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Endomorphism and subalgebra structure; a concrete characterization

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§ 1. Introduction

In [5] the following abstract structure problem is solved: For what semigroups S and what lattices L does there exist an algebra $\mathfrak{A}$ with $S \cong \text{End}\,\mathfrak{A}$, the endomorphisms of $\mathfrak{A}$, and $L \cong \text{Su}\,\mathfrak{A}$ the lattice of subalgebras of $\mathfrak{A}$? Here we provide a solution to the corresponding concrete representation problem, where isomorphism is replaced by equality. Thus let $S \subseteqq A^A$ be a given transformation monoid and $L \subseteqq 2^A$ a set lattice. It is well known that $L = \text{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A}$ over the set A iff L is complete and compactly generated [2]; such lattices are called *algebraic.* In [4] necessary and sufficient conditions for $S = \text{End}\,\mathfrak{A}$ for some algebra $\mathfrak{A}$ over the set A are given; such transformation semigroups are called *algebraic.* A similar characterization is given in [4] for semigroups of partial functions. We make use of the latter result by representing subalgebras with partial identity functions to derive a simultaneous characterization for S and L. Our characterization, like that for the endomorphisms alone involves the solutions to systems of linear equations.

If M is a set of partial functions on A to A with $\text{id} \in M$ the identity function on A, a *system of linear equations* Σ *over* M is a set of functional equations each of the general form: $fx = y$, or $fx = g$ with $f, g \in M$, together with a specified solution variable X^Σ. An *assignment* α for Σ is a map from the variables of Σ to partial functions on A to A with a common domain. The assignment α *satisfies* Σ *at* $d \in A$. provided $f(\alpha x(d)) = \alpha y(d)$ whenever $fx = y \in \Sigma$ and $f(\alpha x(d)) = g(d)$ whenever $fx = g \in \Sigma$. The assignment α *satisfies* Σ *on* $D \subseteqq A$ iff α satisfies Σ at d for each $d \in D$. If X^Σ is the specified solution variable we say f *is a solution to* Σ *on* D provided there is

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an assignment α which satisfies Σ on D and $\alpha(X^\Sigma)=f$. A solution f to Σ on D is *unique* provided $f\restriction D=h\restriction D$ whenever h is any solution to Σ on D. The *support* of a system Σ is the set of all $d\in A$ for which there exists a solution to Σ at d. We write $B=\mathrm{Spt}\ \Sigma$ if B is the support of Σ.

Denote by $\mathfrak{A}_M$ the algebra of all finitary operations which admit each $f\in M$ a a homomorphism. $\tilde{M}$ is the set of all partial endomorphisms of $\mathfrak{A}_M$ and $\overline{M}$ is the set of all (total) functions which are endomorphisms of $\mathfrak{A}_M$. As usual a partial function g is a homomorphism with respect to an operation P of rank ν provided $gP(x)$ is defined and equals $P(gx)$ whenever gx is defined for $x\in A^\nu$. A total function is one whose domain is all of A. We will use:

Proposition 1. *$g\in A^B$ belongs to $\tilde{M}$ iff $B\in\mathrm{Su}\,\mathfrak{A}_M$ and for each finite $D\subseteq B$ there is a system Σ over M with g a unique solution to Σ on D.*

Proof. Take $\mu=\aleph_0$ in Theorem 2 of [4]. □

§ 2. The subalgebras of $\mathfrak{A}_M$

We first establish some easy facts about the support of systems Σ over M.

Lemma 1. *If $C=\mathrm{Spt}\ \Sigma$ then there is an assignment α which satisfies Σ on C.*

Proof. For each $d\in C$ there is an assignment α_d which satisfies Σ at d. Define α for a variable x of Σ by:

$$\alpha x(d)=\begin{cases}\alpha_d x(d) & \text{if } d\in C\\ d & \text{otherwise.}\end{cases}$$

It is straightforward to verify that α satisfies Σ on C. □

Lemma 2. *If $C=\mathrm{Spt}\ \Sigma$ then there is a system Γ and an assignment β which satisfies Γ on C* and *$C=\mathrm{Spt}\ \Gamma$ and $\beta(X^\Gamma)=\mathrm{id}\restriction C$ is a unique solution to Γ on C.*

