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Entropy of states of a gage space

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Let (H, A, m) be a regular gage space. Let ϱ, σ, and $\psi=\lambda\varrho+(1-\lambda)\sigma$, $0<\lambda<1$, be regular states. The density operator D_ϱ of a regular state is a non-negative (possibly unbounded) self-adjoint measurable operator. Let F be a continuous convex function on $[0, \infty)$ and define the entropy of ϱ by $e(\varrho)=m(F(D_\varrho))$. Conditions are obtained, in terms of $e(\varrho)$ and $e(\sigma)$, for $e(\psi)$ to be $-\infty$, finite, ∞, or undefined. If both ϱ and σ have finite entropy, then ψ has finite entropy and $e(\psi)\geqq\lambda e(\varrho)+(1-\lambda)e(\sigma)$; if $A=B(H)$, F is strictly convex, and $\varrho\neq\sigma$, then strict inequality is obtained. These results are restated as inequalities concerning the trace of a convex function of an operator.

1. Introduction

We work in the context of a regular gage space (H, A, m); H is a Hilbert space, A is a von Neumann algebra on H, and m is a faithful semi-finite normal trace on A. (See [4] for definitions and notation.) A regular state of A is a positive linear functional ϱ on A with $\varrho(I)=1$, where I is the identity operator on H, which is strongly continuous on the unit ball of A. If ϱ is a regular state of A, then by [4] Theorem 14 there is a unique operator $D_\varrho\in L^1(H, A, m)$ with $D_\varrho\geqq 0$, $m(D_\varrho)=1$, and $\varrho(T)=m(D_\varrho T)$ for all $T\in A$; D_ϱ is called the density operator of ϱ.

The entropy of a regular state ϱ is usually defined by $e(\varrho)=m(-D_\varrho \ln D_\varrho)$, cf. [3] Chapter V and [5]. Both von Neumann and Segal suggested defining the entropy by $e(\varrho)=m(F(D_\varrho))$, where F is an arbitrary continuous convex function on $[0, \infty)$; we use this definition for the remainder of this paper. The results basically say that the mixing of states cannot reduce entropy.

Bendat and Sherman [1] determined when a continuous convex function defined on an interval is operator convex; i.e., when $F(\lambda K+(1-\lambda)L)\geqq\lambda F(K)+(1-\lambda)F(L)$ holds for bounded self-adjoint operators K and L whose spectra

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are contained in the domain of F. Below we show that $m(F(\lambda K+(1-\lambda)L))\geqq \geqq\lambda m(F(K))+(1-\lambda)m(F(L))$ holds under suitable hypotheses for self-adjoint measurable operators K and L; this is merely a restatement of the fact that mixing of states cannot reduce entropy.

2. Statement of the results

Theorem 1. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$, and $\psi=\lambda\varrho+(1-\lambda)\sigma$. Assume* $\liminf_{x\to\infty} F(x)/F(kx)>0$ *for each $k>1$. Then:*

A. *$e(\psi)$ is defined iff both $e(\varrho)$ and $e(\sigma)$ are defined and $\{e(\varrho), e(\sigma)\}\neq\{-\infty, \infty\}$.*

B. *$e(\psi)$ is finite iff both $e(\varrho)$ and $e(\sigma)$ are finite.*

C. *$e(\psi)=\infty$ iff $\{\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq R\cup\{\infty\}$, where R is the set of real numbers.*

D. *$e(\psi)=-\infty$ iff $\{-\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq\{-\infty\}\cup R$.*

Corollary 1. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$, and $\psi=\lambda\varrho+(1-\lambda)\sigma$. Then*

A. *$e(\psi)$ is defined if both $e(\varrho)$ and $e(\sigma)$ are defined and $\{-\infty, \infty\}\neq\{e(\varrho), e(\sigma)\}$.*

B. *$e(\psi)$ is finite if both $e(\varrho)$ and $e(\sigma)$ are finite.*

C. *$e(\psi)=\infty$ if $\{\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq R\cup\{\infty\}$, and $\lim_{x\to\infty} F(x)=-\infty$.*

