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Multiparameter strong laws of large numbers. I (Second order moment restrictions)

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

§ 1. Notations and a preliminary result

Let Z^d be the set of d-tuples $\mathbf{k}=(k_1, k_2, \ldots, k_d)$ with non-negative integers for coordinates, where $d \geqq 1$ is a fixed integer. If the coordinates k_j are positive integers, we write $\mathbf{k} \in Z_+^d$. Two tuples $\mathbf{k}=(k_1, k_2, \ldots, k_d)$ and $\mathbf{m}=(m_1, m_2, \ldots, m_d)$ are said to be distinct if for at least one j we have $k_j \neq m_j$. Z^d is partially ordered by agreeing that $\mathbf{k} \leqq \mathbf{m}$ iff $k_j \leqq m_j$ for each j. If $\mathbf{k} \leqq \mathbf{m}$ and $\mathbf{k} \neq \mathbf{m}$, then write $\mathbf{k} < \mathbf{m}$.

Let $\mathbf{k}+\mathbf{m}$ and $\mathbf{km}$ denote the usual coordinatewise sums and products, respectively. Let $2^{\mathbf{k}}=(2^{k_1}, 2^{k_2}, \ldots, 2^{k_d})$ and let $|\mathbf{k}|$ stand for the product $k_1 k_2 \ldots k_d$. Further, let us write $\mathbf{0}$ and $\mathbf{1}$ for the points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^d, respectively.

Let $(X, \mathcal{A}, \mu)$ be a (not necessarily σ-finite) positive measure space. Let $\{\zeta_{\mathbf{k}}\}=\{\zeta_{\mathbf{k}}: \mathbf{k} \in Z_+^d\}$ be a set of measurable functions defined on $(X, \mathcal{A}, \mu)$ and having finite second moments:

$$\sigma_{\mathbf{k}}^2 = \int \zeta_{\mathbf{k}}^2 \, d\mu < \infty$$

for all $\mathbf{k} \in Z_+^d$, where for the sake of simplicity we write $\int \cdot \, d\mu$ instead of $\int_X \cdot \, d\mu$. Consider the d-multiple series

$$\sum_{\mathbf{k} \geqq \mathbf{1}} \zeta_{\mathbf{k}} = \sum_{j=1}^{d} \sum_{k_j=1}^{\infty} \zeta_{k_1, k_2, \ldots, k_d}.^{1)} \tag{1.1}$$

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¹) Here $\sum_{j=1}^{d} \sum_{k_j=1}^{\infty}$ means the d-fold summation $\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_d=1}^{\infty}$.

For $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$ set

$$S(\mathbf{b},\mathbf{m}) = \sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} \zeta_{\mathbf{k}} = \sum_{j=1}^{d}\sum_{k_j=b_j+1}^{b_j+m_j} \zeta_{k_1,k_2,\dots,k_d}.\ ^{2)}$$

In case $\mathbf{b}=\mathbf{0}$ the abbreviated notation $S(\mathbf{m})=S(\mathbf{0},\mathbf{m})$ is used.

Convergence properties of the following types will be discussed:

(i) $S(\mathbf{m})$ converges a. e. as $\mathbf{m}\to\infty$, which expresses the convergence of the d-multiple series (1.1);

(ii) $S(\mathbf{m})/|\mathbf{m}|$ converges to 0 a.e. as $\mathbf{m}\to\infty$, which expresses a strong law of large numbers (SLLN) for the d-multiple sequence $\{\zeta_{\mathbf{k}}\}$.

We want to emphasize that the term "$\mathbf{m}\to\infty$" in (i) and (ii) has different meanings. By the limit $\mathbf{m}\to\infty$ in statements of type (i) we mean $\min_{1\leqq j\leqq d} m_j\to\infty$, while in statements of type (ii) we mean $\max_{1\leqq j\leqq d} m_j\to\infty$. In other words, the neighbourhood of ∞ defined by a positive number K in the first case is $\bigcap_{j=1}^{d}\{\mathbf{k}\in Z_+^d\colon k_j>K\}$, whereas in the second case is $\bigcup_{j=1}^{d}\{\mathbf{k}\in Z_+^d\colon k_j>K\}$.

As is well-known from the theory of multiple Fourier series, the notion "partial sum" is used in several ways. If there are no restrictions on the ratios m_i/m_j, then $S(\mathbf{m})$ is called *unrestricted rectangular* partial sum, while if there are positive constants C_1 and C_2 such that for each i and j we have $C_1\leqq m_i/m_j\leqq C_2$, then $S(\mathbf{m})$ is called *restricted rectangular* partial sum. If here $C_1=C_2=1$, that is if $m_1=m_2=\dots=m_d=m$, then $S(m,m,\dots,m)$ is called *square* partial sum. In this paper $S(\mathbf{m})$ always means unrestricted rectangular partial sum. It is obvious that the requirement of a.e. convergence for the rectangular partial sums is stronger than for the square partial sums. The same observation is true concerning a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$. Finally, the *spherical* partial sum $\tilde{S}(r)$ is defined by

$$\tilde{S}(r) = \sum_{k_1^2+k_2^2+\dots+k_d^2\leqq r^2} \zeta_{k_1,k_2,\dots,k_d},$$

where r is a positive integer. Clearly, the notions of rectangular, square, and spherical partial sums coincide only for $d=1$.

The asymptotic behaviour of both square and spherical partial sums will be studied in the following more general setting. Let $Q_1\subset Q_2\subset\dots$ be an arbitrary sequence of finite regions in Z_+^d such that either $\bigcup_{r=1}^{\infty} Q_r=Z_+^d$ in statements of

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2) $\sum_{j=1}^{d}\sum_{k_j=b_j+1}^{b_j+m_j}$ also denotes a d-fold summation: $\sum_{k_1=b_1+1}^{b_1+m_1}\sum_{k_2=b_2+1}^{b_2+m_2}\dots\sum_{k_d=b_d+1}^{b_d+m_d}$.

type (i) or $\bigcup_{r=1}^{\infty} Q_r$ contains infinitely many points from Z_+^d in statements of type (ii). For $r=1,2,\dots$ set

$$T(r) = \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}}.$$

The next two particular choices of $\{Q_r\}$ provide both square and spherical partial sums.

