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On injections, intertwining operators of class C_0

BÉLA SZ.-NAGY and CIPRIAN FOIAŞ

1. An operator T on a (complex, separable) Hilbert space $\mathfrak{H}$ is of class C_0 if it is a completely non-unitary contraction and if $m(T)=0$ for some inner function $m(\lambda)$ on the unit disc: $|\lambda|<1$. T is of class $C_0(N)$ with some integer $N\geqq 0$ if, moreover, its defect indices are $\leqq N$. For a first introduction to the study of these classes see [*H*]. These investigations have already lead to many new concepts and methods in the theory of Hilbert space operators, and in particular to generalizations of the "Jordan model" for finite matrices.

In the present Note we are going to make use of these models for establishing some further properties of class C_0 operators.

Let us recall some further definitions and facts.

An operator $X:\mathfrak{H}'\to\mathfrak{H}$ is called an *injection* if $\ker X=\{0\}$, and a *quasi-surjection* if $\overline{X\mathfrak{H}'}=\mathfrak{H}$ or, equivalently, if $\ker X^*=\{0\}$.

Given two operators, T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$, we say that T' can be *injected in* T, or *quasi-surjected on* T if there exists an operator $X:\mathfrak{H}'\to\mathfrak{H}$ satisfying $TX=XT'$ and which is an injection, or a quasi-surjection, respectively.

An operator X which is both an injection and a quasi-surjection, is called a *quasi-affinity*, and if $TX=XT'$ holds with such an operator X then T' is called a *quasi-affine transform* of T, in notation $T\succ T'$. If both $T\succ T'$ and $T'\succ T$ hold then T and T' are called *quasi-similar*, $T\succ T'$.

Every operator $T\in C_0$ is quasi-similar to a unique "Jordan operator"

$$(1)\qquad S(M)=S(m_1)\oplus S(m_2)\oplus\ldots \quad \text{on} \quad \mathfrak{H}(M)=\mathfrak{H}(m_1)\oplus\mathfrak{H}(m_2)\oplus\ldots$$

where $M=(m_1, m_2, \ldots)$ is a sequence if inner functions each of which is a divisor of the preceding one. Here $S(m)$ means, for any inner function $m(\lambda)$, the operator on the function space $\mathfrak{H}(m)=H^2\ominus mH^2$, defined by $S(m)=P_{\mathfrak{H}(m)}S|\mathfrak{H}(m)$, S denoting the unilateral shift $u(\lambda)\to\lambda u(\lambda)$ on the Hardy—Hilbert space H^2 for

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the disc. We have $\mathfrak{H}(m)=\{0\}$ if (and only if) m is constant, $m=1$.[1]) The number of non-constant functions m_k in (1) is equal to the multiplicity μ_T of the operator T.[2]) For $T \in C_0$ we have $\mu_T=\mu_{T^*}$. For these facts see [1], [2], [3].

2. In the sequel we shall be dealing with operators of class C_0.

Theorem 1. *Let us be given two operators of class C_0, say T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$.* a) *If T and T' can be injected in one another, then they are quasi-similar.* b) *If T' can be injected in, and also quasi-surjected on T, then they are quasi-similar.*

Proof. We have to show that if there exist injections $X: \mathfrak{H}' \to \mathfrak{H}$ and $X': \mathfrak{H} \to \mathfrak{H}'$ such that

$$TX = XT' \tag{2}$$

and

$$\text{(a) } T'X' = X'T, \quad \text{or} \quad \text{(b) } T'^*X' = X'T^*; \tag{3}$$

then $T \sim T'$.

As it is easy to see, there is no loss in generality if we argue with the Jordan models of T and T' instead of T and T' themselves, i.e. if we assume that

$$\mathfrak{H} = \mathfrak{H}(M), \quad \mathfrak{H}' = \mathfrak{H}(M'), \quad T = S(M), \quad T' = S(M'), \tag{4}$$

where $M=(m_1, m_2, \dots)$ and $M'=(m_1', m_2', \dots)$. Following a standard argument (*cf.*, in particular, [4], proof of Theorem 4), we set, for any inner function w,

$$T^w = T\,|\,\overline{w(T)\mathfrak{H}}, \quad T'^w = T\,|\,\overline{w(T')\mathfrak{H}'}, \quad X^w = X\,|\,\overline{w(T')\mathfrak{H}'},$$

and first notice that by condition (2) we also have

$$T^w X^w = X^w T'^w;$$

clearly, X^w is also an injection. Now, T^w and T'^w are unitarily equivalent to $\oplus S(q_i)$ and $\oplus S(q_i')$, respectively, where $q_i=m_i/(m_i \wedge w)$ and $q_i'=m_i'/(m_i' \wedge w)$. Choosing $w=m_k$ for a fixed k we infer that

$$\bigoplus_{i=1}^{k-1} S\left(\frac{m_i}{m_k}\right) \cdot X^{(k)} = X^{(k)} \cdot \bigoplus_{i=1}^{\infty} S\left(\frac{m_i'}{m_i' \wedge m_k}\right), \text{[3])}$$

with some injection $X^{(k)}$. By virtue of [4], Theorem 4, the second direct sum cannot have more non-trivial terms than the first, so we must have $m_i'/(m_i' \wedge m_k)=1$ for $i \geqq k$, and in particular, $m_k'/(m_k' \wedge m_k)=1$, $m_k' | m_k$.