Theorem 2. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$ and $\psi=\lambda\varrho+(1-\lambda)\sigma$. If $e(\varrho)$ and $e(\sigma)$ are finite, then $e(\psi)$ is finite and $e(\psi)\geqq\lambda e(\varrho)+(1-\lambda)e(\sigma)$. If $A=B(H)=$all bounded operators on H, $\varrho\neq\sigma$, and the function F is strictly convex, then $e(\psi)>\lambda e(\varrho)+(1-\lambda)e(\sigma)$.*

Corollary 2. *Let (H, A, m) be a gage space. Let $K, L\in L^1(H, A, m)$. Assume that either $K\geqq 0$ and $L\geqq 0$ or $m(I)<\infty$ and K and L are both bounded from below (or from above). Let F be a continuous convex function defined on an interval which includes the spectra of K and L and let $0<\lambda<1$. If $F(K), F(L)\in L^1(H, A, m)$, then $F(\lambda K+(1-\lambda)L)\in L^1(H, A, m)$, and $m(F(\lambda K+(1-\lambda)L))\geqq\lambda m(F(K))+ +(1-\lambda)m(F(L))$. If $A=B(H)$, $K\neq L$, and F is strictly convex, then $m(F(\lambda K+(1-\lambda)L))>\lambda m(F(K))+(1-\lambda)m(F(L))$.*

Remark. In Theorem 2 and Corollary 2, the restriction that $A=B(H)$ in order to have strict inequality seems unnecessary; this was first suggested by SEGAL [5]. We know of no example which requires this extra hypothesis, but are unable to prove strict inequality without it.

3. Proof of the results

Corollaries 1 and 2 are restatements of Theorems 1 and 2 and require no proof.

We now introduce some notation. The self-adjoint operator T has spectral decomposition $T=\int_{-\infty}^{\infty}\alpha dP_T(\alpha)$; the function P_T is continuous from the left. If S is a Borel measurable set of real numbers, then $P_T(S)$ is the spectral projection of T for the set S. The spectral distribution function Λ_T is defined by $\Lambda_T(x)=\sup\{\lambda : m(P_T[\lambda,\infty))\geqq x\}$; the domain of Λ_T is $(0, m(I)]$ if $m(I)<\infty$ and $(0,\infty)$ if $m(I)=\infty$. $\Lambda_T(x)$ is a nonincreasing function of x and is continuous from the left. $m(P_T(\Lambda_T(x),\infty))=x$ if P has no point mass at $\Lambda_T(x)$ and $T\in L^1(H,A,m)$. The properties of the spectral distribution function are developed in [2]. To simplify the notation, we will frequently write P_ϱ for P_{D_ϱ} and Λ_ϱ for Λ_{D_ϱ}.

Lemma 1. *Let (H,A,m) be a gage space, let $K\in L^1(H,A,m)$ with $K\geqq 0$, and let F be a continuous function on (r,∞), where $P_K\{r\}=0$. Then $\int_r^{\infty} F(\lambda)\,dm(P_K(\lambda))=$ $=\int_0^{m(P_K[r,\infty))} F(\Lambda_K(x))dx$ in the sense that if either integral is defined, then both integrals are defined and are equal. In addition, if F is continuous on $[0,\infty)$, then*

$$\int_{[0,\infty)} F(\lambda)\,dm(P_K(\lambda))=\int_0^{m(I)} F(\Lambda_K(x))\,dx.$$

Proof. Let $s>r$ with $P_K\{s\}=0$. We will show below that $\int_r^s F(\lambda)dm(P_K(\lambda))=$ $=\int_{m(P_K[s,\infty))}^{m(P_K[r,\infty))} F(\Lambda_K(x))dx$. The first conclusion of the theorem will follow by taking the limit as $s\to\infty$. The second conclusion then follows by taking the limit as $r\to 0$.

