{-1})$;

(ii) $C_1+S_1=1$;

(iii) both C_1 and S_1 commute with $\{E_0, E_1\}$;

(iv) C_1 and S_1 are symmetric functions of E_0 and E_1;

(v) $C_1E_0=E_0E_1E_0$ and $C_1E_1=E_1E_0E_1$.

We can think of S_1 as the operator analogue of $\sin^2\theta$ and C_1 as $\cos^2\theta$ where θ is the non-obtuse angle between the ranges of E_0 and E_1. If E_i are one-dimensional ortho-projectors on the Euclidean plane, the analogy is perfect (we identify number 1 with the identity operator if convenient), and higher dimensional pairs of orthoprojectors will essentially decompose into direct sums or integrals of planar pairs, as shown in [1]. Continuing in this analogy, we may give V_1 the meaning of a generalization of the square of the direct rotation that moves the range of E_0 onto the range of E_1. However, the convenient property $0 \leqq S_1 \leqq 1$ and the "angle" $\arcsin(S_1^{1/2})$, which would commute with both E_0 and E_1, lose their meaning in general Banach spaces, as we can see from Example 1. We fill in this gap partially by developing an oblique way of expressing $\sin^2(t\theta)$ for $0 \leqq t \leqq 1$ in terms of $\sin^2\theta$ without having to evaluate θ itself.

5. An auxiliary function

For $z \in (0, 1)$ and t complex, we define

(9) $$f_t(z) = \sin^2(t \arcsin \sqrt{z}).$$

To extend the domain of f, we observe that the function $g(z) = \frac{1}{\sqrt{z}} \arcsin \sqrt{z}$ $(z \neq 0)$, $g(0) = 1$ has a MacLaurin series with radius of convergence equal to 1, and $h(z) = \sin^2 \sqrt{z}$ has a MacLaurin series convergent for all z. With g and h extended by means of their expansions, we can extend $f_t(z)$ as

(10) $$f_t(z) = h(t^2 z g^2(z))$$

for all $|z| < 1$, so that f_t is holomorphic. To guarantee uniqueness of further continuations of f_t, we consider a simple smooth curve Γ connecting 1 with ∞ while missing 0 and define

$$\Delta = \mathbf{C} \setminus \operatorname{range}(\Gamma),$$

so that Δ is a simply connected domain in which f_t is arbitrarily continuable. By the Monodromy Theorem ([5], VI. 6.3), f_t can be continued to a holomorphic function on Δ. We retain, with Γ fixed, the notation of (9) for this holomorphic function.

We remark that for $\Gamma = \Gamma_0 = [1, +\infty]$, the function g from above can be extended from $0 < z < 1$ to $\Delta_0 = \mathbf{C} \setminus \Gamma_0$ by means of the series

$$g(z) = \frac{2}{\sqrt{z}} \arctan \frac{\sqrt{z}}{1+\sqrt{1-z}} = 2 \sum_{n=0}^{\infty} \frac{(-z)^n}{(2n+1)(1+\sqrt{1-z})^{2n+1}}$$

since $|z(1+\sqrt{1-z})^{-2}| < 1$ iff $z \in \Delta_0$. Thus $f_t(z)$ is described by (10) on all of Δ_0.

Here is a list of properties of f_t which will be useful later.

(11) (i) For sufficiently small $|\theta|$, if $z = \sin^2\theta$ then $f_t(z) = \sin^2(t\theta)$;

(ii) $f_0(z)=0, f_1(z)=z, f_2(z)=4z(1-z)$ (in general, f_t is a polynomial in z for every integer t),

(iii) $f_{1/2}(z) = \frac{1}{2}(1-\sqrt{1-z}$; note that $\sqrt{1-z}$ is well-defined on Δ;

(iv) for all s, t and for $z\in\Delta$, $|z|$ sufficiently small, $f_s(f_t(z))=f_{st}(z)$; this gives an unambiguous continuation of $f_s \circ f_t$ to all of Δ; the composition is correct in all of Δ if s is an integer;

(v) if $t=2^{-n}$, $n\geqq 0$ integer, then $f_t(z)\neq 1$ $(z\in\Delta)$; if $n\geqq 1$ then $f_t(z)\neq 1/2$. (This follows by induction.)