Case 1. For each j, $1\leqq j\leqq d$, let $\{m_j(r)\}_{r=1}^{\infty}$ be a non-decreasing sequence of positive integers such that either $\min_{1\leqq j\leqq d} m_j(r)\to\infty$ in statements of type (i) or $\max_{1\leqq j\leqq d} m_j(r)\to\infty$ in statements of type (ii) as $r\to\infty$. Setting $Q_r=\{\mathbf{k}\in Z_+^d\colon k_j\leqq m_j(r)$ for each $j\}$ we have $T(r)=S(m_1(r), m_2(r), \dots, m_d(r))$. In particular, if $m_j(r)=r$ for each j and r, then we get back the square partial sums.

Case 2. The choice $Q_r=\{\mathbf{k}\in Z_+^d\colon k_1^2+k_2^2+\dots+k_d^2\leqq r^2\}$ provides the spherical partial sums: $T(r)=\tilde{S}(r)$.

Thus the sums $T(r)$ can be considered as generalized partial sums of the d-multiple series (1.1), although they form a set $\{T(r)\}_{r=1}^{\infty}$ depending only on one parameter.

Since Z_+^d is a partially ordered set, the main difficulties in studying convergence properties of $S(\mathbf{m})$ arise from the lack of linear ordering when $d\geqq 2$. On the other hand, Z_+^1 has a linear ordering and this explains the better convergence properties of $T(r)$.

Our results will be obtained by making use of a d-multiple maximal inequality of [2] which states bounds on the second moment of

$$M(\mathbf{b}, \mathbf{m}) = \max_{1\leqq \mathbf{k}\leqq \mathbf{m}} |S(\mathbf{b}, \mathbf{k})| = \max_{1\leqq j\leqq d}\ \max_{1\leqq k_j\leqq m_j} |S(\mathbf{b}, \mathbf{k})|\,^{3)}$$

in terms of bounds on the second moment of $S(\mathbf{b}, \mathbf{m})$, whilst $\mathbf{b}$ and $\mathbf{m}$ run over Z^d and Z_+^d, respectively.

We obviously have

$$\int S^2(\mathbf{b}, \mathbf{m})\, d\mu \leqq \sum_{\mathbf{b}+\mathbf{1}\leqq \mathbf{k}, \mathbf{l}\leqq \mathbf{b}+\mathbf{m}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\right| \equiv f(\mathbf{b}, \mathbf{m})\,.^{4)}$$

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3) Here $\max_{1\leqq j\leqq d}\max_{1\leqq k_j\leqq m_j}$ indicates that the maximum has to be taken for all possible integers $k_1, k_2, \dots, k_d$ such that $1\leqq k_1\leqq d_1$, $1\leqq k_2\leqq d_2, \dots$, and $1\leqq k_d\leqq m_d$.

4) $\sum_{\mathbf{b}+\mathbf{1}\leqq \mathbf{k}, \mathbf{l}\leqq \mathbf{b}+\mathbf{m}}$ abbreviates the following $2d$-fold summation:

$$\sum_{k_1=b_1+1}^{b_1+m_1}\ \sum_{l_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d}\ \sum_{l_d=b_d+1}^{b_d+m_d}.$$

The following lemma is the special case of [2, Theorem 8] when $\gamma=2$ and $\lambda_j(m_j)=1$, consequently $\Lambda_j(m_j)=\log 2m_j$ for each j. In this paper all logarithms are of base 2.

Lemma 1 (the Rademacher—Menšov inequality). *For all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z^d_+$ *we have*

$$\int M^2(\mathbf{b},\mathbf{m})\,d\mu \le f(\mathbf{b},\mathbf{m}) \prod_{j=1}^{d} (\log 2m_j)^2. \tag{1.2}$$

For the convenience of using "dyadic blocks" $S(2^{\mathbf{p}}, 2^{\mathbf{p}})$, $\mathbf{p}\in Z^d$, to represent the partial sums $S(\mathbf{m})$ during the proofs below, we may assume that $\zeta_{\mathbf{k}}\equiv 0$ if for at least one j we have $k_j=1$. It is clear that this assumption is of technical character and does not affect generality.

§ 2. A.e. convergence of the rectangular partial sums

On the basis of (1.2) we prove the following

Theorem 1 (the non-orthogonal Rademacher—Menšov theorem). *If*

$$\sum_{\mathbf{m}\ge \mathbf{0}} |\mathbf{m}+\mathbf{1}|^2 \sum_{2^{\mathbf{m}}+\mathbf{1}\le \mathbf{k},\,\mathbf{l}\le 2^{\mathbf{m}+\mathbf{1}}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\,d\mu\right| < \infty, \tag{2.1}$$

then (1.1) *converges a.e. in the sense that* $S(\mathbf{m})$ *converges a.e. as* $\mathbf{m}\to\infty$.

