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Second order Briot-Bouquet differential equations

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1. Introduction. A Briot-Bouquet equation of order k is a DE of the form

$$P[w, w^{(k)}] = 0 \tag{1.1}$$

where $P(x, y)$ is a polynomial in x and y with constant coefficients. In the study of such equations the main problem is to find necessary and, if possible, sufficient conditions in order that the solutions be single-valued functions, holomorphic save for poles in the finite plane.

In 1887 PICARD [7] proved that an algebraic curve

$$P(x, y) = 0 \tag{1.2}$$

admits of a parametric representation

$$x = S(t), \quad y = T(t) \tag{1.3}$$

where S and T are transcendental entire or meromorphic functions of t iff the curve is of *genus* 0 or 1. Since $w(z)$ and $w^{(k)}(z)$ are either both entire or meromorphic or neither has this property we have

Theorem 1. *A necessary condition that* (1.1) *have a single-valued solution, holomorphic save for poles in the finite plane, is that the genus of the curve* (1.2) *be zero or one.*

The condition is not sufficient. Thus the second order DE

$$w'' = w^4 \quad \text{with e.g.} \quad w(z) = \left[w_0^{-3/2} - \frac{3}{2}\sqrt{\frac{2}{5}}(z - z_0)\right]^{-2/3} \tag{1.4}$$

has movable branch-points. The general solution is obtained by inverting a hyper-elliptic integral and has of course infinitely many branch-points.

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For $k=1$ the investigations of Fuchs [1]. Poincaré [8] and Schlesinger [9] have determined the limitations which are put on the polynomial coefficients of the powers of w' by the existence of meromorphic solutions that are non-rational.

Suppose that

$$P(x, y) = P_0(x)y^n+P_1(x)y^{n-1}+\dots+P_n(x) \tag{1.5}$$

and let δ_j be the degree of $P_j(x)$. Fuchs showed that the existence of solutions of the described type requires that $P_0(x)$ be a constant, say $P_0(x)=1$, and that

$$\delta_j = 2j, \quad j = 1, 2, \dots, n. \tag{1.6}$$

These conditions apply to first order BB equations. If they are satisfied and the genus is 0 or 1, then (1.1) has single-valued solutions which are rational functions of z or of e^{az} for some constant a, or of the Weierstrass $\wp$-function and its first derivative. Thus the solutions belong to the class of functions for which Weierstrass has shown the existence of algebraic addition theorems.

The present note is devoted to the case $k=2$. Here the analogue of the conditions of Fuchs read (see [3] Theorem 3)

$$P_0(w) \equiv 1, \quad \delta_j \leqq 3j, \quad j = 1, 2, \dots, n. \tag{1.7}$$

If the solutions are to be entire functions of z, the inequalities become more restrictive:

$$\delta_j \leqq j \tag{1.8}$$

and this inequality holds for all values of k when the solutions are entire functions. Cf. [3] Theorem 4.

In the present note we use the method of Fuchs as presented by Schlesinger to the case $k=2$. We also lean heavily on the results of Painlevé and Gambier concerning second order DE's with fixed critical points. It will be found that the solutions are either of the same three types as for $k=1$ or reducible to such types by a change of variables.

2. Euqations of genus zero. I. Suppose now that $k=2$, conditions (1.7) hold and the curve (1.2) is of genus zero. Then a rational function of x and y exists such that

$$t = R(x, y) \tag{2.1}$$

leads to

$$x = R_1(t), \quad y = R_2(t) \tag{2.2}$$

where R_1 and R_2 are rational functions of t. The DE then becomes

$$w''(z) = R_2(t) \quad \text{with} \quad w(z) = R_1(t). \tag{2.3}$$

Differentiation of the second equation with respect to z gives

$$w'(z) = R_1'(t)\frac{dt}{dz}, \quad w''(z) = R_1''(t)\left(\frac{dt}{dz}\right)^2 + R_1'(t)\frac{d^2t}{dz^2}. \tag{2.4}$$

Thus t as a function of z satisfies the DE

$$\frac{d^2t}{dz^2} + \frac{R_1''(t)}{R_1'(t)}\left(\frac{dt}{dz}\right)^2 = \frac{R_2(t)}{R_1'(t)}, \tag{2.5}$$

where R_1 and R_2 are rational in t. There are two distinct possibilities according as $R_1''(t) \equiv 0$ or not. The first case is by far the simpler one.

