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On the strong summability of Fourier series and the classes H^ω

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1. Let f be a 2π-periodic integrable function and let $\{s_\varkappa\}$ be the sequence of the partial sums of the Fourier series of this function.

FREUD [1] proved that if $1<p<\infty$ and

$$\left\|\sum_{\varkappa=0}^{\infty} |f-s_\varkappa|^p\right\| < \infty \text{ [1]} \tag{1}$$

then $f \in \operatorname{Lip} \frac{1}{p}$. LEINDLER and NIKIŠIN [3] proved that under the condition (1) with $p=1$,

$$\omega(x, f) = O\left(x \log \frac{1}{x}\right) \quad \text{as} \quad x \to 0,$$

but no estimate better than this can be given. OSKOLKOV [7] and SZABADOS [9] (independently) proved that condition (1) with $0<p<1$ implies $f \in \operatorname{Lip} 1$. This is an answer to a problem of LEINDLER [4] in connection with the above result of LEINDLER and NIKIŠIN.

In this paper we investigate the problem to find a necessary and sufficient condition for a monotonic sequence $\{\lambda_\varkappa\}$ such that the condition

$$\left\|\sum_{\varkappa=0}^{\infty} \lambda_\varkappa |f-s_\varkappa|^p\right\| < \infty, \quad 0<p<\infty$$

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[1]) $\|f\| = \sup |f(x)|, \quad 0 \leqq x \leqq 2\pi.$

should imply $f\in H^\omega$, where ω is a fixed modulus of continuity and H^ω denotes the set of functions f having modulus of continuity $\omega(f,\delta)$ with $\omega(f,\delta)=O(\omega(\delta))$. For a monotonic sequence $\{\lambda_\varkappa\}$ and $0<p<\infty$ we denote

$$S_p\{\lambda_\varkappa\}=\left\{f:\left\|\sum_{\varkappa=0}^{\infty}\lambda_\varkappa|f-s_\varkappa|^p\right\|<\infty\right\}.$$

We prove the following

Theorem. *Let $\{\lambda_\varkappa\}$ be a positive monotonic (nondecreasing or nonincreasing) sequence, furthermore let ω be a modulus of continuity and $0<p<\infty$. Then*

i) *condition*

$$\sum_{\varkappa=1}^{n}(\varkappa\lambda_\varkappa)^{-1/p}=O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2}$$

implies

$$S_p\{\lambda_\varkappa\}\subset H^\omega; \tag{3}$$

ii) *if there exists a number θ such that $0\leqq\theta<1$ and*

$$\varkappa^\theta\lambda_\varkappa\uparrow \tag{4}$$

then, conversely, (3) *implies* (2).

Obviously, this Theorem includes all the results mentioned above and, hereby, we give an answer to a problem raised in [6]. Furthermore, our Theorem includes some results of Leindler [2].

2. To prove our Theorem we require the following lemmas.

Lemma 1. *If $\{a_m\}$ is a nonincreasing positive sequence and if $q>0$, then there exists a constant $C_q>0$ not depending on n such that*

$$\sum_{m=0}^{n}2^m a_m\leqq C_q\sum_{m=0}^{n}2^m a_m\left(\frac{a_{m+1}}{a_m}\right)^q\quad(n=1,2,\ldots).$$

Proof. Let $\{m_i\}$ and $\{M_i\}$ $(i=1,2,\ldots)$ be two sequences of natural numbers such that

$$a_{m+1}>\frac{1}{4}a_m\quad\text{for}\quad M_i\leqq m<m_{i+1} \tag{5}$$

and

$$a_{m+1}\leqq\frac{1}{4}a_m\quad\text{for}\quad m_i\leqq m<M_i. \tag{6}$$

By (6) we obtain

$$a_{m_i+r}\leqq 4^{-r}a_{m_i}\quad(r=0,\ldots,M_i-m_i-1;\ i\geqq 2),$$

therefore, if $i \geqq 2$, then

$$\sum_{m=m_i}^{M_i-1} 2^m a_m = \sum_{r=0}^{M_i-m_i-1} 2^{m_i+r} a_{m_i+r} \leqq 2^{m_i} a_{m_i} \sum_{r=0}^{\infty} 2^{-r} \leqq$$
$$\leqq 4^{1+q} 2^{m_i-1} a_{m_i-1} \left(\frac{a_{m_i}}{a_{m_i-1}}\right)^q .$$

Furthermore, (5) implies

$$\sum_{m=M_i}^{m_{i+1}-1} 2^m a_m \leqq 4^q \sum_{m=M_i}^{m_{i+1}-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

and the last two inequalities give for $i \geqq 2$

$$\sum_{m=m_i}^{m_{i+1}-1} 2^m a_m \leqq 4^{1+q} \sum_{m=m_i-1}^{m_{i+1}-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q . \tag{6}$$

If $m_i \leqq n < m_{i+1}$ and $i \geqq 2$, then

$$\sum_{m=m_i}^{n} 2^m a_m \leqq 4^{1+q} \sum_{m=m_i-1}^{n} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q . \tag{7}$$

The proof runs exactly as before.