If the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal, i.e., if for all distinct pairs $\mathbf{k}$ and $\mathbf{l}$ we have

$$\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\,d\mu = 0,$$

then the general term of (2.1) may be simplified as follows

$$|\mathbf{m}+\mathbf{1}|^2 f(2^{\mathbf{m}}, 2^{\mathbf{m}}) \le \sum_{2^{\mathbf{m}}+\mathbf{1}\le \mathbf{k}\le 2^{\mathbf{m}+\mathbf{1}}} \sigma_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2.$$

Hence Theorem 1 yields

Corollary 1 (the Rademacher—Menšov theorem). *If the functions* $\zeta_{\mathbf{k}}$ *are mutually orthogonal and if*

$$\sum_{\mathbf{k}\ge \mathbf{1}} \sigma_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 < \infty, \tag{2.2}$$

then (1.1) *converges a.e.*

Condition (2.2) is satisfied if, for example,

$$\sigma_{\mathbf{k}}^2 = O\left\{\prod_{j=1}^{d} k_j^{-1}(\log 2k_j)^{-3}(\log\log 4k_j)^{-1-\varepsilon}\right\}$$

or

$$\sigma_{\mathbf{k}}^2 = O\{|\mathbf{k}|^{-1}(\log 2|\mathbf{k}|)^{-3d}(\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \tag{2.3}$$

with an $\varepsilon>0$. The fulfilment of (2.2) in the second case can be verified by repeated use of the estimation

$$\sum_{j=1}^{\infty} j^{-1}(\log 2aj)^{-i}(\log\log 4aj)^{-1-\varepsilon} = O\{(\log 2a)^{-i+1}(\log\log 4a)^{-1-\varepsilon}\},$$

where $a\geqq 1$ and $i\geqq 2$ are integers, and $\varepsilon>0$.

We remark that Theorem 1 for $d=1$ was essentially proved by SZÉP [6] (although it is stated there in a slightly weaker form), while Corollary 1 for $d=2$ was proved by AGNEW [1] (see also PANDZAKIDZE [3], where the proof of Step 2 is not complete).

Proof of Theorem 1. By the above remark it is enough to treat the case $d\geqq 2$.

Step 1. We begin with proving that $S(2^{\mathbf{p}})$ converges a.e. as $\mathbf{p}\to\infty$. By the Cauchy convergence criterion it is sufficient to show that

$$S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}}) \quad \text{tends to 0 a.e.} \quad \text{as} \quad \mathbf{p}\to\infty \quad \text{and} \quad \mathbf{q}>\mathbf{0}. \tag{2.4}$$

To this end let us represent the difference in (2.4) as follows

$$S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}}) = \Big\{\sum_{\mathbf{1}\leqq\mathbf{k}\leqq 2^{\mathbf{p}+\mathbf{q}}} - \sum_{\mathbf{1}\leqq\mathbf{k}\leqq 2^{\mathbf{p}}}\Big\}\zeta_{\mathbf{k}} = \Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}S(2^{\mathbf{m}}, 2^{\mathbf{m}}),$$

where $\mathbf{p}\geqq\mathbf{1}$ and $\mathbf{q}>\mathbf{0}$. Applying the Cauchy inequality hence we get that

$$\big(S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}})\big)^2 \leqq \Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}|\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\times$$

$$\times\Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}\frac{1}{|\mathbf{m}+\mathbf{1}|^2}.$$

Taking into account that the second factor on the right is uniformly bounded for all $\mathbf{p}\geqq\mathbf{1}$ and $\mathbf{q}>\mathbf{0}$,

$$\big(S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}})\big)^2 = O(1)\Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}|\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}}). \tag{2.5}$$

Since by (2.1)

$$\sum_{\mathbf{m}\geqq\mathbf{0}} |\mathbf{m}+\mathbf{1}|^2 \int S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\,d\mu \leqq \sum_{\mathbf{m}\geqq\mathbf{0}} |\mathbf{m}+\mathbf{1}|^2 f(2^{\mathbf{m}}, 2^{\mathbf{m}}) < \infty,$$

the B. Levi theorem implies the a.e. convergence of the d-multiple series

$$\sum_{\mathbf{m}\geqq\mathbf{0}} |\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}}).$$

Consequently, the right-hand side of (2.5) can be made as small as needed by choosing $\min_{1\leqq j\leqq d} p_j$ large enough. This proves (2.4).

Step 2. It has remained to prove that the maximal deviation

$$\max_{1\leqq \mathbf{m}\leqq 2^{\mathbf{p}}} |S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}})| \quad \text{tends to 0 a.e. as} \quad \mathbf{p}\to\infty. \tag{2.6}$$

Let $\mathbf{p}\geqq 0$ and $1\leqq \mathbf{m}\leqq 2^{\mathbf{p}}$ be fixed. It is not hard to check that

$$S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}}) = \sum_{\varepsilon} S(\varepsilon 2^{\mathbf{p}}, \varepsilon\mathbf{m}+(\mathbf{1}-\varepsilon)2^{\mathbf{p}}),$$

where the summation $\sum_{\varepsilon}$ is extended over all possible 2^d-1 choices of $\varepsilon=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)$, $\varepsilon_j=0$ or 1, the case $\varepsilon_1=\varepsilon_2=\dots=\varepsilon_d=0$ excluded. From this representation it follows immediately that

$$\max_{1\leqq \mathbf{m}\leqq 2^{\mathbf{p}}} |S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}})| \leqq \tag{2.7}$$

$$\leqq \sum_{\varepsilon} \max_{\substack{1\leqq j\leqq d\\ \varepsilon_j=1}} \max_{1\leqq m_j\leqq 2^{p_j}} |S(\varepsilon 2^{\mathbf{p}}, \varepsilon\mathbf{m}+(\mathbf{1}-\varepsilon)2^{\mathbf{p}})| \equiv \sum_{\varepsilon} M_{\varepsilon}(\mathbf{p}),$$

i.e. $M_{\varepsilon}(\mathbf{p})$ is the maximum of all $|S(\varepsilon 2^{\mathbf{p}}, \varepsilon\mathbf{m}+(\mathbf{1}-\varepsilon)2^{\mathbf{p}})|$, where those coordinates m_j run between 1 and 2^{p_j} whose subscript j is such that $\varepsilon_j=1$ in ε.

