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A quasimilarity model for algebraic operators

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In this note, all Hilbert spaces $\mathfrak{H}$ will be understood to be complex. We denote by $\mathcal{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. For A in $\mathcal{L}(\mathfrak{H})$, $\sigma(A)$ denotes the spectrum of A, $\mathcal{K}(A)$ the kernel of A, and $\mathcal{R}(A)$ the range of A. An operator A in $\mathcal{L}(\mathfrak{H})$ is said to be *algebraic* if there exists a nonzero polynomial $p(z)$ with complex coefficients such that $p(A)=0$. If $A^m=0$ for some positive integer m, then we say that A is *nilpotent*. If n is a positive integer, then the nilpotent operator acting on the direct sum of n copies of $\mathfrak{H}$ and defined by the $n\times n$ matrix $[A_{ij}]$, where

$$A_{i,i+1} = 1_{\mathfrak{H}} \text{ for } i = 1, 2, \dots, n-1, \text{ and } A_{ij}=0 \text{ for all other entries,}$$

is called a *Jordan block* operator of order n. (By definition, the zero operator on $\mathfrak{H}$ is a Jordan block operator of order one.) Suppose that $\mathfrak{H}_1, \dots, \mathfrak{H}_m$ are Hilbert spaces and $n_1, \dots, n_m$ are positive integers. Let $\tilde{\mathfrak{H}}_k$ be the direct sum of n_k copies of $\mathfrak{H}_k$ and J_k be the Jordan block operator of order n_k acting on $\tilde{\mathfrak{H}}_k$, $k=1, 2, \dots, m$. An operator of the form $J_1\oplus\dots\oplus J_m$ acting on $\tilde{\mathfrak{H}}_1\oplus\dots\oplus\tilde{\mathfrak{H}}_m$ is called a *Jordan operator*.

We recall that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are Hilbert spaces and $X: \mathcal{K}_1\to\mathcal{K}_2$ is a bounded linear transformation such that $\mathcal{K}(X)=\mathcal{K}(X^*)=\{0\}$, then X is called a *quasi-affinity*. If $A_1\in\mathcal{L}(\mathcal{K}_1)$ and $A_2\in\mathcal{L}(\mathcal{K}_2)$ and there exist quasiaffinities $X: \mathcal{K}_1\to\mathcal{K}_2$ and $Y: \mathcal{K}_2\to\mathcal{K}_1$ such that $XA_1=A_2X$ and $A_1Y=YA_2$, then A_1 and A_2 are said to be *quasisimilar*. In case that there exists an invertible bounded linear transformation $Z: \mathcal{K}_1\to\mathcal{K}_2$ such that $ZA_1=A_2Z$, then A_1 and A_2 are said to be *similar*.

It is well-known that every operator on a finite dimensional Hilbert space is algebraic and similar to its Jordan canonical form. Hence it is natural to ask

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whether there exists an analogous model for the class of algebraic operators on an infinite dimensional Hilbert space. Apostol, Douglas, and Foiaş proved in [1] that every nilpotent operator on a Hilbert space is quasisimilar to a Jordan operator. (This author provided a different proof of this theorem in [2].) The first purpose of this note is to show that there exists such a model for the class of algebraic operators also.

Necessary and sufficient conditions that a nilpotent operator is similar to a Jordan operator were also presented in [2]. We proved that a nilpotent operator A is similar to a Jordan operator if and only if the range of A^i is closed, $i=1, 2, \dots$. The following theorem generalizes this result also.

Theorem. (a) *Suppose that A is an algebraic operator on a Hilbert space $\mathfrak{H}$ and $\sigma(A)=\{\lambda_1, \dots, \lambda_n\}$. Then there exist Jordan operators $J_1, \dots, J_n$ acting on Hilbert spaces $\mathfrak{H}_1, \dots, \mathfrak{H}_n$, respectively, such that A is quasisimilar to $B=\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$.*

(b) *A is similar to B if and only if the range of $(A-\lambda_j)^i$ is closed $(i=1, 2, \dots, j=1, 2, \dots, n)$.*

Note that as a result of the spectral mapping theorem, the spectrum of every algebraic operator is a finite set. Thus the operator A in the Theorem is the most general algebraic operator. (Of course, in the Theorem and throughout this note, we assume that if $i\neq j$, then $\lambda_i\neq\lambda_j$.)

We begin with the following lemma.

Lemma 1. *Suppose that A is an algebraic operator on a Hilbert space $\mathfrak{H}$, say $p(A)=0$, and let $\sigma(A)=\{\lambda_1, \dots, \lambda_n\}$. Then there are operators A_k with $p(A_k)=0$ and $\sigma(A_k)=\{\lambda_k\}$ $(k=1, 2, \dots, n)$, such that A is similar to $A_1\oplus\dots\oplus A_n$.*

Proof. We prove the lemma by induction on the number of points n in $\sigma(A)$. If $n=1$, the lemma is obviously true. Suppose that $n>1$ and that the lemma is true for every algebraic operator which has $n-1$ points in its spectrum. Let f_1 be an analytic function which is identically one in a neighborhood of $\{\lambda_1, \dots, \lambda_{n-1}\}$ and identically zero in a neighborhood of $\{\lambda_n\}$. Let $f_2(z)=1-f_1(z)$ for each z where f_1 is defined. The idempotent operators $f_1(A)$ and $f_2(A)$ are defined by the Riesz functional calculus. Let $\mathfrak{M}=\mathfrak{R}(f_1(A))$ and $\mathfrak{N}=\mathfrak{R}(f_2(A))$. According to the theory of the Riesz functional calculus, $\mathfrak{M}$ and $\mathfrak{N}$ are hyperinvariant subspaces for A, $\sigma(A|\mathfrak{M})=\{\lambda_1, \dots, \lambda_{n-1}\}$, and $\sigma(A|\mathfrak{N})=\{\lambda_n\}$. The matrices of A, $f_1(A)$, and $f_2(A)$ with respect to the decomposition $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{M}^{\perp}$ are respectively