I. $R_1''(t) \equiv 0$. We may assume $R_1'(t) \equiv 1$. The DE (2.5) now reduces to

$$\frac{d^2t}{dz^2} = R_2(t). \tag{2.6}$$

Since in this case $w(z) = t(z) + t_0$, the requirement that $w(z)$ shall have no branch-points implies that $R_2(t)$ is a polynomial in t of degree $\leqq 3$ by (1.7). It is necessary to distinguish between a number of subcases.

I:1. $\delta[R_2] = 3$. A first integral of (2.6) takes the form

$$\left(\frac{dt}{dz}\right)^2 = A(t-a_1)(t-a_2)(t-a_3)(t-a_4). \tag{2.7}$$

Here there are essentially five different possibilities.

I:11. The a_j's are distinct. Then there exists an affine transformation $z = as$, $t = bv + c$ which takes (2.7) into the Jacobi normal form

$$\left(\frac{dv}{dz}\right)^2 = (1-v^2)(1-k^2v^2) \tag{2.8}$$

where the modulus k is determined by the a_j's. The solutions of (1.1) are thus elliptic functions of z.

I:12. $a_1 = a_2 = a$, $a_3 \neq a_4$, $(a-a_3)(a-a_4) \neq 0$. Set

$$t = a + \frac{1}{v} \tag{2.9}$$

which reduces (2.7) to the form

$$\left(\frac{dv}{dz}\right)^2 = B(v-v_1)(v-v_2). \tag{2.10}$$

The corresponding solution $w(z)$ is a rational function of e^{az} for some a. It is simply periodic.

I:13. $a_1=a_2=a,\ a_3=a_4=b,\ a\neq b.$ Thus

$$\frac{dt}{dz}=B(t-a)(t-b). \tag{2.11}$$

Here also $w(z)$ is simply-periodic and a rational function of $\exp[(a-b)Bz]$.

I:14. $a_1=a_2=a_3=a,\ a_4=b\neq a$ so that

$$\left(\frac{dt}{dz}\right)^2=A(t-a)^3(t-b). \tag{2.11}$$

The substitution (2.9) leads to a DE of the form

$$\left(\frac{dv}{dz}\right)^2=B(v-c) \tag{2.12}$$

which is satisfied by a quadratic polynomial so that $w(z)$ is a rational function of z.

I:15. All the a_j's are equal to a. The equation may be reduced to the form

$$\frac{dv}{dz}=v^2 \quad \text{with} \quad v(z)=v_0-(z-z_0)^{-1} \tag{2.13}$$

so that $w(z)$ is also in this case a rational function of z.

This exhausts the possibilities when $\delta(R_2)=3,\ R_1(t)\equiv 1$ and $p=0$.

I:2. $\delta(R_2)=2$ gives the first integral

$$\left(\frac{dt}{dz}\right)^2=A(t-a_1)(t-a_2)(t-a_3). \tag{2.14}$$

Here we have the following subcases:

I:21. The a's are distinct. An affine transformation leads to Weierstrass's normal form

$$\left(\frac{dv}{ds}\right)^2=4v^3-g_2v-g_3 \tag{2.15}$$

so that $w(z)$ is of the form

$$w(z)=b\wp(cz-s_0;\,g_2,g_3)+v_0 \tag{2.16}$$

with s_0 and v_0 as arbitrary constants.