Let us fix an ε. If for each j we have $\varepsilon_j=1$, then the corresponding maximum on the right of (2.7) is

$$M_{\mathbf{1}}(\mathbf{p}) = \max_{1\leqq j\leqq d} \max_{1\leqq m_j\leqq 2^{p_j}} |S(2^{\mathbf{p}}, \mathbf{m})| = M(2^{\mathbf{p}}, 2^{\mathbf{p}}).$$

In virtue of Lemma 1 we have

$$\int M^2(2^{\mathbf{p}}, 2^{\mathbf{p}})\, d\mu \leqq |\mathbf{p}+\mathbf{1}|^2 f(2^{\mathbf{p}}, 2^{\mathbf{p}}).$$

By (2.1) hence

$$\sum_{\mathbf{p}\geqq 0} \int M^2(2^{\mathbf{p}}, 2^{\mathbf{p}})\, d\mu < \infty,$$

which implies via the B. Levi theorem that $M(2^{\mathbf{p}}, 2^{\mathbf{p}})$ tends to 0 a.e. as $\mathbf{p}\to\infty$.

Now consider an ε such that for at least one j we have $\varepsilon_j=0$. For the sake of simplicity we assume that $\varepsilon_1=\varepsilon_2=\dots=\varepsilon_e=1$ and $\varepsilon_{e+1}=\dots=\varepsilon_d=0$, where $1\leqq e<d$. Then for the corresponding maximum $M_{\varepsilon}(\mathbf{p})$ we have

$$M_{\varepsilon}(\mathbf{p}) = \max_{1\leqq j\leqq e} \max_{1\leqq m_j\leqq 2^{p_j}} |S(\varepsilon 2^{\mathbf{p}}, \varepsilon\mathbf{m}+(\mathbf{1}-\varepsilon)2^{\mathbf{p}})| \leqq$$

$$\leqq \sum_{i=e+1}^{d} \sum_{n_i=0}^{p_i-1} \Big(\max_{1\leqq j\leqq e} \max_{1\leqq m_j\leqq 2^{p_j}} |S(\varepsilon 2^{\mathbf{p}}+(\mathbf{1}-\varepsilon)2^{\mathbf{n}}, \varepsilon\mathbf{m}+(\mathbf{1}-\varepsilon)2^{\mathbf{n}})|\Big),\ ^{5)}$$

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[5] We remind that $\max\limits_{1\leqq j\leqq e} \max\limits_{1\leqq m_j\leqq 2^{p_j}}$ abbreviates $\max\limits_{1\leqq m_1\leqq 2^{p_1}} \max\limits_{1\leqq m_2\leqq 2^{p_2}} \dots \max\limits_{1\leqq m_e\leqq 2^{p_e}}$, and $\sum\limits_{i=e+1}^{d} \sum\limits_{n_i=0}^{p_i-1}$ abbreviates $\sum\limits_{n_{e+1}=0}^{p_{e+1}-1} \dots \sum\limits_{n_d=0}^{p_d-1}$.

where $\mathbf{n}=(n_1, n_2, \ldots, n_d)$, although the first e coordinates $n_1, n_2, \ldots, n_e$ of $\mathbf{n}$ play no role on the right-hand side of the last inequality. By the Cauchy inequality,

$$M_\varepsilon^2(\mathbf{p}) = O(1) \sum_{i=e+1}^{d} \sum_{n_i=0}^{p_i-1} \Big\{ \prod_{i=e+1}^{d} (n_i+1)^2 \times \tag{2.8}$$
$$\times \max_{1\leqq j\leqq e} \max_{1\leqq m_j\leqq 2^{p_j}} S^2(\varepsilon 2^{\mathbf{p}}+(1-\varepsilon)2^{\mathbf{n}}, \varepsilon\mathbf{m}+(1-\varepsilon)2^{\mathbf{n}})\Big\}.$$

We have to apply the e-parameter version of Lemma 1 for all sets

$$\Big\{\xi_{k_1,\ldots,k_e} = \sum_{i=e+1}^{d} \sum_{k_i=2^{n_i}+1}^{2^{n_i+1}} \zeta_{k_1,\ldots,k_e,k_{e+1},\ldots,k_d} : 2^{p_j}+1 \leqq k_j \leqq 2^{p_j+1};\quad j=1,2,\ldots,e\Big\},$$

where n_i may take on the values $0, 1, \ldots, p_i-1$ for $i=e+1, \ldots, d$. By virtue of (1.2) we come to the inequality

$$\int \Big\{\max_{1\leqq j\leqq e} \max_{1\leqq m_j\leqq 2^{p_j}} S^2(\varepsilon 2^{\mathbf{p}}+(1-\varepsilon)2^{\mathbf{n}}, \varepsilon\mathbf{m}+(1-\varepsilon)2^{\mathbf{n}})\Big\}\, d\mu \leqq \tag{2.9}$$
$$\leqq \prod_{j=1}^{e} (p_j+1)^2 \sum_{j=1}^{e} \sum_{2^{p_j}+1\leqq k_j, l_j\leqq 2^{p_j+1}} \Big|\int \xi_{k_1,\ldots,k_2}\xi_{l_1,\ldots,l_e}\, d\mu\Big| \leqq$$
$$\leqq \prod_{j=1}^{e} (p_j+1)^2 \sum_{2^{\mathbf{q}}+1\leqq \mathbf{k},\mathbf{l}\leqq 2^{\mathbf{q}+1}} \Big|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\Big|$$

with $\mathbf{q}=\varepsilon\mathbf{p}+(1-\varepsilon)\mathbf{n}$.[6]

Combining inequalities (2.8) and (2.9), we obtain that

$$\int \{\max_{e+1\leqq i\leqq d} \sup_{p_i\geqq 0} M_\varepsilon^2(\mathbf{p})\}\, d\mu = O(1) \sum_{i=e+1}^{d} \sum_{n_i=0}^{\infty} |\mathbf{q}+\mathbf{1}|^2 f(2^{\mathbf{q}}, 2^{\mathbf{q}}).\,^{7)}$$

By (2.1) we can establish that

$$\sum_{j=1}^{e} \sum_{p_j=0}^{\infty} \int \{\max_{e+1\leqq i\leqq d} \sup_{p_i\geqq 0} M_\varepsilon^2(\mathbf{p})\}\, d\mu = O(1) \sum_{\mathbf{q}\geqq \mathbf{0}} |\mathbf{q}+\mathbf{1}|^2 f(2^{\mathbf{q}}, 2^{\mathbf{q}}) < \infty,$$

whence via the B. Levi theorem it follows that $M_\varepsilon(\mathbf{p})$ tends to 0 a.e. as $\mathbf{p}\to\infty$. Since this is true for each $M_\varepsilon(\mathbf{p})$ on the right-hand side of (2.7), statement (2.6) holds true.