Proof. Let Γ have one additional new variable X^Γ not among those of Σ and let the equations of Γ consist of those of Σ together with the new equation $X^\Gamma=\mathrm{id}$. By Lemma 1 there is an assignment α which satisfies Σ on C. Let β extend α by assigning $\mathrm{id}\restriction C$ to X^Γ. Clearly β satisfies Γ on C and $C=\mathrm{Spt}\ \Gamma$. If g is any solution to Γ on C then for $d\in C$, $g(d)=d$ so $g\restriction C=\beta(X^\Gamma)=\mathrm{id}\restriction C$ thus $\mathrm{id}\restriction C$ is a unique solution to Γ on C. □

Lemma 3. *Let each $C\in\mathcal{F}$ be the support of some system Γ_C. Then $\bigcap_{C\in\mathcal{F}} C$ is also the support of some system Γ.*

Proof. Assume without loss of generality that each pair of systems Γ_C, Γ_D have no variables in common for $C \neq D$ and let X^Γ be a new variable distinct from all of those of the Γ_C. By Lemma 2 we may further assume that $\mathrm{id} \restriction C$ is a unique solution to Γ_c on C for each $C \in \mathcal{F}$. Form $\Gamma = \left(\bigcup_{C \in \mathcal{F}} \Gamma_C\right) \cup \{X^\Gamma = \mathrm{id}\}$. We claim $\bigcap_{C \cap \mathcal{F}} C = \mathrm{Spt}\, \Gamma$. If $d \in \mathrm{Spt}\, \Gamma$, say α satisfies Γ at d, then clearly α_C, the restriction of α to the variables of Γ_C, satisfies Γ_C at d for each $C \in \mathcal{F}$ so $d \in \bigcap_{C \in \mathcal{F}} C$. If on the other hand $d \in \bigcap_{C \in \mathcal{F}} C$ and α_C satisfies Γ_C at d then let $\alpha X = \alpha_C X \restriction \bigcap_{C \in \mathcal{F}} C$ for a variable X in Γ_C and let $\alpha X^\Gamma = \mathrm{id} \restriction \bigcap_{C \in \mathcal{F}} C$. Clearly α satisfies Γ on $\bigcap_{C \in \mathcal{F}} C$ so $d \in \mathrm{Spt}\, \Gamma$. Thus $\mathrm{Spt}\, \Gamma = \bigcap_{C \in \mathcal{F}} C$.

Lemma 4. *For $D \subseteq A$ the operation defined by $\bar{D} = \bigcap_{D \subseteq \mathrm{Spt}\, \Sigma} \mathrm{Spt}\, \Sigma$ is a closure operator.*

Proof. Clevarly $D \subseteq \bar{D}$, and $[C \subseteq D \Rightarrow \bar{C} \subseteq \bar{D}]$. To show $\bar{\bar{D}} = \bar{D}$ it is only necessary to see then that $\bar{\bar{D}} \subseteq \bar{D}$. By Lemma 3 there is some system Γ with $\bar{D} = \bigcap_{D \subseteq \mathrm{Spt}\, \Sigma} \mathrm{Spt}\, \Sigma = \mathrm{Spt}\, \Gamma$. Clearly $\bar{\bar{D}} = \bigcap_{\bar{D} \subseteq \mathrm{Spt}\, \Sigma} \mathrm{Spt}\, \Sigma = \mathrm{Spt}\, \Gamma = \bar{D}$.

Lemma 5. *For $D \subseteq A$ the operation defined by $\tilde{D} = \bigcup_{C \text{ finite}, C \subseteq D} \bar{C}$ is a closure operator.*