$$\begin{bmatrix} A_0 & B \\ 0 & C \end{bmatrix}, \quad \begin{bmatrix} 1_{\mathfrak{M}} & D \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & -D \\ 0 & 1_{\mathfrak{M}^{\perp}} \end{bmatrix},$$

where $A_0=A|\mathfrak{M}$. Since the operators A and $f_1(A)$ commute, we have $A_0D=B+DC$. Now we have

$$\begin{bmatrix}1_{\mathfrak{M}} & D\\ 0 & 1_{\mathfrak{M}^\perp}\end{bmatrix}\begin{bmatrix}A_0 & B\\ 0 & C\end{bmatrix}\begin{bmatrix}1_{\mathfrak{M}} & -D\\ 0 & 1_{\mathfrak{M}^\perp}\end{bmatrix}=\begin{bmatrix}A_0 & -A_0D+B+DC\\ 0 & C\end{bmatrix}=\begin{bmatrix}A_0 & 0\\ 0 & C\end{bmatrix}.$$

Thus A is similar to $A_0\oplus C$. It follows that A_0 and C are algebraic operators; indeed $p(A_0)=p(C)=0$.

We now show that C is similar to $A|\mathfrak{N}$, and thus $\sigma(C)=\sigma(A|\mathfrak{N})=\{\lambda_n\}$. Indeed, from the matrix of $f_2(A)$ we have $\mathfrak{N}=\left\{\begin{bmatrix}-Dy\\ y\end{bmatrix}\in\mathfrak{M}\oplus\mathfrak{M}^\perp : y\in\mathfrak{M}^\perp\right\}$. Define a linear transformation $S\colon\mathfrak{M}^\perp\to\mathfrak{N}$ by setting $Sy=\begin{bmatrix}-Dy\\ y\end{bmatrix}$ for each y in $\mathfrak{M}^\perp$. Then S is invertible and

$$(A|\mathfrak{N})Sy=\begin{bmatrix}A_0 & B\\ 0 & C\end{bmatrix}\begin{bmatrix}-Dy\\ y\end{bmatrix}=\begin{bmatrix}(-A_0D+B)y\\ Cy\end{bmatrix}=\begin{bmatrix}-DCy\\ Cy\end{bmatrix}=SCy$$

for each y in $\mathfrak{M}^\perp$. Hence $(A|\mathfrak{N})S=SC$, and thus C is similar to $A|\mathfrak{N}$.

We observe that $p(A_0)=0$ and $\sigma(A_0)=\{\lambda_1, \dots, \lambda_{n-1}\}$. By the induction hypothesis, there exist operators A_k with $p(A_k)=0$ and $\sigma(A_k)=\{\lambda_k\}$, $k=1, 2, \dots, n-1$, such that A_0 is similar to $A_1\oplus\dots\oplus A_{n-1}$. Hence A is similar to $A_1\oplus\dots\oplus A_n$ where $A_n=C$. The proof is complete since $p(A_n)=0$ and $\sigma(A_n)=\{\lambda_n\}$.

Lemma 2. *Suppose that A is an algebraic operator on $\mathfrak{H}$ and $\sigma(A)=\{\lambda\}$. Then there exists a Jordan operator J acting on a Hilbert space $\mathfrak{H}_0$ such that A is quasisimilar to $\lambda 1_{\mathfrak{H}_0}+J$.*

Proof. Apply Theorem 1 of [2] to the operator $T=A-\lambda$ to get that T is quasisimilar to a Jordan operator J acting on a Hilbert space $\mathfrak{H}_0$. Hence $A=\lambda+T$ is quasisimilar to $\lambda 1_{\mathfrak{H}_0}+J$.

Proof of the Theorem. (a) This follows immediately from Lemma 1 and Lemma 2.

(b) Suppose that there exist Jordan operators $J_1, \dots, J_n$ acting on Hilbert spaces $\mathfrak{H}_1, \dots, \mathfrak{H}_n$, respectively, such that A is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$. Then, for positive integers i and j, $1\leqq j\leqq n$, the operator $(A-\lambda_j)^i$ is similar to $\sum_{k=1}^{n}\oplus((\lambda_k-\lambda_j)1_{\mathfrak{H}_k}+J_k)^i$, which has closed range. Thus the range of $(A-\lambda_j)^i$ is also closed. On the other hand, suppose that the range of each $(A-\lambda_j)^i$ is closed. According to Lemma 1, there exist algebraic operators A_k with $\sigma(A_k)=\{\lambda_k\}$ such that A is similar to $\sum_{k=1}^{n}\oplus A_k$; and hence $(A-\lambda_j)^i$ is similar to $\sum_{k=1}^{n}\oplus(A_k-\lambda_j)^i$. So for each positive integer i and for each integer k, $1\leqq k\leqq n$, the range of $(A_k-\lambda_j)^i$

is closed. In particular, the operator $A_j-\lambda_j$ is nilpotent and the range of $(A_j-\lambda_j)^i$ is closed. Thus, by Theorem 2 of [2], there exists a Jordan operator J_j acting on a Hilbert space $\mathfrak{H}_j$ such that $A_j-\lambda_j$ is similar to J_j. Hence it follows that A_j is similar to $\lambda_j 1_{\mathfrak{H}_j}+J_j$, $j=1, 2, \ldots, n$. Thus $\sum_{k=1}^{n}\oplus A_k$ is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$. Therefore, A is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$.

References

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