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Convexoid operators and generalized growth conditions associated with unitary ϱ-dilations of Sz.-Nagy and Foiaş

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Dedicated to the memory of the late Professor H. Hiruta

An operator on a complex Hilbert space is said to be convexoid if the closure of its numerical range coincides with the convex hull of its spectrum. We shall consider some generalized growth conditions associated with unitary ϱ-dilations defined by B. Sz.-Nagy and C. Foiaş and as an application of these generalized growth conditions we shall give some characterization of convexoid operators which is an improvement form of the already known criterions due to G. H. Orland, C.-S. Lin and S. M. Patel.

Subsequently we shall give some generalizations of both theorems of S. K. Berberian and S. M. Patel for operators implying the equation $\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re}T)$ and we shall give some characterization of the class R introduced by G. R. Luecke.

1. Introduction

In this paper an operator T means a bounded linear operator on a complex Hilbert space $\mathfrak{H}$. The class $C_\varrho(\varrho>0)$ denotes the set of all operators with unitary ϱ-dilation [20]: there exist a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and a unitary operator U on $\mathfrak{K}$ such that

$$T^n x = \varrho P U^n x \quad \text{for} \quad x\in\mathfrak{H}\ (n=1,2,\ldots) \tag{1}$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$.

It is well known that $C_1=\{T:\|T\|\leqq 1\}$ [21] and $C_2=\{T:w(T)\leqq 1\}$ [2], where $w(T)$ indicates the numerical radius of T, i.e. $w(T)=\sup\{|\lambda|:\lambda\in W(T)\}$ and $W(T)$ denotes the numerical range of T defined by $W(T)=\{(Tx,x):\|x\|=1,\ x\in\mathfrak{H}\}$. In [21] there are given several characterizations of the operators belonging to C_ϱ and one of them is as follows:

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Theorem A [21]. *In order that T belong to the class C_ϱ it is necessary and sufficient that the condition*

$$(2)\qquad (\varrho-2)\|(I-zT)x\|^2+2\operatorname{Re}((I-zT)x, x)\geqq 0$$

be satisfied for all $x\in\mathfrak{H}$ and $|z|\leqq 1$.

In [9] an operator radius $w_\varrho(T)$ is defined by

$$(3)\qquad w_\varrho(T)=\inf\{u: u>0,\ u^{-1}T\in C_\varrho\}.$$

$w_\varrho(T)$ is non-increasing function of ϱ, in particular $w_1(T)=\|T\|$, $w_2(T)=w(T)$ and $w_\infty(T)=r(T)$ ($r(T)$ denotes the spectral radius of T) [9]. Moreover,

$$(4)\qquad \text{if}\quad 0<\beta<\varrho\leqq\infty \quad\text{and}\quad w_\varrho(T)=w_\beta(T),\quad\text{then}\quad w_\alpha(T)=w_\beta(T)$$

whenever $\beta\leqq\alpha\leqq\infty$ [9, Theorem 5.3], [10, (e)].

In [9] the following characterization of C_ϱ is given in term of operator radii:

$$(5)\qquad C_\varrho=\{T: w_\varrho(T)\leqq 1\}.$$

An operator T is called to be ϱ-oid [4], [5] if

$$(6)\qquad w_\varrho(T^k)=(w_\varrho(T))^k\quad (k=1,2,\ldots).$$

For each $\varrho\geqq 1$, $w_\varrho(T)=r(T)$ if and only if T is ϱ-oid and for each $0<\varrho<1$ there exists no non-zero ϱ-oid which is included in the class of normaloids [4]. Clearly 1-oid is normaloid and 2-oid is spectraloid (recall that an operator T is said to be normaloid if $\|T\|=r(T)$ and spectraloid if $w(T)=r(T)$).

We shall define generalized growth conditions associated with unitary ϱ-dilations as follows.

Definition 1. An operator T is called to satisfy the condition $(\varrho-G_1)$ for (M, N), in symbol $T\in(\varrho-G_1)$ for (M, N), if T satisfies the following inequality:

$$(7)\qquad w_\varrho((T-\mu)^{-1})\leqq\frac{1}{d(\mu, M)}\quad\text{for all complex}\quad \mu\notin N,$$

where M and N are two closed and bounded sets satisfying $N\supset M\supset\sigma(T)$.