Let $P=\{x_1,x_2,\dots,x_{n+1}\}$ be a partition of $[r,s]$ with $m(P_K\{x_i\})=0$ for $1\leqq i\leqq n+1$. Then $\int_r^s F(\lambda)dm(P_K(\lambda))\sim\sum_{i=1}^{n} F(\Lambda_K(m(P_K[x_i,\infty))))\,m(P_K[x_i,x_{i+1}])=$ $=\sum_{i=1}^{n} F(\Lambda_K(m(P_K[x_i,\infty))))(m(P_K[x_i,\infty))-m(P_K[x_{i+1},\infty)))\sim\int_{m(P_K[s,\infty))}^{m(P_K[r,\infty))} F(\Lambda_K(x))\,dx.$

Note that, although $m(P_K[x_i,\infty))-m(P_K[x_{i+1},\infty))$ may be large due to the spectrum of K having point masses in the interval (x_i,x_{i+1}), $F(\Lambda_K(\alpha))$ is nearly constant on the interval $m(P_K[x_{i+1},\infty))<\alpha\leqq m(P_K[x_i,\infty))$ since for α in this interval, $x_i\leqq\Lambda_K(\alpha)<x_{i+1}$.

Proof of Theorem 1. There are essentially four different non-trivial possibilities for F:

A. $F'(0)>0,\ \lim\limits_{x\to\infty} F(x)=-\infty.$

B. $F'(0)>0,\ \lim\limits_{x\to\infty} F(x)=\infty.$

C. $F'(0)>0,\ \lim\limits_{x\to\infty} F(x)=k,\quad \text{where } 0<k<\infty.$

D. $F'(0)\leqq 0,\ \lim\limits_{x\to\infty} F(x)=-\infty.$

Theorem 1 will be proved for case A since this is the most difficult case; the proofs for the other cases are trivial modifications and parts of the results are vacuous in the other cases. For the sake of simplicity, we assume $F(0)=0$; if $F(0)\neq 0$, little change is needed if $m(I)$ is finite and the results become essentially vacuous if $m(I)=\infty$. We further assume that F has a relative maximum at $x=1$, $F(1)=1$, and that $F(2)=0$. We will prove the "if" parts of B, C and D. The remainder of the proof is essentially redundant.

Assume now that $e(\varrho)$ and $e(\sigma)$ are both finite. $e(\psi)$ can be infinite in two ways: ψ can be highly concentrated so that D_ψ is unbounded and $e(\psi)=-\infty$, or ψ can be so spread out that D_ψ has very large support and $e(\psi)=\infty$.

Let $\alpha>0$ and $x\in H, x\neq 0$. If $P_\psi[\alpha,\infty)x=x$, then $\lambda(D_\varrho x,x)+(1-\lambda)(D_\sigma x,x)\geqq \geqq\alpha\|x\|^2$, so that either $P_\varrho[\alpha,\infty)x\neq 0$ or $P_\sigma[\alpha,\infty)x\neq 0$. By [2, lemma 2], $m(P_\psi[\alpha,\infty))\leqq m(P_\varrho[\alpha,\infty))+m(P_\sigma[\alpha,\infty))$. Then

$$\int_2^\infty F(\alpha)\,dm(P_\psi(\alpha))\geqq\int_2^\infty F(\alpha)\,dm(P_\varrho(\alpha))+\int_2^\infty F(\alpha)\,dm(P_\sigma(\alpha))>-\infty,$$

have finite entropy.