[1]) Up to a constant factor of modulus one. It is convenient not to distinguish two inner functions which differ in such a factor only.

[2]) For any operator T, μ_T is defined as the smallest cardinal of a set of vectors which, together with its transforms by T, T^2, etc., span the whole space of T.

[3]) If $k=1$ the first direct sum should be meant as the trivial operator on the space $\{0\}$.

In case condition (3a) holds the same argument applies with the rôles of T and T' interchanged, and we have $m_k | m'_k$ for every k. We conclude that $m_k = m'_k$.

If it is condition (3b) which is assumed, we arrive at the same result as follows. It is well known that for any inner m, $S(m)^*$ is unitarily equivalent to $S(m^\sim)$, where $m^\sim(\lambda) = \overline{m(\bar{\lambda})}$. So (4) implies that T^* and T'^* are unitarily equivalent to $S(M^\sim)$ and $S(M'^\sim)$, respectively, and then we deduce from (3b) that $m_k^\sim | m_k'^\sim$ for every k, in the same way as we deduced $m'_k | m_k$ from (2). But $m_k^\sim | m_k'^\sim$ obviously implies $m_k | m'_k$ and we conclude again that $m_k = m'_k$.

Thus in both cases T and T' have the same Jordan model so they are quasi-similar. This concludes the proof.

Remark. The quasi-similarity of T and T' is, in general, not effectuated by the operators X, X' figuring in (2), (3a) or (3b), since they need not be quasi-affinities. Example: $T = T' = 0$ on an infinite dimensional Hilbert space $\mathfrak{H}$, and $X = X' =$ (a unilateral shift on $\mathfrak{H}$).

However, such a phenomenon cannot occur if the operators T, T' are of finite multiplicity. This will be proved in the rest of this paper.

3. First we prove the following

Lemma. *Let T be an operator of class $C_0(N)$ on $\mathfrak{H}$, with some finite N. Then every injection X on $\mathfrak{H}$, commuting with T, is a quasi-affinity.*

Proof. Let T_1 be the restriction of T to the subspace $\mathfrak{H}_1 = \overline{X\mathfrak{H}}$, which is invariant for T_1 because

$$T_1 X = XT. \tag{5}$$

Let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let $\Theta_T = \Theta_2 \Theta_1$ be the corresponding regular factorization of the characteristic function Θ_T of T; Θ_{T_1} and Θ_{T_2} coincide then with the purely contractive parts Θ_1^0 and Θ_2^0 of Θ_1 and Θ_2, respectively; *cf.* [*H*] Chapter VII. Since T is of class $C_0(N)$, all these functions are finite square-matrix valued so we have

$$\det \Theta_T = \det \Theta_2 \cdot \det \Theta_1 = \det \Theta_2^0 \cdot \det \Theta_1^0 = \det \Theta_{T_2} \cdot \det \Theta_{T_1}, \tag{6}$$

up to constant factors of modulus one.

Since X can be regarded as a quasi-affinity $\mathfrak{H} \to \mathfrak{H}_1$, from (5) it follows that T is a quasi-affine transform of T_1; $T_1 \succ T$. Hence, T and T_1 are quasi-similar to the same Jordan operator $S(M)$, $M = (m_1, m_2, \ldots, m_k)$, $K \leqq N$; *cf.* [1] Theorem 2 (and also [2] Theorem 3). Hence we have, by the formula (1.7) of [1],

$$\det \Theta_T = m_1 \ldots m_k = \det \Theta_{T_1};$$

comparing this with (6) we conclude that det Θ_{T_2} is a constant (of modulus one). The minimal function m_{T_2}, being a divisor of det Θ_{T_2} (*cf.* [*H*] Sec. VI. 5), is also constant, and therefore we have $\mathfrak{H}_2=\{0\}$, $\mathfrak{H}_1=\mathfrak{H}$, as asserted.

4. Every operator $T\in C_0(N)$ is of finite multiplicity, $\mu_T\leqq N$, but not every operator $T\in C_0$ with finite multiplicity belongs to some class $C_0(N)$. Therefore, the following theorem is an extension of the Lemma (even if $T=T'$).

Theorem 2. *Let T and T' be operators of class C_0 on the spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, and suppose T and T' are quasi-similar and have finite multiplicity, $\mu_T=\mu_{T'}=K$. Then every injection operator intertwining T and T' is a quasi-affinity.*[4])

Proof. Let X be an injection $\mathfrak{H}'\to\mathfrak{H}$ such that $TX=XT'$. Since $T\sim T'$, there exists a quasi-affinity $Q:\mathfrak{H}\to\mathfrak{H}'$ such that $T'Q=QT$, and hence $TX'=X'T$ with $X'=XQ$. Clearly X' is an injection; furthermore, X' is a quasi-surjection iff so is X. Hence it suffices to show that every injection X' on $\mathfrak{H}$ satisfying

$$TX' = X'T \tag{7}$$

is a quasi-surjection, i.e. such that $\overline{X'\mathfrak{H}}=\mathfrak{H}$.