6. Similarity and interpolation

For a set $M\subset B(X)$, denote by $\mathcal{A}\{M\}$ the norm-closed sub-algebra of $\mathcal{B}(X)$ generated by M and the identity and closed under the inversion if defined. Recall also the notation in (7).

Theorem 1. *Let E_0, E_1 be projectors in $\mathcal{B}(X)$, $E_0\neq E_1$.*

(i) *If the number* 1 *lies in the unbounded component of the complement of $\sigma(S_1)$ then there exists an involution $T_{1/2}$ in $\mathcal{A}\{E_0, E_1\}$ such that $E_1=T_{1/2}E_0T_{1/2}$ and there exists $W\in\mathcal{A}\{V_1\}$ such that the projector-valued path*

$$t \to E_t = e^{-itW} E_0 e^{itW}, \quad 0 \leqq t \leqq 1, \tag{12}$$

connects E_0 with E_1. Moreover, $V_1=e^{2iW}$ and $T_tW=-WT_t$ where $T_t=2E_t-1$.

(ii) *(Poor man's path): If E_0+E_1-1 is invertible then there exists Z which is a product of two involutions from $\mathcal{A}\{E_0, E_1\}$ such that $E_1=Z^{-1}E_0Z$, and there exists a projector-valued path $t\to E_t$, $0\leqq t\leqq 1$, connecting E_0 with E_1 and consisting of two straight line segments.*

Remarks.

(a) The condition $\|E_1-E_0\|<1$ clearly implies the assumption (i) of the theorem, for then $\|S_1\|<1$.

(b) Assumption (i) implies assumption (ii) because $C_1=(E_0+E_1-1)^2=1-S_1$ does not have 0 in its spectrum (recall (7) and (8) (ii)). We shall see a counter-example demonstrating that (ii) does not imply (i).

(c) In (ii), the inequality $E_0\neq E_1$ implies that E_0 does not commute with E_1. In fact, more than that is true: If $E_0E_1=E_1E_0$ and (E_0+E_1-1) is either left- or right-cancellable then $E_0=E_1$. Indeed, look at the identities:

$$(E_0+E_1-1)(E_0-E_1) = (E_1E_0-E_0E_1) = (E_1-E_0)(E_0+E_1-1).$$

(d) The operator W from (i) is an operator analogue of the angle between E_0 and E_1. Indeed, from (8i) and the equation $V_1=e^{2iW}$ there follows that $\sin^2 W = S_1$, except that W need not be in $\mathcal{A}\{S_1\}$ because $T_0W=-WT_0$ while S_1 commutes with T_0.

Proof. (Theorem 1). In part (i), let us choose a simple smooth curve Γ which connects 1 with ∞ within the complement of $\sigma(S_1)\cup\{0\}$. As in section 5, we can construct the function f relative to Γ. For the construction of W, as well as for future use, we define

$$S_t = f_t(S_1) \quad \text{for all (complex) } t. \tag{13}$$

According to (11v), for $t=2^{-n}$ ($n\geqq 0$ integer) the operator $1-2S_{t/2}$ is invertible, and since $S_t=f_2(S_{t/2})=4S_{t/2}(1-S_{t/2})$ by (11ii and 11iv), we have

$$(1-2S_{t/2})^2 = 1-S_t. \tag{14}$$

All S_t commute with T_0, T_1 by (8iii).