Now let $0<\alpha<1$. $m(P_\psi(\alpha,1])=m(P_\psi(\alpha,\infty))-m(P_\psi(1,\infty))\leqq m(P_\varrho(\alpha,\infty))+ +m(P_\sigma(\alpha,\infty))-m(P_\psi(1,\infty))=m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+m(P_\varrho(1,\infty))+m(P_\sigma(1,\infty))- -m(P_\psi(1,\infty))$. Let $c=m(P_\varrho(1,\infty))+m(P_\sigma(1,\infty))-m(P_\psi(1,\infty))$. Then $0\leqq c<\infty$, and $m(P_\psi(\alpha,1])\leqq m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+c$. Let M be the unique Borel measure on $(0,1]$ such that $M(\alpha,1]=m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+c$. Then $\int_{(0,1]} F(\alpha)\,dm(P_\psi(\alpha))\leqq \leqq\int_{(0,1]} F(\alpha)\,dM(\alpha)$, since F is non-negative and non-decreasing on $(0,1]$, and $\int_{(0,1]} F(\alpha)\,dM(\alpha)<\infty$ since ϱ and σ have finite entropy.

We now prove part C. Assume $e(\varrho)=\infty$. Then $\int_0^1 F(\alpha)\,dm(P_\varrho(\alpha))=\infty$, and by lemma 1, $\int_c^\infty F(\Lambda_\varrho(\alpha))\,d\alpha=\infty$ for some c such that $\Lambda_\varrho(c)\leqq 1$ and $\Lambda_\psi(c)\leqq 1$. Note that F is non-negative and non-decreasing on $[0,1]$. Since $\psi=\lambda\varrho+(1-\lambda)\sigma$,

$D_\psi \geqq \lambda D_\varrho$, so by [2] Corollary 1, $\Lambda_\psi(\alpha) \geqq \Lambda_{\lambda D_\varrho}(\alpha) = \lambda \Lambda_\varrho(\alpha)$. Then for $\alpha \geqq c$, $F(\Lambda_\psi(\alpha)) \geqq F(\lambda \Lambda_\varrho(\alpha)) \geqq \lambda F(\Lambda_\varrho(\alpha))$ by convexity. Then

$$\int_c^\infty F(\Lambda_\psi(\alpha))\, d\alpha \geqq \lambda \int_c^\infty F(\Lambda_\varrho(\alpha))\, d\alpha = \infty.$$

We now prove part D. Assume $e(\varrho) = -\infty$, so that $\int_0^\infty F(\alpha)\, dm(P_\varrho(\alpha)) = -\infty$. Choose $\varepsilon > 0$ and $q > 0$ so that $F(x)/F(x/\lambda) \geqq \varepsilon$ for $x \geqq \lambda q$. Since $D_\psi \geqq \lambda D_\varrho$, $m(P_\psi[\alpha, \infty)) \geqq m(P_{\lambda D_\varrho}[\alpha, \infty)) = m(P_\varrho[\alpha/\lambda, \infty))$. Then $\int_0^\infty F(\alpha)\, dm(P_\varrho(\alpha)) = -\infty$ implies $\int_q^\infty F(\alpha)\, dm(P_\varrho(\alpha)) = -\infty$, so that $\int_{\lambda q}^\infty F(\alpha/\lambda)\, dm(P_\varrho(\alpha/\lambda)) = -\infty$. Then $\int_{\lambda q}^\infty F(\alpha/\lambda)\, dm(P_\psi(\alpha)) = -\infty$ so $\int_{\lambda q}^\infty F(\alpha)\, dm(P_\psi(\alpha)) = -\infty$ and $e(\psi) = -\infty$.

Lemma 2. *Let R and S be either finite sequences with the same number of members or countable sequences. Assume $r_k \geqq r_{k+1} \geqq 0$, $s_k \geqq s_{k+1} \geqq 0$, $\sum_k r_k = \sum_k s_k$, and $\sum_{k=1}^j r_k \geqq \sum_{k=1}^j s_k$ for $j \geqq 1$. Then there is a doubly stochastic matrix M with $s_j = \sum_k m_{jk} r_k$ for $j \geqq 1$.*

Proof. If R and S are finite sequences the result is well known; our proof will contain this case if R and S are extended to countable sequences by adding a string of zeroes at the end. Let R and S be countable sequences and assume $r_k \neq 0$ for all k. M will be constructed one row at a time; each row of M will have finitely many non-zero entries. Let $w(1)$ be the smallest integer such that $s_1 \geqq r_{w(1)}$. Express s_1 as a convex combination of $\{r_i : 1 \leqq i \leqq w(1)\}$ to obtain the first row of M.