I:22. $a_1=a_2=a,\ a_3=b\neq a.$ The substitution (2.9) reduces the DE to the form (2.10).

I:23. All the a_j's are equal to a so that the solution $w(z)$ is a rational function of z.

The only remaining case is that where $\delta(R_2)=1$ so that

$$\frac{d^2t}{dz^2} = c^2t-a, \quad c \neq 0 \tag{2.17}$$

with

$$t(z) = K_1e^{cz}+K_2e^{-cz}+ac^{-2} \tag{2.18}$$

so that $w(z)$ is a rational function of e^{cz}. This ends the case $R_1''(t)\equiv 0$.

3. General case with $p=0$. We have now equation (2.5) with $R_1''(t)\not\equiv 0$. Suppose that $w(z)$ has a pole at a finite point z_0. Since $w(z)=R_1(t)$ is a rational function of t, it is seen that $R_1(t)\to\infty$ as $t\to\infty$ and this says that at $t=\infty$

$$R_1(t) = a_0t^\mu+o(t^\mu), \quad a_0 \neq 0, \tag{3.1}$$

where μ is a positive integer. The cases $\mu=1$ and $\mu>1$ require separate treatment. If $\mu=1$, then

$$R_1(t) = a_0t+a_1+a_2t^{-\lambda}+O(t^{-\lambda-1}), \quad a_0a_2 \neq 0 \tag{3.2}$$

where λ is a positive integer. Hence

$$\frac{R_1''(t)}{R_1'(t)} = \frac{a_2}{a_0}\lambda(\lambda+1)t^{-\lambda-2}+O(t^{-\lambda-3}). \tag{3.3}$$

In the second case $\mu>1$ the ratio equals

$$(\mu-1)t^{-1}+O(t^{-2}). \tag{3.4}$$

Further

$$R_2(t) = b_0t^\nu+O(t^{\nu-1}). \tag{3.5}$$

Since $R_2(t)=w''(z)$ and $w''(z)/w(z)$ becomes infinite as z approaches a pole, one concludes that

$$\nu \geqq \mu+1 \tag{3.6}$$

and in

$$\frac{R_2(t)}{R_1'(t)} = \frac{b_0}{\mu a_0}t^{\nu+1-\mu}+O(t^{\nu-\mu}) \tag{3.7}$$

the leading exponent is at least 2.

We can start to whittle down the exponent. Some of this work is elementary but ultimately we have to fall back on the results of Painlevé and Gambier. Suppose that $w(z)$ has a pole of order α at $z=z_0$ where

$$w(z) = a(z-z_0)^{-\alpha}[1+o(1)] \tag{3.8}$$

so that

$$w'(z) = -\alpha a(z-z_0)^{-\alpha-1}[1+o(1)], \quad w''(z) = \alpha(\alpha+1)a(z-z_0)^{-\alpha-2}[1+o(1)]. \tag{3.9}$$

Set

$$\frac{R_1''(t)}{R_1'(t)} = Q_1(t), \quad \frac{R_1(t)}{R_1'(t)} = Q_2(t) \tag{3.10}$$

so that (2.5) becomes

$$t''(z)+Q_1(t)[t'(z)]^2 = Q_2(t). \tag{3.11}$$

Now

$$w(z) = R_1[t(z)] = a_0[t(z)]^{\mu}[1+o(1)]$$

in a neighborhood of a pole and $t(z)$ is a rational function of z, $w(z)$ and $w''(z)$ by (2.1) so any infinitude of $t(z)$ must be a pole, say of z order β at $z=z_0$ and here

$$\beta = \frac{\alpha}{\mu} \tag{3.12}$$

so that μ is a divisor of α. At $z=z_0$ the three terms of (3.11) have poles of order

$$\beta+2, \quad \beta+2 \quad \text{or} \quad 2-\lambda\beta, \quad \text{and} \quad (\nu+1-\mu)\beta,$$