By induction, we will now construct involutions T_t for $t=2^{-n}$, $n\geqq 0$ integer. They too will commute with all S_t. T_1 is already given and $1/4(T_0+T_1)^2=1-S_1$ by (8). Assume that for $t=2^{-n}$, T_t has been defined and

$$T_t^2 = 1, \tfrac{1}{4}(T_0+T_t)^2 = 1-S_t \tag{15}$$

holds. Along with T_t, we consider $V_t=T_0T_t$, and define

$$T_{t/2} = \tfrac{1}{2}(T_0+T_t)(1-2S_{t/2})^{-1}. \tag{16}$$

Note that $1-2S_{t/2}$ is invertible. Using (14) and (15), we verify mechanically that $T_{t/2}^2=1$ and $\frac{1}{4}(T_0+T_{t/2})^2=1-S_{t/2}$. For the associated V_t, we have

$$V_{t/2} = \tfrac{1}{2}(1+V_t)(1-2S_{t/2})^{-1} = T_0T_{t/2} = T_{t/2}T_t, \tag{17}$$

so that

$$V_{t/2}^2 = T_0T_{t/2}T_{t/2}T_t = V_t. \tag{18}$$

Thus, T_t and V_t are constructed for all $t=2^{-n}$ and, in addition, $V_t\in\mathcal{A}\{V_1\}$ and $V_{2^{-n}}^{2^n}=V_1$. Also, for $t=1$, (17) implies

$$T_1 = T_{1/2}T_0T_{1/2}, \quad \text{so that} \quad E_1 = T_{1/2}E_0T_{1/2},$$

as claimed in the theorem.

The next task is to show that $\lim_{n\to\infty} V_{2^{-n}}=1$ in norm. Indeed, writing $V_{2^{-n}}=1+K_n$, we re-write (17) as

$$K_{n+1} = \tfrac{1}{2}(1-2S_{t/2})^{-1}K_n+2S_{t/2}(1-2S_{t/2})^{-1} \quad \text{with} \quad t=2^{-n}.$$

Since $\lim_{t\to 0} S_t = \lim_{t\to 0} S_t(1-2S_t)^{-1}=0$ in norm, we have for all sufficiently large n,

$$\|\tfrac{1}{2}(1-2S_t)^{-1}\| \leqq \frac{2}{3}, \quad \text{and} \quad \|K_{n+1}\| \leqq \frac{2}{3}\|K_n\|+p_n \quad \text{where} \quad \lim_{n\to\infty} p_n = 0.$$

This implies $\lim_{n\to\infty} \|K_n\|=0$, as claimed.

We select an n_0 such that $\|V_{2^{-n}}-1\|<1$ for all $n\geqq n_0$ and define $W_n=-i2^{n-1}\cdot \ln V_{2^{-n}}$. Due to (18), W_n is independent of $n\geqq n_0$, and we claim that the common value is the desired W. The relations $V_1=e^{2iW}$ and $E\in\mathscr{A}\{V_1\}$ are immediate. Further, (17) implies

$$T_t = V_{t/2}^{-1}T_0V_{t/2} = e^{-itW}T_0e^{itW} \tag{19}$$

for $t=2^{-n}$, $n\geqq 0$ integer. We can therefore use (19) to define T_t and $V_t=T_0T_t$ for all t, so that T_t are involutions, $E_t=e^{-itW}E_0e^{itW}$ are projectors and the equation is correct for both $t=0$ and $t=1$.

For $n\geqq n_0$, $t=2^{-n}$, we also have $T_0V_tT_0=V_t^{-1}$ (since $T_0V_t=T_t$ is an involution), and taking logarithms on both sides, we obtain $T_0WT_0=-W$, or $T_0W=-WT_0$, hence

$$T_tW+WT_t = e^{-itW}(T_0W+WT_0)e^{itW} = 0,$$

completing the proof of part (i).

Proof of part (ii): Since E_0+E_1-1 has an inverse in $\mathscr{B}(X)$, so does $C_1=(E_0+E_1-1)^2$, and we claim:

$F=E_0C_1^{-1}E_1$ is a projector, $(E_1-F)^2=0=(F-E_0)^2$, $E_0F=FE_1=F$, $FE_0=E_0$, $E_1F=E_1$. Indeed, by (8 iii and v), $F^2=C_1^{-2}(E_0E_1E_0)E_1=C_1^{-2}C_1E_0E_1=F$, and the remaining statements follow similarly.