Finally, we set

$$C = \max_{1 \leqq n \leqq m_2} \sum_{m=0}^{n} 2^m a_m \Big/ \sum_{m=0}^{n-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q ;$$

then

$$\sum_{m=0}^{n} 2^m a_m \leqq C \sum_{m=0}^{n-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

for $n=1, \dots, m_2$ and (6) and (7) imply

$$\sum_{m=0}^{n} 2^m a_m = \sum_{m=0}^{m_2-1} + \sum_{i=2}^{\varkappa-1} \sum_{m=m_i}^{m_{i+1}-1} + \sum_{m=m_\varkappa}^{n} \leqq$$
$$\leqq C \sum_{m=0}^{m_2-2} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q + 8 \cdot 4^q \sum_{m=m_2-1}^{n} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

for $m_\varkappa \leqq n < m_{\varkappa+1}$ $(\varkappa \geqq 2)$. Therefore, our inequality is true with

$$C_q = \max(C, 8 \cdot 4^q).$$

and Lemma 1 is proved.

Lemma 2. *Let $\{\lambda_\varkappa\}$ and p be as in the Theorem. Then $f \in S_p\{\lambda_\varkappa\}$ implies*

$$\sum_{m=0}^{n} 2^m E_{2^m}(f) \leqq C_{p,\lambda}(f) \sum_{m=0}^{n} 2^m (2^m \lambda_{2^m})^{-1/p} \quad (n = 1, 2, \dots), \tag{8}$$

where $C_{p,\lambda}(f)$ is a positive constant and $E_\varkappa(f)$ is the best approximation of f by trigonometric polynomials of degree at most $\varkappa$.

Proof. First we assume $p \geqq 1$. Then by Hölder's inequality we have

$$E_{2n}(f) \leqq \left\| \frac{1}{n+1} \sum_{\varkappa=n}^{2n} s_\varkappa - f \right\| \leqq \frac{1}{n+1} \left(\sum_{\varkappa=n}^{2n} 1 \right)^{1-(1/p)} \left\| \left\{ \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \leqq$$

$$\leqq \left\| \left\{ \frac{1}{n} \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \leqq C(f)(n\lambda_n^*)^{1/p} \quad (n = 1, 2, \ldots),$$

where $\lambda_n^* = \min(\lambda_n, \lambda_{2n})$. This implies (8) for $p \geqq 1$.

In the case $0<p<1$ we require the following result of [5]:

$$E_n(f) \left[\frac{E_{2n}(f)}{E_n(f)} \right]^{1/p_2} \leqq C_p \left\| \left\{ \frac{1}{n} \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \quad (n = 1, 2, \ldots),$$

where C_p depends only on p. Using this inequality, by Lemma 1 we obtain (8).

Lemma 3. *If $a_\varkappa \geqq 0$ and the function*

$$f \sim \sum_{\varkappa=1}^{\infty} a_\varkappa \sin \varkappa x$$

belongs to the class H^ω, then

$$\sum_{\varkappa=1}^{n} \varkappa a_\varkappa = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Proof. Since $f(0)=0$, $f \in H^\omega$ implies

$$\max_{0<t\leqq x} |f(x)| \leqq C\omega(x), \quad 0 < x < \pi.$$

Therefore,

$$2 \sum_{\varkappa=1}^{\infty} \frac{a_\varkappa}{\varkappa} \sin^2 \frac{\varkappa x}{2} = \int_0^x f(t)\, dt \leqq Cx\omega(x).$$

If we take $x = \frac{\pi}{n}$, then

$$n^{-2} \sum_{\varkappa=0}^{n} \varkappa a_\varkappa = \sum_{\varkappa=1}^{n} \frac{a_\varkappa}{\varkappa} \left(\frac{\varkappa}{n}\right)^2 \leqq \sum_{\varkappa=1}^{n} \frac{a_\varkappa}{\varkappa} \sin^2 \frac{\varkappa \pi}{2n} \leqq \frac{C}{n} \omega\left(\frac{1}{n}\right)$$

for $n=1, 2, \ldots$ and Lemma 3 is proved.