Definition 2. An operator T is called to satisfy the condition $E-(\varrho-G_1)$ for (M, N), in symbol $T\in E-(\varrho-G_1)$ for (M, N), if there is equality in (7).

$T\in(\varrho-G_1)$ for M (resp. $T\in E-(\varrho-G_1)$ for M) means $T\in(\varrho-G_1)$ for (M, M) (resp. $T\in E-(\varrho-G_1)$ for (M, M)).

Remark 1. Since $r(T)\leqq w_\varrho(T)$ holds for any $\varrho>0$ [9] and $1/d(\mu, \sigma(T))=$ $=r((T-\mu)^{-1})$ is always valid for all $\mu\notin\sigma(T)$, so that we remark that $T\in(\varrho-G_1)$ for $(\sigma(T), N)$ is equivalent to $T\in E-(\varrho-G_1)$ for $(\sigma(T), N)$, namely $(T-\mu)^{-1}$ is ϱ-oid for all complex $\mu\notin N$.

Remark 2. T is called an operator of class $M_\varrho(\varrho \geqq 1)$ [14] if $(T-\mu)^{-1}$ is ϱ-oid for all $\mu \notin \sigma(T)$, so that we remark $T \in M_\varrho(\varrho \geqq 1)$ coincides with $T \in E-(\varrho-G_1)$ for $\sigma(T)$. $T \in (G_1)$ for M [18] means $T \in (1-G_1)$ for (M, M) and $T \in (G_1)$ means $T \in (1-G_1)$ for $\sigma(T)$, equivalently, $T \in E-(1-G_1)$ for $\sigma(T)$.

An operator T is said to be convexoid [8] if $\overline{W(T)} = \operatorname{co} \sigma(T)$, where $\overline{M}$ denotes the closure of a set M in the complex plane and co M means the convex hull of M. It is well known [12] that T is convexoid if and only if $T \in (G_1)$ for co $\sigma(T)$. A new class designed by R of convexoid operators was introduced in [11] as follows: $T \in R$ if

$$\|(T-\mu)^{-1}\| = \frac{1}{d(\mu, W(T))} \quad \text{for all} \quad \mu \notin \overline{W(T)}, \tag{8}$$

that is, $T \in R$ if and only if $T \in E-(1-G_1)$ for $\overline{W(T)}$.

Generalized numerical ranges $W_\alpha(T)$ $(\alpha \geqq 1)$ is defined in [10] as follows:

$$W_\alpha(T) = \bigcap_\mu \{\lambda : |\lambda-\mu| \leqq w_\alpha(T-\mu)\}. \tag{9}$$

$W_\alpha(T)$ is a compact convex set containing co $\sigma(T)$. In case $1 \leqq \alpha \leqq 2$ $W_\alpha(T)$ coincides with $\overline{W(T)}$ [10] and $W_\infty(T) = \operatorname{co} \sigma(T)$ [6], [7], [10]. Since $w_\alpha(T-\mu)$ is a non-increasing function of α [9], $W_\alpha(T) \supset W_\beta(T)$ if $1 \leqq \alpha < \beta$. The function $w_\varrho^0(T)$ is defined by $w_\varrho^0(T) = \sup\{|\lambda| : \lambda \in W_\varrho(T)\}$ for $1 \leqq \varrho \leqq \infty$. $w_\varrho^0(T)$ satisfies the following properties [10];

$$r(T) \leqq w_\varrho^0(T) \leqq w_\varrho(T), \quad w_\infty^0(T) = r(T),$$
$$w_\varrho^0(\mu T) = |\mu|\, w_\varrho^0(T) \quad \text{for all complex } \mu,$$
$$w_2(T) = w_\varrho^0(T) \quad \text{for} \quad 1 \leqq \varrho \leqq 2.$$

The hen-spectrum $\tilde{\sigma}(T)$ is defined by $\tilde{\sigma}(T) = [[\sigma(T)^c]_\infty]^c$ in [3], where M^c is the complement of M, and $[M]_\infty$ is the unbounded component of M. $\tilde{\sigma}(T)$ is a compact set containing $\sigma(T)$ in the complex plane [3]. Using this notion of $\tilde{\sigma}(T)$, another new class denoted by (H_1) of convexoid operators was introduced in [3]: $T \in (H_1)$ if

$$\|(T-\mu)^{-1}\| \leqq \frac{1}{d(\mu, \tilde{\sigma}(T))} \quad \text{for all} \quad \mu \notin \tilde{\sigma}(T), \tag{10}$$

i.e. $T \in (H_1)$ if and only if $T \in (G_1)$ for $\tilde{\sigma}(T)$. (H_1) properly contains both (G_1) and R [3].