Consequently, E_0+F-1 and E_1+F-1 are involutions, and we can set $Z=(E_0+F-1)(E_1+F-1)$ which makes $E_1=Z^{-1}E_0Z$. The straight line segment from E_0 to F consists of projectors since

$$(E_0+t(F-E_0))^2 = E_0+tE_0(F-E_0)+t(F-E_0)E_0+t^2(F-E_0)^2 = E_0+t(F-E_0)$$

by the above equations, and similarly the line segment from F to E_1 consists of projectors. The proof is complete.

More remarks. (a) In part (i), the projectors E_t move from E_0 to E_1 at a constant angular velocity in the sense that for s and t sufficiently close, the angle operator between E_s and E_t is

$$\frac{1}{2i}\ln(T_sT_t) = \frac{1}{2i}\ln e^{2i(t-s)W} = (t-s)W,$$

and the separation is $(E_t-E_s)^2=S_{t-s}$, as we can verify from the relations (19) defining T_t and from $T_0W=-WT_0$ which implies $T_0e^{-itW}=e^{-itW}T_0$. It may be interesting that

$$E_sE_t-E_tE_s=\tfrac{1}{2}\,i\sin 2(t-s)W.$$

(b) From $T_0WT_0=-W$ it follows that the spectrum of W is centrally symmetric, including its fine structure. For example, if ω is an eigenvalue of W with an eigenvector x then T_0x is an eigenvector corresponding to $(-\omega)$.

(c) If $S_1=0$, as between E_0 and F in the proof of part (ii) of the theorem, then the angle operator $W=-i(E_0E_1-E_1E_0)$, but $(E_0E_1-E_1E_0)^2=0$ and hence $W^2=0$. In this case, the path (12) becomes the straight line segment from E_0 and E_1.

We can describe those projector pairs (E_0, E_1) for which $S_1=0$ as follows. E_0 defines a direct decomposition $X=\text{Range } E_0\oplus\text{Ker } E_0$, so that E_0 is represented by $1\oplus 0$. Then E_1 can be represented by

$$\begin{bmatrix}1 & A\\ B & 0\end{bmatrix}\quad\text{where}\quad AB=0\quad\text{and}\quad BA=0,$$

as we can verify by writing $E_1^2=E_1$ and $(E_1-E_0)^2=0$ in components.

(d) The possible non-uniqueness of the part Γ in part (i) may cause non-uniqueness of the path (12). It would be interesting to find conditions on the pair (E_0, E_1) which would allow to characterize at least $T_{1/2}$ or $V_{1/2}$ before it is constructed, in a similar way as its unitary counterpart for self adjoint pairs was described in [2, Prop. 3.3] and [1, Theorem 4.1].

(e) The projector $F=E_0C_1^{-1}E_1$ can be obtained as the spectral projector of E_0E_1 corresponding to the complement of $\{0\}$. Indeed, using property $(E_0E_1)^2=$ $=C_1E_0E_1$ from (8v), we can verify a partial fraction decomposition

$$(\lambda-E_0E_1)^{-1}=\lambda^{-1}(1-F)+(\lambda-C_1)^{-1}F$$

and obtain $1-F$ as $(2\pi i)^{-1}\int_\Gamma(\lambda-E_0E_1)^{-1}d\lambda$ where Γ is a small circle around 0.

F is the unique projector which shares its range with E_0 and its nullspace with E_1. An equivalent construction of F is implied in [6, Problem I. 4.12], namely

$$F=(1-E_1+E_0E_1)E_1(1-E_1+E_0E_1)^{-1}.$$

7. Examples

Example 1. For every complex λ_0, there exist two 2×2 idempotent matrices e_0, e_1 such that $(e_1-e_0)^2=\lambda_0 1$ and their Euclidean bound norms are

$$\|e_0\|=\|e_1\|=[\tfrac{1}{2}(1+|\lambda_0|+|\lambda_0-1|)]^{1/2}. \tag{20}$$

For a construction, take any δ satisfying $(\delta-\delta^{-1})^2=-4\lambda_0$, and define

$$e_0=1/2\begin{bmatrix}1 & \delta^{-1}\\ \delta & 1\end{bmatrix}\quad\text{and}\quad e_1=1/2\begin{bmatrix}1 & \delta\\ \delta^{-1} & 1\end{bmatrix}.$$

It can be proved that for every pair of idempotent 2×2 matrices with a prescribed separation λ_0, the maximum of their norms is not less than the quantity in (20).