To put (2.4) and (2.6) together, we can conclude the assertion of Theorem 1.

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6) We remind that $\sum_{j=1}^{e} \sum_{2^{p_j}+1\leqq k_j, l_j\leqq 2^{p_j+1}}$ abbreviates $\sum_{k_1=2^{p_1}+1}^{2^{p_1+1}} \sum_{l_1=2^{p_1}+1}^{2^{p_1+1}} \ldots \sum_{k_e=2^{p_e}+1}^{2^{p_e+1}} \sum_{l_e=2^{p_e}+1}^{2^{p_e+1}}$ (a $2e$-fold summation).

7) $\max_{e+1\leqq i\leqq d} \sup_{p_i\geqq 0} M_\varepsilon^2(\mathbf{p})$ is understood as the supremum of all $M_\varepsilon^2(\mathbf{p})$, when the last $d-e$ coordinates $p_{e+1}, \ldots, p_d$ of $\mathbf{p}\in Z^d$ run, independently of each other, over the non-negative integers.

§ 3. A.e. convergence of the square and the spherical partial sums

Let $Q_1 \subset Q_2 \subset \dots$ be an arbitrary sequence of finite regions in Z_+^d such that $\bigcup_{r=1}^{\infty} Q_r = Z_+^d$, and let $Q_0 = \emptyset$. Set

$$T(r) = \sum_{\mathbf{k} \in Q_r} \zeta_{\mathbf{k}} \quad (r = 1, 2, \dots).$$

The one-parameter versions of Theorem 1 and Corollary 1 read as follows.

Theorem 2. *If*

$$\sum_{t=0}^{\infty} (t+1)^2 \sum_{\mathbf{k}, \mathbf{l} \in Q_{2^{t+1}-1} \setminus Q_{2^t - 1}} \left| \int \zeta_{\mathbf{k}} \zeta_{\mathbf{l}} \, d\mu \right| < \infty,$$

then $T(r)$ *converges a.e. as* $r \to \infty$.

Corollary 2. *If the functions* $\zeta_{\mathbf{k}}$ *are mutually orthogonal and if*

$$\sum_{r=1}^{\infty} \Big(\sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 \Big) \log^2 2r < \infty, \tag{3.1}$$

then $T(r)$ *converges a.e. as* $r \to \infty$.

By setting $\xi_r = \sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} \zeta_{\mathbf{k}}$ for $r = 1, 2, \dots$, Theorem 2 follows from Theorem 1 in the case $d = 1$, while Corollary 2 is a consequence of Theorem 2.

It is worth going into details in connection with the square partial sums, i.e., when $Q_r = \{\mathbf{k} \in Z_+^d \colon k_j \leqq r \text{ for each } j\}$. Then $\mathbf{k} \in Q_r \setminus Q_{r-1}$ iff $\max(k_1, k_2, \dots, k_d) = r$, further, $|Q_r \setminus Q_{r-1}| = O(r^{d-1})$. Here $|Q|$ denotes the number of the points of Z_+^d contained in Q. Condition (3.1) is satisfied if, e.g., for $\mathbf{k} \in Q_r \setminus Q_{r-1}$ we have

$$\sigma_{\mathbf{k}}^2 = O\{r^{-d} (\log 2r)^{-3} (\log\log 4r)^{-1-\varepsilon}\}$$

or

$$\begin{aligned} \sigma_{\mathbf{k}}^2 &= O\{|\mathbf{k}|^{-1} (\log 2r)^{-d-2} (\log\log 4r)^{-1-\varepsilon}\} = \\ &= O\{|\mathbf{k}|^{-1} (\log 2|\mathbf{k}|)^{-d-2} (\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \end{aligned} \tag{3.2}$$

with an $\varepsilon > 0$. The first relation in (3.2) ensures the fulfilment of (3.1) since

$$\sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} |\mathbf{k}|^{-1} = O\{r^{-1} (\log 2r)^{d-1}\}.$$

The second relation in (3.2) follows from

$$r = \max(k_1, k_2, \dots, k_d) \leqq |\mathbf{k}| \leqq r^d.$$

Condition (3.2) is clearly weaker than (2.3) for $d \geqq 2$.

We note that in the more general situation when e coordinates of $\mathbf{m} \in Z_+^d$ depend on a parameter r, while the other $d-e$ coordinates vary independently of each other where $1 \leqq e < d$, then the following result can be achieved. For the sake of simplicity we consider only the case when the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal. Let $\{m_j(r)\}_{r=1}^{\infty}$ be non-decreasing sequences of positive integers such that $m_j(1)=1$ and $m_j(r) \to \infty$ as $r \to \infty$ for each $j=1, 2, \ldots, e$. If

$$\sum_{\mathbf{k} \geqq \mathbf{1}} \sigma_{\mathbf{k}}^2 \lambda^2(k_1, k_2, \ldots, k_e) \prod_{i=e+1}^{d} (\log 2k_i)^2 < \infty,$$

where $\lambda(k_1, k_2, \ldots, k_e) = \log 2r$ if $m_j(r) \leqq k_j < m_j(r+1)$ for each $j=1, 2, \ldots, e$, then $S(m_1(r), \ldots, m_e(r), m_{e+1}, \ldots, m_d)$ converges a.e. as $r \to \infty$ and $m_i \to \infty$ for each $i=e+1, \ldots, d$.