Assume $k-1$ rows of M have been obtained. If $s_k \geqq r_{w(k-1)}$, let $w(k) = 1 + w(k-1)$; otherwise, let $w(k)$ be the smallest integer such that $s_k \geqq r_{w(k)}$. We will show that s_k can be expressed as a convex combination of $\{r_i : 1 \leqq i \leqq w(k)\}$ such that $\sum_{i=1}^k m_{ij} \leqq 1$ for $1 \leqq j \leqq w(k)$ by showing that there is such a convex combination which is $\geqq s_k$ and that there is such a convex combination (namely, $\sum_{i=1}^{w(k)-1} 0 r_i + 1 r_{w(k)}$) which is $\leqq s_k$.

When $\sum_i m_{ij} = 1$, we will say r_j is "used up". Let the number $c = \sum_{i=1}^k c_i r_i$ be formed as follows: c_1 is chosen so that r_1 is used up; i.e., $c_1 = 1 - \sum_{i=1}^{k-1} m_{i1}$. Choose c_2 so that $c_1 + c_2 \leqq 1$ and r_2 is used up if possible; $c_2 = \min\left(1 - c_1, 1 - \sum_{i=1}^{k-1} m_{i2}\right)$. Continue this process until c_k is chosen. Then $c \geqq s_k$ follows from the hypothesis that $\sum_{i=1}^k r_i \geqq \sum_{i=1}^k s_i$.

This completes the construction of the matrix M. Clearly $s_j=\sum_k m_{jk}r_k$ for all j, $m_{jk}\geqq 0$ for all j, k, and the sum of the elements of any row of M is 1. It remains to show that the sum of the elements of any column of M is 1. $1=\sum_i s_i=\sum_i\sum_j m_{ij}r_j=$ $=\sum_j\sum_i m_{ij}r_j$; since all terms are non-negative the interchange of order of summation is valid. Since $1=\sum_j r_j$, $0=\sum_j(1-\sum_i m_{ij})r_j$. By the construction of M, $(1-\sum_i m_{ij})\geqq 0$ for each j. Since $r_j\neq 0$ for all j, $1=\sum_i m_{ij}$.

If $r_j=0$ for some j, then $r_k=0$ for all $k\geqq j$. The construction of M must then be modified so that, for $k\geqq j$, r_k is used up before one begins to use r_{k+1}.

Lemma 3. *Let (H, A, m) be a gage space, let $T\in L^1(H, A, m)$ with $T\geqq 0$, let $\gamma>0$, and let $q=m(P_T(\gamma,\infty))$. Let P be any projection in A with $m(P)=q$. Then $m(PT)\leqq m(P_T(\gamma,\infty)T)$.*

Proof. By lemma 1, $m(PT)=m(PTP)=\int_0^{m(I)}\Lambda_{PTP}(x)\,dx=\int_0^{q}\Lambda_{PTP}(x)\,dx$. By [2] Theorem 4, $\Lambda_{PTP}(x)\leqq\Lambda_T(x)$ for $0<x\leqq m(I)$. Note that $\Lambda_{P_T(\gamma,\infty)T}(x)=$ $=\Lambda_T(x)$ for $0<x\leqq q$ so that $\Lambda_{PTP}(x)\leqq\Lambda_{P_T(\gamma,\infty)T}(x)$ for $0<x\leqq q$. Then

$$\int_0^q\Lambda_{PTP}(x)\,dx\leqq\int_0^q\Lambda_{P_T(\gamma,\infty)T}(x)\,dx=\int_0^{m(I)}\Lambda_{P_T(\gamma,\infty)T}(x)\,dx=m(P_T(\gamma,\infty)T).$$

Proof of Theorem 2. Assume first that $A=B(H)$. Let ϱ_i be the i^{th} eigenvalue of D_ϱ, where the eigenvalues of D_ϱ are arranged in decreasing order and are counted according to multiplicity.