Theorem B [14]. *T is convexoid if and only if there exists $\varrho \geqq 1$ such that*

$$w_\varrho((T-\mu)^{-1}) \leqq \frac{1}{d(\mu, \operatorname{co} \sigma(T))} \quad \textit{for all} \quad \mu \notin \operatorname{co} \sigma(T). \tag{11}$$

Theorem B is an improvement of the well-known criterion for convexoidity due to [12].

C. R. PUTNAM considered conditions on an operator T implying

$$(*) \qquad \operatorname{Re}\sigma(T)=\sigma(\operatorname{Re}T).$$

This equation $(*)$ holds for normal and also seminormal operators [16] and moreover $(*)$ has played a role in the proofs in [16], [17] which state that a seminormal operator whose spectrum has zero area is normal. S. K. BERBERIAN has not only given a simple proof of this Putnam's result, but he also has proved the following theorem.

Theorem C [1]. *If $T\in(G_1)$ and $\sigma(T)$ is connected, then $(*)$ holds.*

Related to Theorem C, S. M. PATEL [13] has established that the equation $(*)$ also holds for operations in the class R without any restriction on the spectrum as follows:

Theorem D [13]. *If $T\in R$, then $(*)$ holds.*

S. M. PATEL shows the following characterization of operators in the class R.

Theorem E [15]. *$T\in R$ if and only if there exist $\varrho\geqq 1$ and $\alpha\geqq 1$ such that*

$$(12) \qquad w_\varrho\big((T-\mu)^{-1}\big)=\frac{1}{d\big(\mu,W_\alpha(T)\big)} \quad \text{for all} \quad \mu\notin W_\alpha(T).$$

Our Theorem 1 below is an improvement of Theorem B. Theorem 2 implies Corollary 2 which is a generalization of Theorem C and Theorem D. Finally, Theorem 3 is an improvement of Theorem E.

2. Statement of the results

Theorem 1. *Any one of the following conditions is necessary and sufficient in order that T be convexoid:*

(i) *$T-\mu$ is spectraloid for all complex μ* ([6], [7], [10]),

(ii) *$T-\mu$ is spectraloid for all complex μ whose absolute values are sufficiently large,*

(iii) *there exist $\varrho\geqq 1$ and $2<\alpha\leqq\infty$ such that $T\in(\varrho-G_1)$ for $\big(W_\alpha(T),N\big)$, where N runs over the closed and bounded sets containing $W_\alpha(T)$.*

Theorem 2. *If there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $\big(\sigma(T),\tilde\sigma(T)\big)$ and* $\operatorname{Re}\sigma(T)$ *is connected, then $(*)$ holds.*

Corollary 1. *If $T\in M_\varrho$ and* $\operatorname{Re}\sigma(T)$ *is connected, then $(*)$ holds.*

Corollary 2. *If $T\in(H_1)$ and* $\operatorname{Re}\sigma(T)$ *is connected, then $(*)$ holds.*

Theorem 3. *$T\in R$ if and only if there exist $\varrho\geqq 1$ and $1\leqq\beta\leqq\alpha\leqq\infty$ such that $T\in E-(\varrho-G_1)$ for $(W_\alpha(T), W_\beta(T))$.*

Take $N=W_\alpha(T)$ and $\alpha=\infty$ in (iii) of Theorem 1. Since $W_\infty(T)=\operatorname{co}\sigma(T)$, Theorem 1 implies Theorem B. The class (H_1) properly contains (G_1) [3], consequently Corollary 2 contains Theorem C.

$T\in R$ if and only $\partial W(T)\subset\sigma(T)$ by [11] (that is, $\overline{W(T)}=\tilde{\sigma}(T)$ [3]). The convex set $\overline{W(T)}$ contains $\sigma(T)$, consequently $T\in R$ implies that $\operatorname{Re}\sigma(T)=\operatorname{Re}\overline{W(T)}$ is connected. The class (H_1) properly contains R [3], so Corollary 2 contains Theorem D.