Proof. Since $d \in D \Rightarrow d \in \{\bar{d}\} \subseteq \tilde{D}$ we have $D \subseteq \tilde{D}$. Further $[C \subseteq D \Rightarrow \tilde{C} \subseteq \tilde{D}]$ since each finite subset of C is also a finite subset of D. To show $\tilde{\tilde{D}} = \tilde{D}$ it remains only to see $\tilde{\tilde{D}} \subseteq \tilde{D}$. Suppose $\tilde{\tilde{D}} \nsubseteq \tilde{D}$; then there is some $a \in \tilde{\tilde{D}}$ with $a \notin \tilde{D}$. We will show this leads to a contradiction. Since $a \in \tilde{\tilde{D}}$ there is some $B \subseteq \tilde{D}$, B finite, with $a \in \bar{B}$. Thus $a \in \bigcap_{B \subseteq \mathrm{Spt}\, \Sigma} \mathrm{Spt}\, \Sigma$, and say $B = \{b_1, \ldots, b_n\}$. From $B \subseteq \tilde{D}$ we have each $b_K \in \tilde{D}$, say $b_K \in \bar{C}_K$ for some $C_K \subseteq D$, C_K finite, $K = 1, \ldots, n$. Then $C = \bigcup_{K=1}^{n} C_K$ is a finite subset of D, so $\bar{C} \subseteq \tilde{D}$. Now $B \subseteq \bigcup_{K=1}^{n} \bar{C}_K \subseteq \overline{\bigcup_{K=1}^{n} C_K}$ so $a \in \bar{B} \subseteq \overline{\overline{\bigcup_{K=1}^{n} C_K}} = \overline{\bigcup_{K=1}^{n} C_K} = \bar{C} \subseteq \tilde{D}$. Thus $a \in \tilde{D}$, contrary to the original choice $a \notin \tilde{D}$, the desired contradiction.

We can now describe explicitly the subalgebras of $\mathfrak{A}_M$:

Theorem 1. $B \in \mathrm{Su}\, \mathfrak{A}_M$ *iff* $B = \bigcup_{D \text{ finite}, D \subseteq B} \bar{D}$.

Proof. Let $B = \bigcup_{D \text{ finite}, D \subseteq B} \bar{D}$. We first consider the case $B = \emptyset$. Thus for each $D \subseteq B$, $D = \emptyset$ and $\bar{D} = \emptyset$ so $\bar{D} = \bigcap_{\emptyset \subseteq \mathrm{Spt}\, \Sigma} \mathrm{Spt}\, \Sigma$. If $\mathfrak{A}_M$ has any nullary operations (constants) $a \in A$ we claim $a \in \mathrm{Spz}\, \Sigma$ for every system Σ. To see this let α be the assignment which associates with every variable the constant function $\mathrm{f}: A \to \{a\}$.

Note α satisfies arbitrary Σ at a, since each $g \in M$ must have a as a fixed point. Thus from $\bar{D}=\emptyset$ we conclude $\mathfrak{A}_M$ has no nullary operations, whence $B=\emptyset \in \in \mathrm{Su}\,\mathfrak{A}_M$. Now if $B \neq \emptyset$, fix an operation P of $\mathfrak{A}_M$ of rank n, and $a_1, a_2, \ldots, a_n \in B$. It suffices to show that $P(a_1, \ldots, a_n) \in \overline{\{a_1, \ldots, a_n\}}$ since $\overline{\{a_1, \ldots, a_n\}} \subseteq B$. Let $D= =\{a_1, \ldots, a_n\}$. If $P(a_1, \ldots, a_n) \notin \bar{D}$ then there is some Σ with $D \subseteq \mathrm{Spt}\,\Sigma$ and $P(a_1, \ldots, a_n) \notin \mathrm{Spt}\,\Sigma$. By Lemma 1 we may assume that there is an assignment α which satisfies Σ on Spt Σ. We use α to produce an assignment α' which satisfies Σ at $d=P(a_1, \ldots, a_n)$ and thus obtain $P(a_1, \ldots, a_n) \in \mathrm{Spt}\,\Sigma$ contradicting the hypothesis that $P(a_1, \ldots, a_n) \notin \bar{D}$. For a variable x in Σ let $\alpha' x$ be defined by

$$\alpha' x(d) = \begin{cases} \alpha x(d) & \text{if } d \neq P(a_1, \ldots, a_n) \\ P(\alpha x(a_1), \ldots, \alpha x(a_n)) & \text{if } d = P(a_1, \ldots, a_n). \end{cases}$$

We claim α' satisfies Σ at $d=P(a_1, \ldots, a_n)$. To see this consider an equation $fx=y$ in Σ:

$$f\alpha' x(d) = P(f\alpha x(a_1), \ldots, f\alpha x(a_n)) = P(g(a_1), \ldots, g(a_n)) = gP(a_1, \ldots, a_n) = g(d).$$