Setting $\mathfrak{H}_1=\overline{X'\mathfrak{H}}$ and $T_1=T|\mathfrak{H}_1$ we deduce from (7), as in the proof of the Lemma, that $T_1\succ T$, $T^*\succ T_1^*$. Since, on the other hand, $T^*\sim S(M)^*$ with some $M=(m_1, m_2, \ldots, m_K)$, we conclude that there exist quasi-affinities

$$A:\mathfrak{H}(M)\to\mathfrak{H},\quad B:\mathfrak{H}\to\mathfrak{H}(M),\quad B_1:\mathfrak{H}_1\to\mathfrak{H}(M)$$

such that

$$T^*A = A\,S(M)^*,\quad S(M)^*B = BT^*,\quad S(M)^*B_1 = B_1T_1^*. \tag{8}$$

Set $Y=B_1P_1A$, where P_1 denotes the orthogonal projection of $\mathfrak{H}$ onto its subspace $\mathfrak{H}_1$. Then $P_1T^*=T_1^*P_1$ and by (8):

$$YS(M)^* = B_1P_1A\,S(M)^* = B_1P_1T^*A = B_1T_1P_1A = S(M)^*B_1P_1A = S(M)^*Y.$$

Furthermore, by the quasi-surjectivity of A and B_1,

$$\overline{Y\mathfrak{H}(M)} = \overline{B_1P_1A\mathfrak{H}(M)} = \overline{B_1P_1\mathfrak{H}} = \overline{B_1\mathfrak{H}_1} = \mathfrak{H}(M).$$

It follows that Y^* is an injective operator on $\mathfrak{H}(M)$, commuting with $S(M)$. As we have $S(M)\in C_0(K)$ it follows from the Lemma that Y^* is quasi-surjective. Hence, Y $(=B_1P_1A)$ is injective. This implies that P_1A is injective also.

Let us now assume that $\mathfrak{H}_1\neq\mathfrak{H}$, and consider in $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ a cyclic subspace for T_2^* $(=T^*|\mathfrak{H}_2)$. The restriction of T_2^* to this subspace is then quasi-similar to an operator $S(n)$ associated with a non-constant inner function n (*cf.* [1], Theorem 2 applied to a C_0-class operator of multiplicity 1). Since $S(n)^*$ is unitarily equivalent

[4]) This was conjectured by P. Y. Wu in connection with his investigations [5] on commutants of class C_0 operators (communication to the first author on October 1, 1975). As far as we know the following one is the first proof.

to $S(m)$, where $m=n^{\sim}$, there exists in particular an *injection* $C\colon \mathfrak{H}(m)\to\mathfrak{H}_2$ such that

$$CS(m)^* = T^*C. \tag{9}$$

Next consider the operator Z on $\mathfrak{H}(M)\oplus\mathfrak{H}(m)$ defined by

$$Z(h_M\oplus h_m) = B(Ah_M+Ch_m)\oplus 0 \quad (h_M\in\mathfrak{H}(M),\ h_m\in\mathfrak{H}(m)). \tag{10}$$

From (8), (9), (10) we obtain

$$B(A\,S(M)^*h_M + C\,S(m)^*h_m) = B(T^*Ah_M+T^*Ch_m) =$$
$$= BT^*(Ah_M+Ch_m) = S(M)^*B(Ah_M+Ch_m),$$

and hence,

$$Z(S(M)\oplus S(m)^*)(h_M\oplus h_m) = (S(M)^*\oplus S(m)^*)Z(h_M\oplus h_m),$$

i.e. Z commutes with the operator $S(M)^*\oplus S(m)^*$, which clearly belongs to $C_0(K+1)$. Furthermore, Z is injective. Indeed, $Z(h_M\oplus h_m)=0$ implies $Ah_m=-Ch_m\in\mathfrak{H}_2$, and hence $P_1Ah_M=0$, $h_M=0$ because P_1A is an injection. Since C is also injective, $Ch_m=-Ah_M=0$ implies $h_m=0$.

Applying the Lemma we get that Z is also quasi-surjective. This contradicts the fact that, by (10), its range lies in $\mathfrak{H}(M)\oplus\{0\}$.

This contradiction proves that $\mathfrak{H}_1=\mathfrak{H}$ so X is a quasi-affinity.

5. To end let us venture the following conjecture, which would largely generalize Theorem 2 in case $T=T'$:

Conjecture. For any contraction T on $\mathfrak{H}$ of class C_0 and of finite multiplicity, and for any operator X on $\mathfrak{H}$ such that $TX=XT$, the operators

$$T|\ker X \quad\text{and}\quad (T^*|\ker X^*)^*$$

are quasi-similar.

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BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

FACULTY OF MATH.
UNIVERSITY BUCHAREST
STR. ACADEMIEI 14
BUCHAREST, ROMANIA