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Quasi-varieties of binary relations

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A *quasi-variety* of groups is the class of all groups satisfying a certain set of laws of the form $\wedge u_i = v_i \Rightarrow u = v$ for all values of the variables in the groups, where u, v, u_i, v_i are terms. Likewise quasi-varieties of semigroups, groupoids, etc. have been studied. Here we define an analogous concept for binary relations.

Definition. A *quasi-variety* of binary relations is the class of all binary relations R for which a class of laws of the following forms holds:

I. $$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_a \mathrm{R} x_b,$$

II. $$(\wedge (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_a \overline{\mathrm{R}} x_b,$$

III. $$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_a = x_b.$$

Such a law is specified by giving K, a, b. Here K can be any subset of the Cartesian product of any set with itself. The notation $x_a \overline{\mathrm{R}} x_b$ means "$x_a \mathrm{R} x_b$ is false".

Definition. Let S_a be a family of sets and let R_a be a binary relation defined on S_a for each a. Then the *direct product* of the R_a is the relation R on ΠS_a such that $x \mathrm{R} y$ if and only if $x_a \mathrm{R}_a y_a$ for each a, where x_a and y_a are the components in factor a of x and y.

Definition. Two binary relations $\mathrm{R}_1, \mathrm{R}_2$ on sets S_1, S_2 are *isomorphic* if and only if there exists an isomorphism f from S_1 to S_2 such that $x \mathrm{R}_1 y$ iff $f(x) \mathrm{R}_2 f(y)$.

Theorem 1. *A non-empty class of binary relations is a quasi-variety if and only if it is closed under direct products, restrictions, and isomorphisms.*

Proof. It follows from the form of the laws that quasi-varieties are in fact closed under direct products, restrictions, and isomorphisms.

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Let V be a non-empty class of binary relations closed under direct products, restrictions, and isomorphisms. Let T be a binary relation which satisfies every law of the forms I, II, III which holds for every member of V. Let S be the set on which T is defined. We will show there exists a member Q of V and a mapping $s \to x_s$ of S into the set on which Q is defined, such that $i\mathrm{T}j$ implies $x_i \mathrm{Q} x_j$ for all $i, j \in S$. Suppose not. Then $a\mathrm{T}b$ for some a, b. Let $K = \{(i, j): i\mathrm{T}j, i, j \in S$ and $(i, j) \neq (a, b)\}$.

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_a \overline{\mathrm{R}} x_b$$

is a law in V but is not true for T. This is contrary to assumption.

Let A be the set of all relations Q defined on subsets of S, and having the above property; let, furthermore, elements x_s be of the set on which Q is defined, such that $i\mathrm{T}j$ implies $x_i \mathrm{Q} x_j$ for all $i, j \in S$. Given a Q defined on some set as above, by restriction and isomorphism we obtain a Q′ defined on a subset of S. Thus A is non-empty. Let π be the direct product of all the relations of A. Let a_s for $s \in S$ be the element of π which is x_s in each factor.

Suppose $a_c = a_d$ for $c \neq d$. Then put $K = \{(i, j): i\mathrm{T}j\}$.

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_c = x_d$$

is a law which holds for all relations of V defined on subsets of S. This means that it is a law of V. But it does not hold for T. This is contrary to assumption. Therefore the a_s are distinct.

Suppose $c\overline{\mathrm{T}}d$ but $a_c \pi a_d$. Then with the same K

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_c \mathrm{R} x_d$$

is a law in V but not for T. This is contrary to assumption. Therefore $c\overline{\mathrm{T}}d$ implies $a_c \bar{\pi} a_d$. It follows by construction that $c\mathrm{T}d$ implies $a_c \pi a_d$. Therefore the restriction of π to the a_s is isomorphic to T. Therefore T is isomorphic to a member of V. Therefore $\mathrm{T} \in V$. This shows V is a quasi-variety and proves the theorem.

Remark. Theorem 1 can be proved for any relational structure.

In the following we represent a binary relation R by the *Boolean matrix* $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $(i, j) \in \mathrm{R}$; otherwise $a_{ij} = 0$.

Theorem 2. *Every* 3×3 *Boolean matrix belongs to a proper sub-quasi-variety of the quasi-variety of all binary relations.*

Proof. Suppose A is a 3×3 Boolean matrix which does not belong to any proper sub-quasi-variety of the quasi-variety of all binary relations.

The set of binary relations R such that $x\mathrm{R}x$ holds for at most one x is a quasi-variety. So A has at least two 1's on its diagonal. Since reflexive matrices are a quasi-variety, A has exactly two ones on its diagonal.

Then if A does not belong to the quasi-variety given by the law

$$(iRi \text{ and } jRj \text{ and } iRj \text{ and } jRi) \Rightarrow i=j,$$

it will belong to the quasi-variety given by

$$(iRi \text{ and } jRj \text{ and } iRj) \Rightarrow jRi.$$

Theorem 3. *The binary relation corresponding to a* 4×4 *Boolean matrix of the form*

$$\begin{bmatrix} 0 & 1 & * & * \\ 1 & 1 & 1 & 0 \\ * & 1 & 1 & 1 \\ * & 1 & 1 & 1 \end{bmatrix}$$

does not belong to any proper sub-quasi-variety of the quasi-variety of all binary relations.

Proof. It suffices to show no non-trivial law of the form I, II or III can hold for this relation. Since 2R2, no law of type II can hold. Since 2R2, 3R3, 2R3, 3R2, no law of type III can hold. Suppose a law of type I holds, $a\neq b$, $(a, b)\notin K$. Then set $x_a=2$, $x_b=4$, all other $x_i=3$. Then for all $(i, j)\in K$, x_iRx_j is true but x_aRx_b is false. So the law does not hold. If a law of type I with $a=b$ is given, $(a, b)\notin K$, let $x_a=1$, all other $x_i=2$. The law will not hold.

Proposition 4. *Any quasi-variety containing all idempotent binary relations also contains all transitive binary relations.*

Proof. Any transitive relation is a restriction of an idempotent relation, by the following construction. Let T be transitive. For each x, y such that xTy add an element $z(x, y)$ to the set on which T is a relation. Define T_1 on the new set by xT_1y if and only if xTy or $x=y=z(u, v)$ for some u, v or $x=z(u, v)$, $y=v$ or $x=u, y=z(u, v)$. Let T_2 be the transitive relation generated by T_1, i.e. xT_2y if and only if xT_1x_1, $x_1T_1x_2, \dots, x_kT_1y$ for some sequence $x_1, \dots, x_k$. Then T_2 is idempotent and T is a restriction of T_2.

Proposition 5. *Any quasi-variety containing only idempotent binary relations is contained in one of the following two quasi-varieties:* (i) *all transitive relations such that* xRy *implies* xRx, *and* (ii) *all transitive relations such that* xRy *implies* yRy.

Proof. Suppose a quasi-variety V contains relations R_1 and R_2 which are transitive but such that (i) fails for R_1 and (ii) fails for R_2. Let aR_1b and $a\overline{R}_1a$ and cR_2d and $d\overline{R}_2d$. Then if we restrict $R_1\times R_2$ to (a, c), (b, d) we have a relation whose matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Neither is idempotent.

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