Corollary 1 easily implies Theorem C. Take $\alpha=\beta$ in Theorem 3, then $T\in R$ if and only if $T\in E-(\varrho-G_1)$ for $W_\alpha(T)$ for $1\leqq\varrho$ and $1\leqq\alpha$, that is, (12) holds, so Theorem 3 contains Theorem E.

3. Proofs of the theorems

In order to prove Theorem 1 we need the following Lemma.

Lemma 1. *If X is a closed convex subset of the complex plane, then $X\supset\overline{W(T)}$ if and only if there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for (X, Y), where Y runs over the closed and bounded sets containing X.*

Proof. The proof is along the same lines as the argument in [14, Theorem 4] and we shall state it for the sake of convenience in the subsequent discussion. If $X\supset\overline{W(T)}$, then there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for X by [14, Theorem 4] so there exists $\varrho\leqq 1$ such that $T\in(\varrho-G_1)$ for (X, Y).

Conversely, assuming that there exists $\varrho\leqq 1$ such that $T\in(\varrho-G_1)$ for (X, Y), we have only to show that every half plane M containing X also contains $\overline{W(T)}$. Without loss of generality we may assume $M=\{\lambda: \operatorname{Re}\lambda\geqq 0\}$. Since $M\supset X$ and the hypothesis holds we have

$$w_\varrho((\mu^{-1}T+I)^{-1})=w_\varrho(\mu(T+\mu)^{-1})\leqq\frac{\mu}{d(-\mu, X)}\leqq 1$$

for all positive μ whose absolute values are sufficiently large. Therefore, by (5), we have $(\mu^{-1}T+I)^{-1}\in C_\varrho$ for all positive μ whose absolute values are sufficiently large. By Theorem A we have

$$(\varrho-2)\|(I-(\mu^{-1}T+I)^{-1})x\|^2+2\operatorname{Re}\big((I-(\mu^{-1}T+I)^{-1})x, x\big)\geqq 0,$$

that is,

$$(\varrho-2)\|\mu^{-1}T(\mu^{-1}T+I)^{-1}x\|^2+2\operatorname{Re}\big(\mu^{-1}T(\mu^{-1}T+I)^{-1}x, x\big)\geqq 0$$

for all x in H. Multiply this above inequality by μ and transfering μ to ∞, we obtain $\operatorname{Re}(Tx, x)\geqq 0$ for all x in H, whence $\overline{W(T)}\subset X$, so the proof is complete.

Proof of Theorem 1. The proof of (i) was shown in [6], [7] and thereafter in [10], so that we have only to show the sufficiency of (ii). If X is any bounded closed set in the complex plane, then $\mathrm{co}\, X$ coincides with the intersection of all the circles with sufficiently large radii which contain the set X, so that

(13) $\quad \mathrm{co}\, X = \bigcap_{\mu} \{\lambda : |\lambda-\mu| \leqq \sup_{x\in X} |x-\mu|$ for all complex μ whose absolute values are sufficiently large$\}$.

Taking $X=\overline{W(T)}$ and $\sigma(T)$ in (13) respectively, we have the following formulas since $\overline{W(T)}$ is convex [8],

(14) $\quad \overline{W(T)} = \bigcap_{\mu} \{\lambda : |\lambda-\mu| \leqq w(T-\mu)$ for all complex μ whose absolute values are sufficiently large$\}$,

(15) $\quad \mathrm{co}\, \sigma(T) = \bigcap_{\mu} \{\lambda : |\lambda-\mu| \leqq r(T-\mu)$ for all complex μ whose absolute values are sufficiently large$\}$.

The sufficiency of (ii) follows from (14) and (15).

