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Jordan model for weak contractions

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SZ.-NAGY and FOIAŞ defined in [10] a class of multiplicity-free operators among C_0 contractions (also *cf.* [8]). Later on in [1] they developed a "Jordan model" for C_0 contractions, which resembles in some respects the usual canonical model of a finite matrix. In the present paper we extend both concepts from the context of C_0 contractions to that of weak contractions.

1. Preliminaries. Let T be a contraction defined on a complex, separable Hilbert space H. Denote by $d_T=\operatorname{rank}(I-T^*T)^{1/2}$, $d_{T^*}=\operatorname{rank}(I-TT^*)^{1/2}$ the defect indices of T.

Recall that T is called a weak contraction if (i) its spectrum $\sigma(T)$ does not fill the open unit disk D, and (ii) $I-T^*T$ is of finite trace. Thus in particular $C_0(N)$ contractions and C_{11} contractions with finite defect indices are weak contractions. For the theory of $C_0(N)$ contractions and C_{11} contractions, we refer the reader to [9]. If T is a completely non-unitary (c.n.u.) weak contraction on H, then $d_T=d_{T^*}$ and we can consider its C_0-C_{11} decomposition. Let H_0 and H_1 be the invariant subspaces for T such that $T_0\equiv T|H_0$ and $T_1\equiv T|H_1$ are the C_0 part and C_{11} part of T. Note that T_0 and T_1 are the operators appearing in the triangulations

$$T=\begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix} T_1 & Y \\ 0 & T_0' \end{bmatrix}$$

of type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix},$$

respectively. These triangulations, in term, correspond to the *-canonical factorization and canonical factorization

$$\Theta(\lambda)=\Theta_{*e}(\lambda)\Theta_{*i}(\lambda), \quad \Theta(\lambda)=\Theta_i(\lambda)\Theta_e(\lambda) \quad (\lambda\in D)$$

of the characteristic function $\Theta(\lambda)$ of T, *cf.* [9], Chap. VIII.

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Let H^2 denote the Hardy space of analytic functions on D. For each inner function φ, S_φ denotes the operator on $H^2 \ominus \varphi H^2$ defined by $(S_\varphi f)(\lambda) = P(\lambda f(\lambda))$ for $\lambda \in D$, where P denotes the (orthogonal) projection of H^2 onto $H^2 \ominus \varphi H^2$. For inner functions φ_1 and φ_2, $\varphi_1 = \varphi_2$ means that φ_1 and φ_2 differ by a constant factor of modulus one; $\varphi_1 | \varphi_2$ means that φ_1 is a divisor of φ_2. $H^2 \ominus \varphi H^2$ reduces to $\{0\}$ if and only if φ is a constant inner function. For a measurable subset E of the unit circle C, M_E denotes the operator of multiplication by e^{it} on the space $L^2(E)$ of square-integrable functions on E, where the measure considered is the (normalized) Lebesgue measure. For measurable subsets E_1 and E_2 of C, $E_1 = E_2$ means that $(E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ is of Lebesgue measure zero. If $E = \emptyset$ then $L^2(E)$ reduces to $\{0\}$.

For arbitrary operators T_1, T_2 on H_1, H_2, respectively, $T_1 \prec T_2$ denotes that T_1 is a quasi-affine transform of T_2, that is, there exists a one-to-one, continuous linear transformation X from H_1 onto a dense linear manifold of H_2 (called quasi-affinity) such that $XT_1 = T_2 X$. T_1 and T_2 are quasi-similar if $T_1 \prec T_2$ and $T_2 \prec T_1$. For an arbitrary operator T on H, let μ_T denote the multiplicity of T, that is, the least cardinal number of a subset K of vectors in H for which $H = \bigvee_{n=0}^{\infty} T^n K$. In particular, if $\mu_T = 1$ then T is cyclic and the vector in K is a cyclic vector for T. Note that both S and M_E are cyclic and that quasi-similar operators have equal multiplicities.