respectively. Since the infinitary terms must balance in the equation, it is seen that $\beta+2 \geqq (\nu+1-\mu)\beta$ or $(\nu-\mu)\beta \leqq 2$. Here both factors on the left are positive integers, at least equal to 1. It follows that

$$1 \leqq \beta \leqq 2, \quad 1 \leqq \nu-\mu \leqq 2. \tag{3.13}$$

Since $\beta=\alpha/\mu$, it is seen that

$$\alpha = \frac{2\mu}{\nu-\mu} \quad \text{and} \quad \nu \leqq 3\mu. \tag{3.14}$$

This is as far as we can get with elementary methods.

P. Painlevé [6] and R. Gambier [2] have determined the DE's of the form

$$v''(z) = L(z, v)[v'(z)]^2+M(z, v)v'(z)+N(z, v) \tag{3.15}$$

which have fixed critical points. Here L, M, N are analytic functions of z and rational in v. An excellent presentation of the theory is given in Ince [5, Chapter XIV]. Painlevé and Gambier found that the equations of type (3.15) with fixed critical points (branch points and essential singular points) could be reduced, possibly by change of variables, to one of 50 different normal forms. We shall apply these results to equation (3.11). This equation does not involve z explicitly, further $M(v)$ is identically zero while $N(v)$ is definitely not. This reduces the types that have to be considered from 50 to 15. These are listed by Ince [5, pp. 337—343] under the headings XII, XVI, XVIII, XIX, XXI—XXIII, XXVI, XXIX, XXX, XXXII, XXXIII, XXXVIII, XLIV and XLIX. We refer to Ince for details.

On the face of it his equation XIV should also be considered, but this equation contains a misprint: a factor $\frac{dw}{dZ}$ is missing so $M(v)$ is not identically zero.

One can set the arbitrary functions $q(Z)$ and $r(Z)$ equal to zero but this also makes $N(v)\equiv 0$ so the equation does not qualify.

The reduction to a normal form may involve a change of variables but in the case of (3.11) and $p=0$ only affine transformations

$$Z = az+b, \quad V = cv+d, \tag{3.16}$$

need be considered.

The function $v\mapsto L(v)$ has at most 3 poles and for XLIX this number is reached and the normal form is

$$L(v) = \frac{1}{2}\left\{\frac{1}{v}+\frac{1}{v-1}+\frac{1}{v-a}\right\}, \quad a \neq 0, 1. \tag{3.17}$$

Two poles occur in XXXVIII and XLIV with $L(v)$ equal to

$$\frac{1}{2v}+\frac{1}{v-1} \quad \text{and} \quad \frac{3}{4}\left\{\frac{1}{v}+\frac{1}{v-1}\right\}, \tag{3.18}$$

respectively. All the other L's are of the form Cv^{-1} where the constant C has only three possible values $\frac{1}{2}\cdot\frac{3}{4}\cdot 1$. At infinity $L(v)$ has a simple zero. Comparison with (3.3) and (3.4) shows that $\mu>1$.

The rational function $N(v)$ is normally of degree $\nu=3$; it is 2 for XVIII, XIX, XXI, XXIII, XXXIII. For XXII $\nu=0$ and -1 for XXXII. The latter two equations are excentional in as much as the solutions are polynomials in z and thus have no finite poles. There is no contradiction with (3.13) and (3.14) since these relations presuppose the existence of poles.

The solutions of (3.11) are elliptic functions when $\nu=2$ or 3. Combining the results of this section with those of the preceding one leads to

Theorem 2. *If the curve* (1.1) *is of genus zero and if* (1.7) *holds then the solutions of the DE*

$$[w''(z)]^n + \sum_{j=1}^{n} P_j[w(z)][w''(z)]^{n-j} = 0 \tag{3.19}$$

are rational functions of z or of e^{az} for some a, or finally of $\wp(az+b; g_2, g_3)$ and its derivative with respect to z for some choice of the parameters a, b, g_2, g_3.