(iii) Assume the hypothesis in (iii), then by Lemma 1 we have $W_\alpha(T)\supset\overline{W(T)}$ for $2<\alpha\leqq\infty$. On the other hand $\overline{W(T)}\supset W_\alpha(T)$ holds in general for $2<\alpha\leqq\infty$, so that $W_\alpha(T)=\overline{W(T)}$ for $2<\alpha\leqq\infty$. This is equivalent to $w(T-\mu)=w_\alpha^0(T-\mu)$ for $2<\alpha\leqq\infty$ [10, Corollary 1] and this implies $w(T-\mu)=w_2(T-\mu)=w_\alpha(T-\mu)$ for $2<\alpha\leqq\infty$ and for all complex μ since $r(T)\leqq w_\alpha^0(T)\leqq w_\alpha(T)\leqq w(T)$ always holds for $2<\alpha\leqq\infty$ [10]. So by (4) we have $w(T-\mu)=w_\infty(T-\mu)=r(T-\mu)$ for all complex μ, hence T is convexoid [6], [7], [10].

Conversely, if T is convexoid, then there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $\mathrm{co}\,\sigma(T)$, by Theorem B and therefore $T\in(\varrho-G_1)$ for $W_\alpha(T)$ $(2<\alpha\leqq\infty)$ since $W_\alpha(T)\supset\mathrm{co}\,\sigma(T)$. Hence there exist $\varrho\geqq 1$ and $2<\alpha\leqq\infty$ such that $T\in(\varrho-G_1)$ for $(W_\alpha(T), N)$, so the proof is complete.

To give the proof of Theorem 2, we shall show the following Lemmas.

Lemma 2. *If there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $(\sigma(T), \tilde\sigma(T))$ and λ is a semibare point of henspectrum $\tilde\sigma(T)$, then*

(i) *λ is a normal approximate eigenvalue of T, i.e. $\emptyset\neq A_\lambda(T)=A_{\lambda^*}(T^*)$*

(ii) *if in addition λ is an eigenvalue of T, then λ is a normal eigenvalue of T, i.e. $N_\lambda(T)=N_{\lambda^*}(T^*)$ where $A_\lambda(T)=\{\{x_n\}:\|x_n\|=1,\ \|Tx_n-\lambda x_n\|\to 0$ as $n\to\infty\}$ and $N_\lambda(T)$ denotes the kernel of $T-\lambda$.*

Lemma 3. *If there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $(\sigma(T), \tilde\sigma(T))$, then* $\mathrm{Re}\,\sigma(T)\subset\sigma(\mathrm{Re}\,T)$ *holds.*

Lemma 4. *If T is convexoid, then*

(i) *if* $\mathrm{Re}\,\sigma(T)\subset\sigma(\mathrm{Re}\,T)$ *and* $\mathrm{Re}\,\sigma(T)$ *is connected, then* $(*)$ *holds,*

(ii) *if* $\sigma(\mathrm{Re}\,T)\subset\mathrm{Re}\,\sigma(T)$ *and* $\sigma(\mathrm{Re}\,T)$ *is connected, then* $(*)$ *holds,*

(iii) *if both* $\mathrm{Re}\,\sigma(T)$ *and* $\sigma(\mathrm{Re}\,T)$ *are connected, then* $(*)$ *holds* [1], [6].

Proof of Lemma 2. If there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, then $T - \lambda$ also belongs to the same class since $\tilde{\sigma}(T + \lambda I) = \tilde{\sigma}(T) + \lambda$ holds for every complex λ, so that we can assume $\lambda = 0$. As $\lambda = 0$ is a semibare point of $\tilde{\sigma}(T)$, we can choose a nonzero complex number $\lambda_0 \notin \tilde{\sigma}(T)$ such that $\{\lambda : |\lambda - \lambda_0| \leqq |\lambda_0|\}$ meets $\tilde{\sigma}(T)$ only at 0. As $\partial\tilde{\sigma}(T) = \tilde{\sigma}(T) \cup \overline{[\tilde{\sigma}(T)^c]} \subset \sigma(T)$ and $\sigma(T) \subset \tilde{\sigma}(T)$, it follows that $d(\lambda_0, \sigma(T)) = d(\lambda_0, \tilde{\sigma}(T)) = |\lambda_0|$. By the assumption there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, consequently we have the following equality by Remark 1:

$$w_\varrho((T - \lambda_0)^{-1}) = \frac{1}{|\lambda_0|}. \tag{16}$$

As $0 \in \partial\tilde{\sigma}(T)$ (that is, $\partial\sigma(T)$), then $\lambda = 0$ is an approximate eigenvalue of T [8, Problem 63], [19, Theorem 66-B] i.e. there exists a sequence $\{x_n\}$ of unit vectors such that $Tx_n \to 0$. Then