Example 2. Given a non-empty compact set K in the plane, there exist two projectors E_0 and E_1 on the space l_2 for which the spectrum of $(E_1-E_0)^2$ is K, and its point spectrum is dense in K. These projectors can be built as direct sums of examples of type 1 over a sequence $\{\lambda_0^{(n)}\}$ whose range is dense in K. Boundedness is guaranted by (20).

A rich source of examples is afforded by a pair (φ,ψ) of formal expressions in x with values in a commutative algebra with identity such that

$$\varphi(-x)=-\varphi(x)\quad\text{and}\quad \varphi^2(x)+\psi(x)\psi(-x)=1, \tag{21}$$

by means of which we can define an involution T:

$$Tf(x)=\varphi(x)f(x)+\psi(x)f(-x). \tag{22}$$

The expressions f, φ and ψ may be functions defined on a centrally symmetric set in the plane, or formal power series with complex coefficients, or elements of suitable subspaces and quotient spaces of the above, and direct sums thereof.

Example 3. A pair E_0, E_1 of projectors on a Banach space for which $\sigma(S_1)$ is a prescribed non-empty compact set K, and every interior point λ of K belongs to the residual spectrum of S_1, i.e. $\lambda-S_1$ is one-to-one but its range is not dense.

Here we first construct $L=\{\lambda \text{ complex}: \lambda^2\in K\}$, take $X=H_\infty(L)$, the space of functions continuous on L and holomorphic in the interior of L, with supremum norm, and define

$$\begin{aligned} T_0f(x)&=-xf(x)+(1+x)f(-x),\\ T_1f(x)&=xf(x)+(1+x)f(-x),\quad x\in L,\quad f\in X.\end{aligned} \tag{23}$$

The conditions (21) are met, $E_i=(1+T_i)/2$ as usual $(i=0,1)$, and $S_1f(x)=$ $=\frac{1}{4}(T_1-T_0)^2f(x)=x^2f(x)$. Consequently $\sigma(S_1)=K$ with residual spectrum as claimed. Starting with an arbitrary nonempty compact centrally symmetric set L, we have an example of $\sigma(E_1-E_0)=L$.

Example 4. If $X=L^2(D)$ where D is the unit disc with the restricted Lebesgue measure and if T_0, T_1 are as in (23) then $\sigma(S_1)=D$ and every $\lambda\in D$ is in the continuous spectrum of S_1 (i.e. $\lambda-S_1$ has a densely defined unbounded inverse).

Example 5. The closeness operator C_1 can be invertible but can fail to have a square root in $\mathscr{A}\{S_1\}$ so that a bisector $E_{1/2}$ from (5) cannot exist.

In Example 3, take

$$L = \{\lambda \text{ complex} : |\lambda| \leqq 2, \quad |\lambda-1| \geqq \tfrac{1}{2}, \quad |\lambda+1| \geqq \tfrac{1}{2}\}.$$

so that from (23) there follows $C_1 f(x)=(1-x^2)f(x)$. The function $(1-x^2)$ has a reciprocal but not a square root in L, and every operator in $\mathscr{A}\{S_1\}$ is of the form (22) with $\psi=0$.

Example 6. Every even and every odd non-constant polynomial p can be the minimal polynomial of (E_1-E_0) for suitable E_0 and E_1 acting on a space of dimension degree (p). Consequently, every nilpotent matrix can be written as the difference of two idempotent matrices.