§ 4. A *d*-parameter version of the SLLN

Application of the results of § 2 to the series $\sum_{\mathbf{k} \geqq \mathbf{1}} \zeta_{\mathbf{k}}/|\mathbf{k}|$ yields, via the d-parameter version of the Kronecker lemma (for $d=1$ see, e.g., [5, p. 35]), criteria for the a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$ as $\mathbf{m} \to \infty$. However, as we emphasized in § 1, the limit $\mathbf{m} \to \infty$ is used in different senses according as the a.e. convergence of a d-multiple series $(\min_{1 \leqq j \leqq d} m_j \to \infty)$ or the a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$ $(\max_{1 \leqq j \leqq d} m_j \to \infty)$ is studied. Since the convergence notion $\max_{1 \leqq j \leqq d} m_j \to \infty$ induces a finer topology than the notion $\min_{1 \leqq j \leqq d} m_j \to \infty$, the application of a generalized form of the widely used Kronecker lemma is not appropriate at present. Thus we follow another way to obtain the following SLLN.

Theorem 3. *If*

$$\sum_{\mathbf{m} \geqq \mathbf{0}} \frac{|\mathbf{m}+\mathbf{1}|^2}{|2^{\mathbf{m}}|^2} \sum_{2^{\mathbf{m}}+\mathbf{1} \leqq \mathbf{k}, \mathbf{l} \leqq 2^{\mathbf{m}+\mathbf{1}}} \left| \int \zeta_{\mathbf{k}} \zeta_{\mathbf{l}} \, d\mu \right| < \infty, \tag{4.1}$$

then

$$\lim_{\mathbf{m} \to \infty} S(\mathbf{m})/|\mathbf{m}| = 0 \quad a.e. \tag{4.2}$$

Corollary 3 (SLLN for orthogonal functions). *If the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal and if*

$$\sum_{\mathbf{k} \geqq \mathbf{1}} \frac{\sigma_{\mathbf{k}}^2}{|\mathbf{k}|^2} \prod_{j=1}^{d} (\log 2k_j)^2 < \infty, \tag{4.3}$$

then (4.2) *follows.*

Condition (4.3) is satisfied if, for example, we have

$$\sigma_{\mathbf{k}}^2 = O\left\{ \prod_{j=1}^{d} k_j (\log 2k_j)^{-3} (\log\log 4k_j)^{-1-\varepsilon} \right\}$$

or

$$\sigma_{\mathbf{k}}^2 = O\{|\mathbf{k}| (\log 2|\mathbf{k}|)^{-3d} (\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \tag{4.4}$$

with an $\varepsilon>0$. We mention that Corollary 3 for $d=1$ was established by TANDORI [7] (see also PETROV [4]).

To prove Theorem 3 we begin with a generalization of the so-called Toeplitz lemma (for $d=1$ see, e.g., [5, p. 36]).

Lemma 2. *Let* $\{w(\mathbf{m},\mathbf{k})\colon \mathbf{m},\mathbf{k}\in Z_+^d\}$ *be a set of non-negative numbers with the following two properties:*

$$\sum_{\mathbf{k}\geqq 1} w(\mathbf{m},\mathbf{k}) \leqq C \tag{4.5}$$

for all $\mathbf{m}\in Z_+^d$ *with a constant* C, *and*

$$\lim_{\mathbf{m}\to\infty} w(\mathbf{m},\mathbf{k}) = 0 \tag{4.6}$$

for all $\mathbf{k}\in Z_+^d$. *If* $\{s(\mathbf{k})\colon \mathbf{k}\in Z_+^d\}$ *is a d-multiple sequence of real numbers such that*

$$s(\mathbf{k})\to 0 \quad as \quad \mathbf{k}\to\infty, \tag{4.7}$$

then

$$t(\mathbf{m}) = \sum_{\mathbf{k}\geqq 1} w(\mathbf{m},\mathbf{k})s(\mathbf{k}) \to 0 \quad as \quad \mathbf{m}\to\infty. \tag{4.8}$$

Proof of Lemma 2. By (4.7) for any $\varepsilon>0$ there exists a $\mathbf{k}_0\in Z_+^d$ such that

$$|s(\mathbf{k})| \leqq \varepsilon \quad \text{if} \quad \mathbf{k} \not\leqq \mathbf{k}_0. \tag{4.9}$$

Consider the decomposition of $t(\mathbf{m})$ into two summands:

$$t(\mathbf{m}) = \Big\{\sum_{1\leqq\mathbf{k}\leqq\mathbf{k}_0} + \sum_{\mathbf{k}\not\leqq\mathbf{k}_0}\Big\} w(\mathbf{m},\mathbf{k})s(\mathbf{k}) \equiv t_1+t_2.$$

On account of (4.5) and (4.9), for all $\mathbf{m}\in Z_+^d$ we have that $|t_2|\leqq C\varepsilon$. By (4.6) we can choose an $\mathbf{m}_0\in Z_+^d$ such that for each $\mathbf{k}\in Z_+^d$ with $1\leqq\mathbf{k}\leqq\mathbf{k}_0$ we have

$$|w(\mathbf{m},\mathbf{k})| \leqq \varepsilon\Big/\sum_{1\leqq\mathbf{k}\leqq\mathbf{k}_0} |s(\mathbf{k})| \quad \text{if} \quad \mathbf{m}\not\leqq\mathbf{m}_0.$$

Hence $|t_1|\leqq\varepsilon$.