4. The case $p=1$. Here we can find four rational functions, each of two arguments, such that

$$s = R_1(x, y), \quad t = R_2(x, y), \quad x = R_3(s, t), \quad y = R_4(s, t) \tag{4.1}$$

with

$$t^2 = 4s^3-g_2s-g_3 \tag{4.2}$$

where the parameters g_2 and g_3 may be determined from the coefficients p_{jk} of (1.2).

Since t^2 is a polynomial in s, the functions R_3 and R_4 may be written as follows

(4.3) $$R_3(s,t) = R_{31}(s)+tR_{32}(s), \quad R_4(s,t) = R_{41}(s)+tR_{42}(s)$$

where the R_{jk} are rational functions of s.

Since by definition $\dfrac{d^2x}{dz^2}=y$ we get

(4.4) $$\frac{d^2x}{dz^2} = \left\{R'_{31}+tR'_{32}+R_{32}\frac{6s^2-\frac{1}{2}g_2}{t}\right\}\frac{d^2s}{dz^2}+$$

$$+\left\{R''_{31}+tR''_{32}+\frac{12s^2-g_2}{t}R'_{32}+R_{32}\frac{1}{t}\left[12s-\left(\frac{12s^2-g_2}{2t}\right)^2\right]\right\}\left(\frac{ds}{dz}\right)^2 = y = R_{41}+tR_{42}.$$

It follows that $z\mapsto s(z)$ satisfies a DE of the form

(4.5) $$\frac{d^2s}{dz^2} = [Q_{11}(s)+tQ_{12}(s)]\left(\frac{ds}{dz}\right)^2+Q_{21}(s)+tQ_{22}(s)$$

where the Q_{jk} are rational functions of s.

Here there are various possibilities.

I. $Q_{12}(s)=Q_{22}(s)\equiv 0$. The equation (4.5) is then essentially of the same nature as (2.5) and the previous results apply. The solution is normally an elliptic function of z but it may degenerate to a rational function of z or of e^{az} for some constant a. This case gives nothing new.

II. At least one of the functions $Q_{12}(s)$ and $Q_{22}(s)$ is not identically zero. We note that at least one of the functions Q_{21} and Q_{22} cannot vanish identically save for the trivial DE $[w''(z)]^n=0$.

Suppose that at $s=\infty$

(4.6) $$Q_{jk}(s) = a_{jk}s^{\delta_{jk}}[1+o(1)]$$

and suppose that a solution $s(z)$ of (4.5) has an infinitude of order β at $z=z_0$ so that

(4.7) $$s(z) = b(z-z_0)^{-\beta}[1+o(1)].$$

Equation (4.5) involves five terms that may become infinite as a negative power of $(z-z_0)$. The orders are respectively

(4.8) $$\beta+2,\quad (\delta_{11}+2)\beta+2,\quad \left(\delta_{12}+\frac{7}{2}\right)\beta+2,\quad \delta_{21}\beta,\quad \left(\delta_{22}+\frac{3}{2}\right)\beta$$

provided the corresponding $a_{jk}\neq 0$. Here the δ_{jk} are integers, >0 or 0 or <0. Since no term can dominate the first term and at least one of the other terms must be of the same order of magnitude we get a set of inequalities which must be satisfied by the δ_{jk}'s:

(4.9) $$\delta_{11}\leqq -1,\quad \delta_{12}\leqq -3,\quad \delta_{21}\leqq 3,\quad \delta_{22}\leqq 1.$$

Let us now bring the known facts to bear on our problem. Painlevé also examined the case where the coefficients L, M, N are algebraic functions of v so that L, M, N are rational functions of the variables v and W where

$$C(w, W) = 0 \tag{4.10}$$

and C is a polynomial with constant coefficients and the curve (4.10) is of genus 0 or 1.

