Acta Sci. Math., 40 (1978), 107—110

# Uniformly distributed sequences on compact, separable, non metrizable groups

V. LOSERT

Let $G$ be a compact topological group with normalized Haar measure $\lambda$. $G^\infty$ shall denote the product of denumerably many copies of $G$, equipped with the product measure $\lambda_\infty$. If $G$ is metrizable, a well-known theorem ([5] Th. 2.2) asserts that the set of uniformly distributed (u.d.) sequences has measure one in $G$. If $G$ is separable (i.e. it contains a countable dense subset) but non-metrizable, it follows from a result of W. A. VEECH [9] that u.d. sequences still exist (a shorter proof of this result has been given by H. RINDLER [7], the abelian case has been treated earlier in [1]). Let $S$ be the set of all u.d. sequences in $G$. In this paper we show that for a compact, separable, non-metrizable group $G$, $S$ is not measurable in $G$, its outer measure is one, the inner measure is zero. By the way of proving this result, we extend a result of [6]: if $G$ is a compact, separable group, $G/H$ a metrizable quotient, then any u.d. sequence in $G/H$ can be lifted to $G$.

For arbitrary separable compact spaces the situation is different. We give an example of a class of spaces for which u.d. sequences exist in trivial cases only, namely if the measure is concentrated on a countable set.

I want to thank W. Maxones and H. Rindler who led my interest to these problems.

For basic notions concerning uniformly distributed sequences see [5].

Lemma 1. *Let $G$ be a compact topological group with normalized Haar-measure $\lambda$, $A$ a measurable subset of $G$. Then there exists a closed normal subgroup $H$ of $G$ and $H$-periodic measurable subsets $B, C$ of $G$* (i.e. $B=BH$, $C=CH$) *such that $B\subseteq A\subseteq C$, $\lambda(C\setminus B)=0$ and $G/H$ is metrizable.*

Proof. It suffices to prove this for open subsets $A$. By the regularity of $\lambda$, there exists an open Baire-set $B\subseteq A$ for which $\lambda(A\setminus B)=0$. $B$ has the form $\{x\in G: f(x)>0$ for some continuous, real valued function $f$ on $G$. Now let $H$ be a closed normal

Received May 10, 1977.

subgroup of $G$ such that $G/H$ is metrizable and $f$ is $H$-periodic, i.e. $f(yx)=f(y)$ for $x \in H$. For $\dot{x} \in G/H$ put

$$T_H(c_A)(\dot{x}) = \int_H c_A(xy)\,dy \quad \text{and} \quad T_H(c_B)(\dot{x}) = \int_H c_B(xy)\,dx.$$

$$T_H(c_B) = c_{\dot{B}}$$

since $B$ is $H$-periodic. Define $\dot{C}=\{\dot{x}\in G/H: T_H(c_A)(\dot{x})>0\}$ and $C=\{x\in G: \dot{x}\in\dot{C}\}$. Then $C$ is open and $H$-periodic, $C\supseteq A$. Since $\dot{C}=\dot{B}$ a.e. on $G/H$ we have $\lambda(C\setminus B)=0$.

Remark. It is essential that the inclusion $B\subseteq A\subseteq C$ and the equations $BH=B$, $CH=C$ hold in the set-theoretical sense and not only $\lambda$-a.e..

One easily concludes that an arbitrary subset $A$ of $G$ has outer measure one if and only if the canonical image of $A$ has outer measure one in any metrizable quotient group $G/H$.

Theorem 1. *Let $G$ be a compact topological group which is separable but non-metrizable, $\lambda$ the normalized Haar measure and $S$ the set of all u.d. sequences in $G$. Then $S$ has interior measure zero in $G^\infty$.*