Collecting the above reasonings we conclude that

$$|t(\mathbf{m})| \leqq (C+1)\varepsilon \quad \text{if} \quad \mathbf{m}\not\leqq\mathbf{m}_0.$$

This is the wanted (4.8).

Lemma 2 just proved makes it possible to show the following simple assertion. Let $\{u_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ be a d-multiple sequence of numbers. Put

$$s(\mathbf{b},\mathbf{m}) = \sum_{\mathbf{b}+1\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} u_{\mathbf{k}} \quad \text{and} \quad s(\mathbf{m}) = s(0,\mathbf{m}),$$

where $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$.

Lemma 3. *The statements*

$$\lim_{\mathbf{m}\to\infty} s(2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \tag{4.10}$$

and

$$\lim_{\mathbf{m}\to\infty} s(2^{\mathbf{m}}, 2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \tag{4.11}$$

are equivalent.

Proof of Lemma 3. From the well-known representation

$$s(\mathbf{b}, \mathbf{m}) = \sum_{\boldsymbol{\varepsilon}} (-1)^{\sum_{j=1}^{d} \varepsilon_j} s(\mathbf{b} + (\mathbf{1} - \boldsymbol{\varepsilon})\mathbf{m}),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is taken for all 2^d choices of $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$ with components $\varepsilon_j = 0$ or 1, the implication (4.10)⇒(4.11) immediately follows.

To facilitate the use of dyadic blocks $s(2^{\mathbf{k}}, 2^{\mathbf{k}})$, we assume that $u_{\mathbf{k}} = 0$ if fo at least one j we have $k_j = 1$. Then

$$\frac{s(2^{\mathbf{m}})}{|2^{\mathbf{m}}|} = \sum_{\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{m}-\mathbf{1}} w(\mathbf{m}, \mathbf{k}) \frac{s(2^{\mathbf{k}}, 2^{\mathbf{k}})}{|2^{\mathbf{k}}|}$$

with $w(\mathbf{m}, \mathbf{k}) = |2^{\mathbf{k}}|/|2^{\mathbf{m}}|$ for $\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{m} - \mathbf{1}$ and $w(\mathbf{m}, \mathbf{k}) = 0$ otherwise. The assumptions of Lemma 2 are clearly satisfied, the application of which gives the implication (4.11)⇒(4.10). This completes the proof.

Proof of Theorem 3. *Step* 1. First we prove that

$$\lim_{\mathbf{m}\to\infty} S(2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \quad \text{a.e.} \tag{4.12}$$

By Lemma 3 it suffices to show that

$$\lim_{\mathbf{m}\to\infty} S(2^{\mathbf{m}}, 2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \quad \text{a.e.} \tag{4.13}$$

For convenience we again assume that $\zeta_{\mathbf{k}} \equiv 0$ if $k_j = 1$ for at least one j. Since by (4.1)

$$\sum_{\mathbf{m} \geqq \mathbf{0}} |2^{\mathbf{m}}|^{-2} \int S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\, d\mu \leqq \sum_{\mathbf{m} \geqq \mathbf{0}} |2^{\mathbf{m}}|^{-2} f(2^{\mathbf{m}}, 2^{\mathbf{m}}) < \infty,$$

where, as before,

$$f(2^{\mathbf{m}}, 2^{\mathbf{m}}) \equiv \sum_{2^{\mathbf{m}}+\mathbf{1} \leqq \mathbf{k}, \mathbf{l} \leqq 2^{\mathbf{m}+\mathbf{1}}} \left| \int \zeta_{\mathbf{k}} \zeta_{\mathbf{l}}\, d\mu \right|,$$

the B. Levi theorem implies (4.13), and consequently (4.12).

Step 2. Now we turn to the proof of the relation

$$\lim_{\mathbf{m}\to\infty} |2^{\mathbf{m}}|^{-1} \max_{\mathbf{1} \leqq \mathbf{p} \leqq 2^{\mathbf{m}}} |S(2^{\mathbf{m}} + \mathbf{p}) - S(2^{\mathbf{m}})| = 0 \quad \text{a.e.} \tag{4.14}$$

As in the proof of Theorem 1, we start with the representation

$$S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}}) = \sum_{\boldsymbol{\varepsilon}} S(\boldsymbol{\varepsilon}2^{\mathbf{m}}, \boldsymbol{\varepsilon}\mathbf{p}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{m}}),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is extended for all $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)$ such that the coordinates ε_j assume the values 0 and 1 independently of each other, excluding the case $\varepsilon_1=\varepsilon_2=\dots=\varepsilon_d=0$. Thus

$$\max_{1 \leqq \mathbf{p} \leqq 2^{\mathbf{m}}} |S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}})| \leqq \tag{4.15}$$

$$\leqq \sum_{\boldsymbol{\varepsilon}} \sum_{\substack{j=1 \\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1} M(\boldsymbol{\varepsilon}2^{\mathbf{m}}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{q}}, \boldsymbol{\varepsilon}2^{\mathbf{m}}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{q}}),$$

where $\sum_{\substack{j=1 \\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1}$ has the following meaning. For given $\boldsymbol{\varepsilon}$, let $\varepsilon_j=0$ iff $j=j_1, j_2, \dots, j_e$, where $1 \leqq j_1 < j_2 < \dots < j_e \leqq d$. Then we have to form the e-fold summation $\sum_{q_{j_1}=0}^{m_{j_1}-1} \dots \sum_{q_{j_e}=0}^{m_{j_e}-1}$.

By virtue of Lemma 1 and (4.1) we have

$$\sum_{\mathbf{k} \geqq \mathbf{0}} |2^{\mathbf{k}}|^{-2} \int M^2(2^{\mathbf{k}}, 2^{\mathbf{k}})\, d\mu \leqq \sum_{\mathbf{k} \geqq \mathbf{0}} \frac{|\mathbf{k}+\mathbf{1}|^2}{|2^{\mathbf{k}}|^2} f(2^{\mathbf{k}}, 2^{\mathbf{k}}) < \infty.$$

Hence the B. Levi theorem implies the a.e. convergence to 0 of $M(2^{\mathbf{k}}, 2^{\mathbf{k}})/|2^{\mathbf{k}}|$ as $\mathbf{k} \to \infty$.