2. Jordan model. The following theorem, gives the Jordan model for C_0 contractions (*cf.* [1] and [10]).

Theorem 1. *Let T be a C_0 contraction on a separable Hilbert space, with defect indices $d_T = d_{T^*}$. Then T is quasi-similar to a uniquely determined operator of the form*

$$S_{\varphi_1} \oplus S_{\varphi_2} \oplus \ldots \oplus S_{\varphi_m} \oplus \ldots,$$

where the φ_j's are inner functions satisfying $\varphi_{j+1} | \varphi_j$ $(j = 1, 2, \ldots)$. Moreover, φ_1 is the minimal function of T, and if there are just m $(0 \leqq m \leqq \infty)$ non-constant φ_j's, then $m = \mu_T = \mu_{T^*} \leqq d_T = d_{T^*}$.

Next we consider C_{11} contractions. In this case a "Jordan model" can also be given.

Theorem 2. *Let T be a c.n.u. C_{11} contraction on a separable Hilbert space, with defect indices $d_T = d_{T^*}$. Then T is quasi-similar to a uniquely determined operator of the form*

$$M_{E_1} \oplus M_{E_2} \oplus \ldots \oplus M_{E_n} \oplus \ldots, \tag{1}$$

where the E_k's are measurable subsets of C satisfying $E_{k+1} \subseteqq E_k$ $(k=1, 2, \ldots)$. If there are just n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure, then $n=\mu_T=\mu_{T^*} \leqq \leqq d_T=d_{T^*}$.

We start the proof with the following

Lemma 1. *Let T_1 and T_2 be operators on H_1 and H_2, respectively. Then* $\max\{\mu_{T_1}, \mu_{T_2}\} \leqq \mu_{T_1 \oplus T_2} \leqq \mu_{T_1} + \mu_{T_2}$.

Proof. Let $K=\{x_\alpha \oplus y_\alpha\}_{\alpha \in A}$ be a subset of vectors in $H_1 \oplus H_2$ such that $H_1 \oplus H_2 = \bigvee_{n=0}^{\infty} (T_1 \oplus T_2)^n K$. Then $K_1 \equiv \{x_\alpha\}_{\alpha \in A}$ is a subset of H_1 satisfying $H_1 = \bigvee_{n=0}^{\infty} T_1^n K_1$. It follows that $\mu_{T_1} \leqq \mu_{T_1 \oplus T_2}$. By symmetry we have $\mu_{T_2} \leqq \mu_{T_1 \oplus T_2}$, and hence $\max\{\mu_{T_1}, \mu_{T_2}\} \leqq \mu_{T_1 \oplus T_2}$.

To prove the second inequality, let $K_1 = \{x_\alpha\}_{\alpha \in A} \subseteqq H_1$ and $K_2 = \{y_\beta\}_{\beta \in \Omega} \subseteqq H_2$ be such that $H_1 = \bigvee_{n=0}^{\infty} T_1^n K_1$ and $H_2 = \bigvee_{n=0}^{\infty} T_2^n K_2$, respectively. Then $K = \{x_\alpha \oplus 0, 0 \oplus y_\beta\}_{\alpha \in A, \beta \in \Omega}$ is a subset of $H_1 \oplus H_2$ satisfying $H_1 \oplus H_2 = \bigvee_{n=0}^{\infty} (T_1 \oplus T_2)^n K$. It follows that $\mu_{T_1 \oplus T_2} \leqq \mu_{T_1} + \mu_{T_2}$.

Note that the inequalities in Lemma 1 actually occur. For example, if $T_1 = T_2$ is a simple unilateral shift then $\mu_{T_1} = \mu_{T_2} = 1$ and $\mu_{T_1 \oplus T_2} = 2$ (*cf.* [15], p. 308); if $T_1 = T_2$ is the adjoint of a simple unilateral shift then $\mu_{T_1} = \mu_{T_2} = 1$ and $\mu_{T_1 \oplus T_2} = 1$ (*cf.* [6], Problem 126).