Lemma 4. *If $\lambda_\varkappa \uparrow$ or $\lambda_\varkappa \downarrow$ and if there exists a number θ, $0 \leqq \theta < 1$, such that $\varkappa^\theta \lambda_\varkappa \uparrow$, then the function*

$$f \sim \sum_{\varkappa=1}^{\infty} \frac{1}{\varkappa} (\varkappa\lambda_\varkappa)^{-1/p} \sin \varkappa x \tag{9}$$

belongs to the class $S_p(\lambda_\varkappa\}$, $0<p<\infty$.

Proof. To prove that $f \in S_p\{\lambda_\varkappa\}$ we fix $0<x<\pi$ and choose N such that

$$\frac{1}{N+1} < x \leqq \frac{1}{N}.$$

We consider the series

$$\sum_{\varkappa=1}^{\infty} \lambda_\varkappa |f(x)-S_\varkappa(x)|^p \leqq C_p \sum_{\varkappa=1}^{N} \lambda_\varkappa \left| \sum_{n=\varkappa+1}^{N+1} \frac{1}{n} (n\lambda_n)^{-1/p} \sin nx \right|^p +$$

$$+\sum_{\varkappa=1}^{N} \lambda_\varkappa \left| \sum_{n=N+2}^{\infty} \frac{1}{n} (n\lambda_n)^{-1/p} \sin nx \right|^p + \sum_{\varkappa=N+1}^{\infty} \lambda_\varkappa \left| \sum_{n=\varkappa+1}^{\infty} \frac{1}{n} (n\lambda_n)^{-1/p} \sin nx \right|^p \equiv$$

$$\equiv c_p(\textstyle\sum_1+\sum_2+\sum_3).$$

First we assume that $\lambda_\varkappa\downarrow$. Then $\varkappa^\theta\lambda_\varkappa\uparrow$ with some $\theta>1-p$. Hence, $\dfrac{\theta-1}{p}>-1$, and we have

$$\textstyle\sum_1 \leqq x^p \displaystyle\sum_{\varkappa=1}^{N} \lambda_\varkappa \left[\sum_{n=\varkappa+1}^{N+1} (n\lambda_n)^{-1/p}\right]^p \leqq x^p \sum_{\varkappa=1}^{N} \varkappa^{-\theta} \left[\sum_{n=1}^{N} n^{(\theta-1)/p}\right]^p =$$

$$= O(x^p N^{1-\theta} N^{\theta-1+p}) = O(1).$$

Furthermore,

$$\textstyle\sum_2 \leqq \displaystyle\sum_{\varkappa=1}^{N} \lambda_\varkappa \left[\sum_{n=N+2}^{\infty} \frac{1}{n} (n\lambda_n)^{-1/p}\right]^p \leqq$$

$$\leqq N^{-\theta} \lambda_N^{-1} \left(\sum_{n=N+2}^{\infty} n^{-1-(1-\theta)/p}\right)^p \sum_{\varkappa=1}^{N} \lambda_\varkappa = O(N^{-\theta}\cdot N \cdot N^{-(1-\theta)}) = O(1).$$

In order to estimate $\sum_3$ we make use of the inequality

$$\left| \sum_{n=\varkappa+1}^{\infty} \frac{1}{n} (n\lambda_n)^{-1/p} \sin nx \right| \leqq \frac{c}{\varkappa x} (\varkappa\lambda_\varkappa)^{-1/p}$$

for $0<x<\pi$. Hence,

$$\textstyle\sum_3 \leqq Cx^{-p} \displaystyle\sum_{\varkappa=N+1}^{\infty} \varkappa^{-1-p} = O(x^{-p}N^{-p}) = O(1).$$

The proof in the case $\lambda_\varkappa\uparrow$ is almost the same as for $\lambda_\varkappa\downarrow$, we only have to replace condition (4) by $\lambda_\varkappa\uparrow$. Therefore, we can omit the details.

The proof is completed.

3. Proof of the Theorem. i) If $f\in S_p\{\lambda_\varkappa\}$ then using (2), (8) and the following inequality of STEČKIN [8]:

$$\omega(2^{-n}, f) \leqq C2^{-n} \sum_{m=0}^{n-1} 2^m E_{2^m}(f) \quad (n = 1, 2, \dots)$$

we obtain $\omega(2^{-n},f)=O(\omega(2^{-n}))$ and $f\in H^\omega$.

ii) If condition (2) is not fulfilled, then, by Lemma 3, the function given in (9) does not belong to H^ω, but, by Lemma 4, it belongs to the class $S_p\{\lambda_\varkappa\}$.

Thus the Theorem is proved.

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