$$\left\|(T - \lambda_0)^{-1}x_n + \frac{1}{\lambda_0}x_n\right\| \leqq \|(T - \lambda_0)^{-1}\| \left\|x_n + (T - \lambda_0)\frac{1}{\lambda_0}x_n\right\| =$$
$$= \|(T - \lambda_0)^{-1}\| \left\|\frac{1}{\lambda_0}Tx_n\right\| \to 0,$$

i.e. $(T - \lambda_0)^{-1}x_n + \dfrac{1}{\lambda_0}x_n \to 0$ and this convergence implies that $(T^* - \lambda_0^*)^{-1}x_n + \dfrac{1}{\lambda_0^*}x_n \to 0$ by S. M. Patel's result [14, Theorem 1] since (16) holds. Whence $T^*x_n \to 0$ by an easy calculation and this means that 0 is an approximate eigenvalue of T^* also. When we replace T by T^* and λ by λ^*, then the above argument is reversible, so we have (i). If we replace x_n by a vector x in the proof of (i), then we have (ii) so the proof is complete.

Proof of Lemma 3. Let $\alpha_0 \in \operatorname{Re}\sigma(T)$. Then there exists $\lambda_0 \in \partial\tilde{\sigma}(T)$ such that $\operatorname{Re}\lambda_0 = \alpha_0$ and λ_0 is an approximate eigenvalue of T by the definition of hen-spectrum $\tilde{\sigma}(T)$. Let $D_n = \left\{\lambda : |\lambda - \lambda_0| \leqq \dfrac{1}{n}\right\}$ for $n = 1, 2, \ldots$, then D_n contains a point $\mu_n \notin \tilde{\sigma}(T)$ such that $|\mu_n - \lambda_0| < \dfrac{1}{2n}$. Clearly it is possible to choose λ_n with the following properties: $\lambda_n \in \tilde{\sigma}(T)$ and $d(\mu_n, \tilde{\sigma}(T)) = d(\mu_n, \sigma(T)) = |\mu_n - \lambda_n|$.

Now $\lambda_n \in \partial\tilde{\sigma}(T)$ lies on the circumference of a closed disc centered at μ_n whose interior contains no point of $\tilde{\sigma}(T)$, whence λ_n is a semibare point of $\tilde{\sigma}(T)$. Since $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, λ_n is a normal approximate eigenvalue of T by Lemma 2, consequently there exists a sequence of unit vectors $\{x_n\}$ such that

$$Tx_n - \lambda_n x_n \to 0 \quad \text{and} \quad T^*x_n - \lambda_n^* x_n \to 0 \quad \text{as} \quad n \to 0.$$

Then we have $Tx_n - \lambda_0 x_n \to 0$ as $n \to \infty$ because

$$\|Tx_n - \lambda_0 x_n\| \leqq \|Tx_n - \lambda_n x_n\| + \|(\lambda_n - \lambda_0)x_n\| \to 0$$

as $n\to\infty$. Similarly $T^*x_n-\lambda_0^*x_n\to 0$ as $n\to\infty$, so that

$$\|(\operatorname{Re}T-\operatorname{Re}\lambda_0)x_n\| \leqq \tfrac{1}{2}\|Tx_n-\lambda_0x_n\|+\tfrac{1}{2}\|T^*x_n-\lambda_0^*x_n\|\to 0$$

as $n\to\infty$, whence $\operatorname{Re}\lambda_0\in\sigma(\operatorname{Re}T)$ and this is the desired relation, so the proof is complete.

We remark that S. K. BERBERIAN has shown Lemma 3 in the case if T satisfies (G_1) for $\sigma(T)$ [1], here we have given the proof of Lemma 3 which is based on (i) of Lemma 2.

Proof of Lemma 4. It is known that T is convexoid if and only if

$$(\Sigma-\theta)\qquad \operatorname{Re}\Sigma(e^{i\theta}T)=\Sigma(\operatorname{Re}e^{i\theta}T)\quad\text{for all}\quad 0\leqq\theta\leqq 2\pi,$$

where $\Sigma(T)$ denotes $\operatorname{co}\sigma(T)$, and this $(\Sigma-\theta)$ is equivalent to $\operatorname{co}\operatorname{Re}\sigma(e^{i\theta}T)=$ $=\operatorname{co}\sigma(\operatorname{Re}e^{i\theta}T)$ for all $0\leqq\theta\leqq 2\pi$ [6].