Proof. We want to show that $M=G^\infty\setminus S$ has outer measure one. Let $p_n: G^\infty\to G$ be the $n$-th coordinate function. If $H_1$ is a closed normal subgroup of $G_1=G^\infty$ for which $G_1/H_1$ is metrizable, there exists a countable set $\{f_n\}$ of continuous functions on $G_1/H_1$ which is dense in $C(G_1/H_1)\subseteq C(G_1)$ (the space of all continuous functions on $G_1/H_1$ resp. $G_1$ with the topology of uniform convergence). By the Stone Weierstrass theorem there exists a countable set $\{g_n\}$ of continuous functions on $G$ such that the closed subalgebra spanned by $\{g_n\circ p_m\}_{m,n=1}^\infty$ contains the $f_n$. Now take a closed normal subgroup $H$ of $G$ for which all $g_n$ are $H$-periodic and such that $G/H$ is metrizable. Then $H^\infty\subseteq H_1$ and $(G/H)^\infty=G^\infty/H^\infty$ is metrizable too. It suffices therefore to prove that the image of $M$ has outer measure zero in $G^\infty/H^\infty$ for all closed normal subgroups $H$ of $G$, for which $G/H$ is metrizable. Since $G$ is not metrizable, $H$ must be a non-trivial subgroup. Take a symmetric neighbourhood $U$ of the unit element in $G$ such that $H$ is not contained in $U^2$. If $(y_n)$ is an arbitrary sequence in $G/H$ there exists a sequence $(x_n)$ in $G\setminus U$ such that $\pi(x_n)=y_n$ ($\pi: G\to G/H$ denotes the canonical projection). $(x_n)$ belongs to $M$ since it is not dense in $G$. Therefore the image of $M$ comprises all of $(G/H)^\infty$.

Lemma 2. *Let $G$ be a compact topological group, $H$ a closed normal subgroup, $\pi: G\to G/H$ the canonical projection. If $(y_n)$ is a sequence in $G/H$ which converges to identity, there exists a sequence $(x_n)$ in $G$, which converges to identity and satisfies $\pi(x_n)=y_n$.*

Proof. Let $(U_\alpha)_{\alpha<\gamma}$ be a well-ordering for the set of all non-equivalent irreducible unitary representations of $G$. For $\alpha<\gamma$ put

$$H_\alpha = H\cap \bigcap_{\beta<\alpha} \ker U_\beta, \quad H_0 = H, \quad G_\alpha = G/H_\alpha = (G/H_{\alpha+1})/(H_\alpha/H_{\alpha+1}).$$

If $\alpha$ is a limit-ordinal, then $G_\alpha$ is the projective limit of the groups $\{G_\beta:\beta<\alpha\}$. It suffices therefore to show that we can lift the sequence from $G_\alpha$ to $G_{\alpha+1}$. Since $U_\alpha$ seperates the points of $H_\alpha/H_{\alpha+1}$, this group is metrizable (in fact a Lie group). This means that we have reduced the proof of the lemma to the case that $H$ is metrizable.

By induction we can define a sequence $\{U_m\}$ of open neighborhoods of the unit element in $G$ with the following properties: $U_{m+1}\subseteqq U_m$, $\{U_m\cap H\}$ is a neighborhood base of the unit element in $H$, if $F_m=H\setminus U_m$, then $U_{m+1}\cap U_{m+1}F_m=\emptyset$. Now we choose elements $x_n\in G$ such that $\pi(x_n)=y_n$ and almost all $x_n$ belong to $U_m$ $(m=1, 2, \ldots)$. We want to show that $\{x_n\}$ tends to zero. Let $V$ be an arbitrary neighborhood of $e$ in $G$ and $W$ an open neighborhood for which $W^2\subseteqq V$. Put $F=H\setminus W$ then there exists an index $m$ such that $U_m\cap U_mF=\emptyset$. If $x\in U_m$ and $\pi(x)\in\pi(W\cap U_m)$ there exists $y\in W\cap U_m$ such that $\pi(y)=\pi(x)$, i.e. $y^{-1}x\in H$. If $y^{-1}x$ would belong to $F$ then $x\in y\subseteqq U_mF$, a contradiction. Therefore $y^{-1}x\in W$ and it follows that $x\in yW\subseteqq V$.