Besides the 50 types found in the rational case $p=0$ Painlevé found only 3 additional types free from movable critical points. If in these equations the arbitrary functions are replaced by arbitrary constants and the conditions $M(v)\equiv\equiv 0, N(v)\not\equiv 0$ are imposed, only two types are found to qualify. These equations may be written

$$s''(z) = \frac{1}{2}\frac{T'(s)}{T(s)}[s'(z)]^2 + r[T(s)]^{1/2} \tag{4.11}$$

and

$$s''(z) = \left\{\frac{1}{2}\frac{T'(s)}{T(s)} - \frac{\pi i}{\omega}[T(s)]^{-1/2}\right\}[s'(z)]^2 + r[T(s)]^{\frac{1}{2}}. \tag{4.12}$$

Here

$$T(s) = 4s^3 - g_2 s - g_3 \tag{4.13}$$

and 2ω is an arbitrary period of $\wp(s; g_2, g_3)$.

Equation (4.11) is equivalent to the system

$$\begin{cases} s'(z) = u(z)\{T[s(z)]\}^{1/2} \\ u'(z) = r \end{cases}, \tag{4.14}$$

with solutions

$$s(z) = \wp\left(\tfrac{1}{2} rz^2 + C_1 z + C_2; g_2, g_3\right). \tag{4.15}$$

By (4.1) the solution $w(z)$ of (1.2) is a rational function of $s(z)$ and $T[s(z)]$, that is expressible in terms of elliptic functions of a quadratic polynomial. Such an elliptic function would necessarily have Nevanlinna order 4. But this contradicts Theorem 6 of [4] according to which the Nevanlinna order of a meromorphic solution of a Briot-Bouquet DE is at most 2. We conclude that an equation of type (4.11) can not arise when the birational transformation (4.1) is applied to a BB equation.

As we shall see in a moment, equation (4.12) can also be dismissed. This equation also leads to a simple system

$$\begin{cases} s'(z) = u(z)T[s(z)]^{1/2} \\ u'(z) = i\dfrac{\pi}{\omega}[u(z)]^2 + r. \end{cases} \tag{4.16}$$

Here $r \neq 0$ since $N(v) \not\equiv 0$. We set $r = i\dfrac{\pi}{\omega}a^2$ so that

$$u(z) = ai\tanh\left[a\frac{\pi}{\omega}(z-z_0)\right], \tag{4.17}$$

$$\int^z u(s)\,ds = i\frac{\omega}{\pi}\log\sinh\left[a\frac{\pi}{\omega}(z-z_0)\right] \tag{4.18}$$

so that

$$s(z) = \wp\left\{i\frac{\omega}{\pi}\log\sinh\left[a\frac{\pi}{\omega}(z-z_0)\right];\, g_2,\, g_3\right\}. \tag{4.19}$$

This solution has singularities at all the points $z_k = z_0 + k\dfrac{\omega}{a}i$. If z describes a positive circuit around one of these points, the logarithm is increased by $2\pi i$ and the argument of the $\wp$-function decreases by 2ω which is a period so the solution returns to its original value. Thus the solution is single-valued but it is not a meromorphic function. In fact, each of the points z_k is a point of accumulation of poles. Thus $z \mapsto s(z)$ takes on every value infinitely often in an arbitrarily small neighborhood of z_k. Now a rational function of $s(z)$ and $T[s(z)]$ inherits these properties of $s(z)$.

According to Theorem 4 of [4] the determinateness theorem of Painlevé holds also for second order BB-equations. This shows that an equation of type (4.11) cannot be obtained as a transform of a BB-equation. Thus we have proved

Theorem 3. *If the curve* (1.1) *is of genus* 1 *and if* (1.7) *holds, then the solutions of* (3.19) *are rational functions of z or of e^{az} for some a or, finally, of $\wp[L(z); g_2, g_3]$ and its z-derivative where $L(z)$ is a linear function of z.*

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