Indeed, take $X=\mathbf{C}[x]/(p)$, the polynomials modulo p. The operators from (23), well-defined on $\mathbf{C}[x]$, leave (p) invariant if p is either even or odd, hence they induce involutions $T_i^{(p)}$ $(i=0, 1)$ on X; note that $\dim X=\text{degree}\,(p)$. For the corresponding projectors, we have $(E_1^{(p)}-E_0^{(p)})[f(x)]=[xf(x)]$, so that the minimal polynomial of $(E_1^{(p)}-E_0^{(p)})$ is indeed p.

The proof for nilpotent matrices can be reduced to the case of one Jordan cell in the normal form and then reconstructed via direct sums. But every Jordan cell of dimension k represents multiplication by x in $\mathbf{C}[x]/(x^k)$, which is the previous case.

Example. 7. Take a non-void compact centrally symmetric set L in the plane and define in $H_\infty(L)$: $T_0 f(x)=f(-x)$ and $T_1 f(x)=e^{-2ix}f(-x)$. It is obvious that $T_t f(x)=e^{-2tix}f(-x)$ $(0\leqq t\leqq 1)$ defines a path from T_0 to T_1 consisting of involutions. Although we can define $Wf(x)=xf(x)$, so that $T_0T_1=e^{2iW}$, $T_tW=-WT_t$ and (12) holds, we cannot always obtain W by a procedure described in Theorem 1. Observe that $S_1f(x)=(\sin^2 x)f(x)$, and the proof breaks down if L contains points $(2k+1)\pi/2$, k integer.

8. Further connections between similarity and interpolation

Theorem 2. *If there exists a continuous projector-valued path $t\to E_t$, $0\leqq t\leqq 1$, then E_0 and E_1 are similar via a finite product of involutions. On the other hand, if two bounded linear operators A_0, A_1 on a complex Hilbert space H are similar (or unitarily equivalent) then there exists a continuous path $t\to A_t$, $0\leqq t\leqq 1$, consisting of operators similar (or unitarily equivalent) to A_0.*

Proof. For the first part, we use uniform continuity of E to subdivide [0, 1] into $0=t_0<t_1<\ldots<t_p=1$ so that $\|E_{t_{k+1}}-E_{t_k}\|<1$, $k=0, \ldots, p-1$, and then we apply Theorem 1 p times.

For the rest, write $A_1=T^{-1}A_0T$ and decompose $T=RV$ with R positive definite and V unitary. Evidently $t\to R^t$, $0\leqq t\leqq 1$, connects 1 with R while $t\to R^{-1}$ connects 1 with R^{-1}. Next, V can be written as the product of at most four self-adjoint involutions if $\dim H=\infty$, as shown in [3]. Every involution $Q=2E-1$ can in turn be connected with 1 by $U_t=E+e^{i\pi t}(1-E)$, $0\leqq t\leqq 1$, so that U_t are unitary. If $\dim H<\infty$, the unitary group of H is known to be arcwise connected. Thus, 1 can be connected with T be a continuous path consisting of invertible operators, and if T is unitary then $R=1$. Hence the conclusion.

Application. Proposition 2 in [4] states that if a pathwise connected subset $\mathcal{T}\subseteqq B(X)$ has the property that the union S of $\sigma(T)$ over $T\in\mathcal{T}$ has a bounded separated subset M then the spectral invariant subspaces of the operators T corresponding to M are mutually homeomorphic and homotopic. Using Theorem 2, we can strengthen the conclusion by asserting the similarity of the corresponding spectral projectors.

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References

[1] C. Davis, Separation of two linear subspaces, *Acta Sci. Math.*, **19** (1958), 172—187.
[2] C. Davis and W. M. Kahan, The rotation of eigenvectors by a perturbation. III, *SIAM J. Numer. Anal.*, (1970).
[3] P. R. Halmos and S. Kakutani, Products of Symmetries, *Bull. Amer. Math. Soc.*, **64** (1958), 77—78.
[4] Z. V. Kovarik, Spectrum localization in Banach spaces, I., *Lin. Alg. and its Appl.*, **8** (1974), 225—236.
[5] S. Saks and A. Zygmund, *Analytic functions*, Polish Scientific Publishers (Warsaw, 1965).
[6] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag (New York, 1966).

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