Theorem 2. *Let $G$ be a compact, separable group, $H$ a closed normal subgroup, $G/H$ metrizable, $(y_n)$ a u.d. sequence in $G/H$. Then there exists a u.d. sequence $(x_n)$ in $G$ such that $\pi(x_n)=y_n$ (where $\pi$ denotes the canonical quotient map).*

Proof. This result was already claimed without proof in [6], but we were informed by the author of that paper that his proof contains a gap. The methods of [6] enable us to show the following: If $G$ is a compact group, $H$ a separable subgroup, $G/H$ metrizable, $(y_n)$ u.d. in $G/H$, then there exists a u.d. sequence $(x_n)$ in $G$ such that $y_n^{-1}\pi(x_n)$ tends to identity. Now it follows from the previous lemma that there exists a sequence $(z_n)$ in $G$ such that $(z_n)$ converges to identity and $\pi(z_n^{-1})= =y_n^{-1}\pi(x_n)$. $(x_nz_n)$ is u.d. in $G$ and $\pi(x_nz_n)=y_n$. According to [4] a compact topological group is separable if and only if it has an open base for its topology of cardinality $\leqq c$ (the power of the continuum). It follows that a closed subgroup of a separable compact group is separable and the proof is finished.

Corollary. *Let $G$ be a separable compact group, $\lambda$ the normalized Haar measure and $S$ the set of all u.d. sequences in $G$. Then $S$ has outer measure one in $G$.*

Now let $X$ be an arbitrary compact Hausdorff space. We write $C(X)$ for the space of continuous scalar-valued functions on $X$ with supremum norm, $M(X)$ for the dual and $M(X)'$ for the bidual of $C(X)$. According to [8] we call $X$ a *G-space* if $\sigma(M(X), C(X))$-convergent sequences are $\sigma(M(X), M(X)')$ convergent.

Proposition. *If $X$ is a $G$-space and $\mu$ is a probability measure on $X$, which is not concentrated on a countable subset, then there exist no $\mu$-u.d. sequences in $X$.*

Proof. Assume that $(x_n)$ is u.d.. Put $\mu_N = N^{-1} \sum_{n \leq N} \varepsilon_{x_n}$, where $\varepsilon_x$ denotes the point measure of mass one concentrated in $x$. $\mu_N$ converges to $\mu$ in the topology $\sigma(M(X), C(X))$ and consequently also for $\sigma(M(X), M(X)')$. It follows that $\mu_N(A)$ converges to $\mu(A)$ for any Borel-subset $A$ of $X$, in particular that $\mu$ is concentrated on the set $\{x_n\}_{n=1}^{\infty}$.

It was first proved in [3] that any extremely disconnected space is a $G$-space ($X$ is called extremely disconnected, if the closure of any open subset is open). For example $\beta$N the Stone—Čech compactification of N is extremely disconnected and clearly separable. The last proposition shows that there exists no u.d. sequence for a measure $\mu$ on $\beta$N which is not concentrated on a countable set.

More generally it has been shown in [8] that any $F$-space is a $G$-space (a compact space $X$ is an $F$-space, if disjoint open $F$-sets have disjoint closures. See [2] for further properties and examples of $F$-spaces).

## References

[1] I. D. Berg, M. Rajagopalan, L. A. Rubel, Uniform distribution in locally compact Abelian groups, *Trans. AMS*, **133** (1968), 435—446.

[2] L. Gillman, M. Jerison, *Rings of Continuous Functions*, Van Nostrand (Princeton, 1960).

[3] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, *Can. J. of Math.*, **5** (1953), 129—173.

[4] G. L. Itzkowitz, On the density character of compact topological groups, *Fund. Math.*, **75** (1972), 201—203.

[5] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Wiley (New York, 1974).

[6] H. Rindler, Uniformly distributed sequences in quotient groups, *Acta Sci. Math.*, **38** (1976), 153—156.

[7] H. Rindler, Uniform distribution on locally compact groups, *Proc. AMS*, **57** (1976), 130—132.

[8] G. L. Seever, Measures on F-spaces, *Trans. AMS*, **133** (1968), 267—280.

[9] W. A. Veech, Some questions of uniform distribution, *Ann. of Math.*, (2) **94** (1971), 125—138.

MATH. INST. DER UNIVERSITÄT WIEN
STRUDLHOFGASSE 4
1090 WIEN
AUSTRIA