Thus $d \in \mathrm{Spt}\,\Sigma$ and we must conclude $P(a_1, \ldots, a_n) \in \overline{\{a_1, \ldots, a_n\}}$, so $B = \bigcup_{D \text{ finite}, D \subseteq B} \bar{D} \Rightarrow \Rightarrow B \in \mathrm{Su}\,\mathfrak{A}_M$. To prove the converse, we suppose $B \in \mathrm{Su}\,\mathfrak{A}_M$. By Lemma 5 $B \subseteq \tilde{B} = = \bigcup_{D \text{ finite}, D \subseteq B} \bar{D}$ so it remains only to show $B \supseteq \tilde{B}$. We proceed again by contradiction. Let $a \in \tilde{B}$ and suppose $a \notin B$. Since $a \in \tilde{B}$ there is some finite $C \subseteq B$ with $a \in \bar{C}$. From the first part of the theorem we know $\bar{C} \in \mathrm{Su}\,\mathfrak{A}_M$. In fact $\bar{C}$ is the subalgebra of $\mathfrak{A}_M$ generated by C, for if $D \in \mathrm{Su}\,\mathfrak{A}_M$ and $C \subseteq D$ then by the first part of our Theorem 1, $D = \bigcup_{G \text{ finite}, G \subseteq D} \bar{G}$ so $\bar{C} \subseteq D$. Thus $\bar{C}$ is the smallest subalgebra of $\mathfrak{A}_M$ which contains C. Now $a \in \bar{C}$ so $a=P(c_1, \ldots, c_n)$ for some operation P in $\mathfrak{A}_M$ and some sequence $c_1, \ldots, c_n$ from C. But $B \in \mathrm{Su}\,\mathfrak{A}_M$ so B is closed under P, and $C \subseteq B$. Thus $a=P(c_1, \ldots, c_n) \in B$. It follows that $B \supseteq \tilde{B}$ and thus $B \in \mathrm{Su}\,\mathfrak{A}_M \Rightarrow B = \bigcup_{D \text{ finite}, D \subseteq B} \bar{D}$. □

§ 3. Characterization Theorems

For $L \subseteq 2^A$ let $\chi(L)=\{f \in A^B | B \in L, f=\mathrm{id} \restriction B\}$ be the set of *characteristic functions of* L. In general a function $f \in A^B$ will be called a *characteristic function* if $f=\mathrm{id} \restriction B$. Recall (§ 1) that when M is a set of partial functions, $\tilde{M}$ denotes the set of all partial endomorphisms of the algebra of all finitary operations which admit each $f \in M$ as a partial endomorphism. In what follows $\mathrm{id} \in S \subseteq A^A$, $L \subseteq 2^A$ and $M=S \cup \chi(L)$.

Theorem 2. $S=\mathrm{End}\,\mathfrak{A}$ *and* $L=\mathrm{Su}\,\mathfrak{A}$ *for some algebra* $\mathfrak{A}$ *iff* $\tilde{M}$ *contains no total functions other than* S *and* $\tilde{M}$ *contains no characteristic functions other than* $\chi(L)$.

Proof. Assume $S = \mathrm{End}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A}$. Let $f \in A^A$, $f \notin S$. Since $S = \mathrm{End}\,\mathfrak{A}$ some operation of $\mathfrak{A}$ destroys f; this same operation must admit each map in $\chi(L)$ since $L = \mathrm{Su}\,\mathfrak{A}$ so the operation is among those of $\mathfrak{A}_M$. Thus the only total functions in $\tilde{M}$ are the members of S. Now let $g \in A^B$ with $g = \mathrm{id} \restriction B$, $B \notin L$, be a characteristic function. Then $B \notin \mathrm{Su}\,\mathfrak{A}$ so there is some operation P in $\mathfrak{A}$ and a finite sequence $b_1, \ldots, b_n \in B$ with $P(b_1, \ldots, b_n) \notin B$. This same operation P is again among the operations of $\mathfrak{A}_M$. But P does not admit g as a partial endomorphism since $gP(b_1, \ldots, b_n)$ is undefined. Thus the only characteristic functions in $\tilde{M}$ are members of $\chi(L)$. This proves one direction of the Theorem. To complete the proof let $M = S \cup \chi(L)$ and assume that $\tilde{M}$ contains no total functions other than S and no characteristic functions other than $\chi(L)$. Let $\mathfrak{A} = \mathfrak{A}_M$. Since $\tilde{M}$ is the set of all partial endomorphisms of $\mathfrak{A}_M$ we have $S = \mathrm{End}\,\mathfrak{A}$. Moreover if $B \in \mathrm{Su}\,\mathfrak{A}_M$ then $\mathrm{id} \restriction B$ is a partial endomorphism of $\mathfrak{A}_M$ so $\mathrm{id} \restriction B \in \tilde{M}$, thus $\mathrm{id} \restriction B \in \chi(L)$ so $B \in L$. Thus $L = \mathrm{Su}\,\mathfrak{A}$. □

We now combine Theorem 2 with Proposition 1 and Theorem 1 to obtain an equational condition for S and L to be jointly algebraic. The characterization theorem which follows says roughly that S must contain all functions which are unique solutions to systems of equations over $M = S \cup \chi(L)$ and that the support of every such system must belong to L. (For A finite the theorem says exactly that; the more general statement involves only additional "compactness" conditions which are "local" analogs of the above properties.)