Define a sequence A by $a_i=\lambda\varrho_i+(1-\lambda)\sigma_i$ and a sequence B by $b_i=\psi_i$. The first three hypotheses of lemma 2 are clearly satisfied. The last hypothesis of lemma 2 follows from lemma 3; a trivial modification of lemma 3 is needed if D_ϱ or D_σ has a repeated eigenvalue. By lemma 2, there is a doubly stochastic matrix M with $\psi_i=\sum_j m_{ij}(\lambda\varrho_j+(1-\lambda)\sigma_j)$. Then $F(\psi_i)\geqq\sum_j m_{ij}(\lambda F(\varrho_j)+(1-\lambda)\,F(\sigma_j))$. Summing this relation yields $m(F(D_\psi))=\sum_i F(\psi_i)\geqq\sum_i\sum_j m_{ij}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=$ $=\sum_j\sum_i m_{ij}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=\sum_j(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=\lambda m(F(D_\varrho))+(1-\lambda)\cdot$ $\cdot m(F(D_\sigma))$; the interchange of the order of summation is valid since ϱ and σ each have finite entropy by hypothesis. If $\varrho\neq\sigma$, then $\psi_{i_0}\neq\lambda\varrho_{i_0}+(1-\lambda)\sigma_{i_0}$ for some i_0, so that M is not the identity matrix. If F is then strictly convex, then $F(\psi_{i_0})>$ $>\sum_j m_{i_0 j}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))$.

We now prove the general case when $m(I)=\infty$; the proof when $m(I)<\infty$ is virtually identical. Let ε be an arbitrary positive number. For n a natural number, let

$$\psi_n=\frac{1}{\varepsilon}\int_{(n-1)\varepsilon}^{n\varepsilon}\Lambda_\psi(x)\,dx;$$

define sequences ϱ_n and σ_n similarly. Assume that D_ψ, D_ϱ, D_σ have no point masses at $\Lambda_\psi(k\varepsilon)$, $\Lambda_\varrho(k\varepsilon)$, $\Lambda_\sigma(k\varepsilon)$ respectively, for all natural numbers k; arbitrarily small ε can always be found so that this holds. By lemma 1 and lemma 3,

$$\varepsilon \sum_{n=1}^{k} \psi_n = \int_0^{k\varepsilon} \Lambda_\psi(x)\,dx = \int_{\Lambda_\psi(k\varepsilon)}^{\infty} \alpha dm(P_\psi(\alpha)) = m(D_\psi P_\psi(\Lambda_\psi(k\varepsilon,\infty))) =$$

$$= \lambda m(D_\varrho P_\psi(\Lambda_\psi(k\varepsilon,\infty))) + (1-\lambda) m(D_\sigma P_\psi(\Lambda_\psi(k\varepsilon,\infty))) \leqq$$

$$\leqq \lambda m(D_\varrho P_\varrho(\Lambda_\varrho(k\varepsilon,\infty))) + (1-\lambda) m(D_\sigma P_\sigma(\Lambda_\sigma(k\varepsilon,\infty))) = \varepsilon\lambda \sum_{n=1}^{k} \varrho_n + \varepsilon(1-\lambda) \sum_{n=1}^{k} \sigma_n .$$

By the first part of the proof of this theorem,

$$\sum_n F(\psi_n) \geqq \lambda \sum_{n=1}^{\infty} F(\varrho_n) + (1-\lambda) \sum_{n=1}^{\infty} F(\sigma_n).$$

To complete the proof, it suffices to show that $\varepsilon \sum_{i=1}^{n} F(\psi_n)$ approximates $\int_0^\infty F(\Lambda_\psi(x))\,dx$ for ε small. This is immediate since Λ_ψ is a non-increasing function implies $\Lambda_\psi((n-1)\varepsilon) \geqq \psi_n \geqq \Lambda_\psi(n\varepsilon)$ and $e(\psi)$ is finite by the hypotheses and Corollary 1.

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