If T is convexoid, then we have the following property by $(\Sigma-\theta)$

$$\operatorname{co}\operatorname{Re}\sigma(T)=\operatorname{co}\sigma(\operatorname{Re}T). \tag{17}$$

On the other hand, by the hypothesis of (i) we have

$$\operatorname{co}\operatorname{Re}\sigma(T)=\operatorname{Re}\sigma(T)\subset\sigma(\operatorname{Re}T)\subset\operatorname{co}\sigma(\operatorname{Re}T) \tag{18}$$

hence we have $(*)$ by (17) and (18). Similarly we have (ii). By (17) and the hypothesis of (iii), we have (iii).

In order to prove Theorem 2 we shall use only (i) of Lemma 4, but here we state (ii) and (iii) for the sake of completeness as some related results.

Proof of Theorem 2. If there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $(\sigma(T),\tilde\sigma(T))$, then $\operatorname{Re}\sigma(T)\subset\sigma(\operatorname{Re}T)$ holds by Lemma 3 and T is convexoid by Theorem B. So we have $(*)$ by the hypothesis and (i) of Lemma 4 and we have finished the proof.

Corollary 1 easily follows from Theorem 2 by the definition of M_ϱ.

Proof of Corollary 2. As stated in the proof of Lemma 2, for all $\mu\notin\tilde\sigma(T)$, $d(\mu,\tilde\sigma(T))=d(\mu,\sigma(T))$ holds, consequently $T\in(\varrho-G_1)$ for $\tilde\sigma(T)$ if and only if $T\in(\varrho-G_1)$ for $(\sigma(T),\tilde\sigma(T))$.

Specially $T\in(H_1)$ if and only if $T\in(G_1)$ for $(\sigma(T),\tilde\sigma(T))$. So Theorem 2 implies Corollary 2.

Proof of Theorem 3. If $T\in R$, then there exist $\varrho\geqq 1$ and $\alpha\geqq 1$ such that $T\in E-(\varrho-G_1)$ for $W_\alpha(T)$ by Theorem E, consequently there exist $\varrho\geqq 1$ and $1\leqq\beta\leqq\alpha\leqq\infty$ such that $T\in E-(\varrho-G_1)$ for $(W_\alpha(T),W_\beta(T))$.

Conversely, suppose that there exist $\varrho\geqq 1$ and $1\leqq\beta\leqq\alpha\leqq\infty$ such that $T\in E-(\varrho-G_1)$ for $(W_\alpha(T),W_\beta(T))$. We remark that the condition $1\leqq\beta\leqq\alpha\leqq\infty$ can be replaced by $2\leqq\beta\leqq\alpha\leqq\infty$ since $W_\alpha(T)=\overline{W(T)}$ for $1\leqq\alpha\leqq 2$ [10]. When $\alpha=\beta=2$, the hypothesis implies $w_\varrho((T-\mu)^{-1})=1/d(\mu,W(T)$ for all $\mu\notin\overline{W(T)}$ and

$\varrho \geqq 1$. On the other hand $w_{\varrho}((T-\mu)^{-1}) \leqq \|(T-\mu)^{-1}\|$ for $\varrho \geqq 1$ [9] and $\|(T-\mu)^{-1}\| \leqq \leqq 1/d(\mu, W(T))$ always holds for all $\mu \notin \overline{W(T)}$ [22, Theorem 6.2-A]. So we have $\|(T-\mu)^{-1} = 1/d(\mu, W(T))$ for all $\mu \notin \overline{W(T)}$, i.e. $T \in R$, consequently we have only to prove Theorem 3 in case $2 < \alpha$. We can apply (iii) of Theorem 1 in this case, then T turns out to be convexoid, hence $\overline{W(T)} = W_\alpha(T) = W_\beta(T) = \mathrm{co}\,\sigma(T)$ so that the proof can be reduced to the case $\alpha = \beta = 2$ in which the theorem is already proved, so the proof is complete.

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