Rewriting (4.15) into the form

$$|2^{\mathbf{m}}|^{-1} \max_{1 \leqq \mathbf{p} \leqq 2^{\mathbf{m}}} |S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}})| \leqq \sum_{\boldsymbol{\varepsilon}} \sum_{\substack{j=1 \\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1} w(\mathbf{m}, \mathbf{k}) \frac{M(2^{\mathbf{k}}, 2^{\mathbf{k}})}{|2^{\mathbf{k}}|}$$

with $\mathbf{k}=\boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{q}$ and $w(\mathbf{m}, \mathbf{k})=|2^{\mathbf{k}}|/|2^{\mathbf{m}}|$ if for at least one j we have $k_j=m_j$ and $w(\mathbf{m}, \mathbf{k})=0$ otherwise, it is enough to apply Lemma 2 in order to get (4.14). This completes the proof of Theorem 3.

§ 5. A one-parameter version of the SLLN

Let $Q_1 \subset Q_2 \subset \dots$ be an arbitrary sequence of finite regions in Z_+^d such that $\bigcup_{r=1}^{\infty} Q_r$ contains infinitely many points of Z_+^d, and let $Q_0=\emptyset$.

Theorem 4. *If*

$$\sum_{t=0}^{\infty} \frac{(t+1)^2}{|Q_{2^t}|^2} \sum_{\mathbf{k}, \mathbf{l} \in Q_{2^{t+1}-1} \setminus Q_{2^t-1}} \left| \int \zeta_{\mathbf{k}} \zeta_{\mathbf{l}}\, d\mu \right| < \infty, \tag{5.1}$$

then

$$|Q_r|^{-1} \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}} \to 0 \quad a.e. \ as \quad r\to\infty. \tag{5.2}$$

Corollary 4. *If the functions* $\zeta_{\mathbf{k}}$ *are mutually orthogonal and if*

$$\sum_{r=1}^{\infty} \Big(\sum_{\mathbf{k}\in Q_r\setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 \Big) \frac{\log^2 2r}{|Q_r|^2} < \infty, \tag{5.3}$$

then (5.2) *follows.*

In fact, set $\xi_t = \sum_{\mathbf{k}\in Q_t\setminus Q_{t-1}} \zeta_{\mathbf{k}}$ for $t=1, 2, \ldots$. Condition (5.1) ensures, owing to Theorem 2 in the case $d=1$, that the series $\sum_{t=1}^{\infty} \xi_t/|Q_t|$ converges a.e. Hence (the usual one-parameter form of) the Kronecker lemma yields

$$|Q_r|^{-1} \sum_{t=1}^{r} \xi_t = |Q_r|^{-1} \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}} \to 0 \quad \text{a.e. as} \quad r\to\infty,$$

as asserted in (5.2).

If $\{Q_r\}$ is chosen as in Case 1 of § 1, then we obtain criteria for the a.e. convergence to 0 of $S(m_1(r), m_2(r), \ldots, m_d(r))/\prod_{j=1}^{d} m_j(r)$, while in Case 2 we obtain criteria for the a.e. convergence to 0 of $\tilde{S}(r)/r^d$ as $r\to\infty$.

It is instructive to specialize condition (5.3) for square partial sums $S(r, r, \ldots, r)$, i.e., when $Q_r=\{\mathbf{k}\in Z_+^d\colon k_j\leqq r \text{ for each } j\}$. Since $|Q_r|=r^d$, (5.3) is surely satisfied if

$$\sum_{\mathbf{k}\in Q_r\setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 = O\{r^{2d-1}(\log 2r)^{-3}(\log\log 4r)^{-1-\varepsilon}\}$$

with an $\varepsilon>0$. Taking into consideration that $\mathbf{k}\in Q_r\setminus Q_{r-1}$ iff $\max(k_1, k_2, \ldots, k_d)=r$ and that

$$\sum_{\mathbf{k}\in Q_r\setminus Q_{r-1}} |\mathbf{k}| = O(r^{2d-1}),$$

condition (5.3) is also satisfied if

$$\begin{aligned} \sigma_{\mathbf{k}}^2 &= O\{|\mathbf{k}|(\log 2r)^{-3}(\log\log 4r)^{-1-\varepsilon}\} = \\ &= O\{|\mathbf{k}|(\log 2|\mathbf{k}|)^{-3}(\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\}, \end{aligned} \tag{5.4}$$

where $\varepsilon>0$ (cf. (3.2)). In case $d\geqq 2$ condition (5.4) is essentially weaker than (4.4).

References

[1] R. P. Agnew, On double orthogonal series, *Proc. London Math. Soc.*, II. s., 33 (1932), 420—434.
[2] F. Móricz, Moment inequalities for the maximum of partial sums of random fields, *Acta Sci. Math.*, 39 (1977), 353—366.
[3] Ш. П. Панджакидзе, Теорема Меньшова-Радемахера для двойных ортогональных рядов, *Соодщения АН Груз ССР*, 39 (1965), 277—282.
[4] В. В. Петров, Об усиленном законе больших чисел для последовательности ортогональных случайных величин, *Вестник Ленинградского ун.*, 7:2 (1975), 52—57.
[5] P. Révész, *The laws of large numbers*, Academic Press (New York, 1968).
[6] A. Szép, The non-orthogonal Menchoff-Rademacher theorem, *Acta Sci. Math.*, 33 (1972), 231—235.
[7] K. Tandori, Bemerkungen zum Gesetz der grossen Zahlen, *Periodica Math. Hungar.*, 2 (1972), 33—39.

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