Theorem 3. $S = \mathrm{End}\,\mathfrak{A}$ *and* $L = \mathrm{Su}\,\mathfrak{A}$ *for some algebra* $\mathfrak{A}$ *iff*

$$\begin{bmatrix} \forall \text{ finite } D \subseteq A\ \exists \text{ system } \Sigma \text{ over } M \text{ with} \\ g \restriction D \text{ the unique solution to } \Sigma \text{ on } D \end{bmatrix} \Rightarrow g \in S. \tag{1}$$

and

$$B = \bigcup_{D \text{ finite } D \subseteq B} \Big(\bigcap_{D \subseteq \mathrm{Spt}\,\Sigma,\ \Sigma \text{ over } M} \mathrm{spt}\,\Sigma \Big) \Rightarrow B \in L. \tag{2}$$

Proof. From Theorem 2 we know $S = \mathrm{End}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A}$ iff

(i) $g \in A^A$ and $g \in \tilde{M} \Rightarrow g \in S$, and (ii) $id \restriction B \in \tilde{M} \Rightarrow B \in L$.

By Proposition 1 of § 1, (i) is equivalent to (1). Again by Proposition 1 of § 1, (ii) is equivalent to:

$$\begin{bmatrix} B \in \mathrm{Su}\,\mathfrak{A}_M \text{ and } \forall \text{ finite } D \subseteq B\ \exists \text{ system } \Sigma \\ \text{over } M \text{ with } (\mathrm{id} \restriction B) \restriction D \text{ the unique solution to } \Sigma \text{ on } D \end{bmatrix} \Rightarrow B \in L. \tag{ii'}$$

Furthermore the system Σ: $\{X^{\Sigma} = \mathrm{id}\}$ has $\mathrm{id} \restriction D$ as a unique solution on each D, thus (ii′) is equivalent to: $[B \in \mathrm{Su}\, \mathfrak{A}_M \Rightarrow B \in L]$, and by Theorem 1 this is equivalent to:

$$B = \bigcup_{D \text{ finite } D \subseteqq B} \Big(\bigcap_{D \subseteqq \mathrm{Spt}\, \Sigma,\, \Sigma \text{ over } M} \Big) \Rightarrow B \in L. \tag{2}$$

Thus $S = \mathrm{End}\, \mathfrak{A}$ and $L = \mathrm{Su}\, \mathfrak{A}$ iff (1) and (2) hold. □

For A finite Theorem 3 can be restated simply and completely as:

Corollary 1. *If A is finite and* $\mathrm{id} \in S \subseteqq A^A$ *and* $L \subseteqq 2^A$ *then* $S = \mathrm{End}\, \mathfrak{A}$ *and* $L = \mathrm{Su}\, \mathfrak{A}$ *for some algebra* $\mathfrak{A}$ *iff*

(1) $g \in S$ *whenever g is the unique solution to some system of equations with coefficients from* $S \cup \chi(L)$, *and*

(2) $B \in L$ *whenever B is the support of any system of equations with coefficients from* $S \cup \chi(L)$.

Bibliography

[1] M. Armbrust and J. Schmidt, Zum Cayleyschen Darstellungsatz, *Math. Annalen*, **154** (1964), 70—72.

[2] G. Birkhoff and O. Frink, Representations of lattices by sets, *Trans. Amer. Math. Soc.*, **64** (1948), 299—316.

[3] G. Grätzer, *Universal Algebra*, Van Nostrand (Princeton, 1968).

[4] N. Sauer and M. G. Stone, The algebraic closure of a semigroup of functions (To appear).

[5] M. G. Stone, Subalgebra and automorphism structure in Universal Algebras, a concrete characterization, *Acta Sci. Math.*, **33** (1972), 45—48.

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