Lemma 2. *If there are just n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure in the operator $T = M_{E_1} \oplus M_{E_2} \oplus \ldots \oplus M_{E_n} \oplus \ldots$, where $E_{k+1} \subseteqq E_k$ $(k=1, 2, \ldots)$, then* $n = \mu_T = \mu_{T^*}$.

Proof. By the first inequality in Lemma 1, it suffices to consider the case $n < \infty$, that is, we have to show that if $T = M_{E_1} \oplus \ldots \oplus M_{E_n}$, $n < \infty$, then $n = \mu_T$. Inequality $\mu_T \leqq n$ is obvious. To prove that $\mu_T \geqq n$, let us make use of the direct integral representation of the Hilbert-space $H = L^2(E_1) \oplus \ldots \oplus L^2(E_n)$, associated with the unitary operator T, that is, let

$$H = \int_{\sigma(T)}^{\oplus} H_\lambda \, dm \quad \text{with} \quad T^{*k} T^h \{x(\lambda)\} = \{\bar{\lambda}^k \lambda^h x(\lambda)\},$$

where m denotes the Lebesgue measure on $\sigma(T) \subseteqq C$. Let $N = \mu_T$. If $K = \{x_1, \ldots, x_N\}$ satisfies $H = \bigvee_{m=0}^{\infty} T^m K$ then $K_\lambda = \{x_1(\lambda), \ldots, x_N(\lambda)\}$ is a set of vectors in H_λ such that $H_\lambda = \bigvee_{m=0}^{\infty} \{\lambda^m x_1(\lambda), \ldots, \lambda^m x_N(\lambda)\}$ for almost all λ in $\sigma(T)$. But for λ in E_n, H_λ is an n-dimensional space. Hence we have $\mu_T = N \geqq n$, completing the proof.

Proof of Theorem 2. Part of this theorem is implicitly contained in the work of SZ.-NAGY and FOIAŞ [9]. Indeed, since T is quasi-similar to the dual residual part of its minimal unitary dilation ([9], p. 72), by [9], pp. 88—89 we can infer that T is quasi-similar to an operator of the form (1) and $n \leqq d_{T^*}$ (also *cf.* [9], pp. 271—273 for the case $d_T = d_{T^*} < \infty$). The uniqueness follows from the multiplicity theory of normal operators [5] and the fact that quasi-similar normal operators are unitarily equivalent. Lemma 2 furnishes the proof of the remaining part.

In light of these results we can generalize the notion of Jordan operators to the following

Definition. An operator T is called a *Jordan operator* if it is of the form

$$S_{\varphi_1} \oplus \ldots \oplus S_{\varphi_m} \oplus \ldots \oplus M_{E_1} \oplus \ldots \oplus M_{E_n} \oplus \ldots,$$

where the φ_j's are inner functions satisfying $\varphi_{j+1} | \varphi_j$ $(j=1, 2, \ldots)$, and the E_k's are measurable subsets of C satisfying $E_{k+1} \subseteq E_k$ $(k=1, 2, \ldots)$.

Combining Theorems 1 and 2 we obtain

Theorem 3. *Let T be a weak contraction on a separable Hilbert space, with defect indices $d_T = d_{T^*}$. Then T is quasi-similar to a uniquely determined Jordan operator*

$$S_{\varphi_1} \oplus \ldots \oplus S_{\varphi_m} \oplus \ldots \oplus M_{E_1} \oplus \ldots \oplus M_{E_n} \oplus \ldots. \tag{2}$$

If there are m $(0 \leqq m \leqq \infty)$ non-constant φ_j's and n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure, then $\mu_T = \mu_{T^*} = \max\{m, n\}$. Moreover, if T is c.n.u., then its corresponding Jordan operator is also a weak contraction and $\mu_T = \mu_{T^*} = \max\{m, n\} \leqq d_T = d_{T^*}$ hold.

We will call the uniquely determined Jordan operator the *Jordan model* for T.

We start the proof of Theorem 3 with the following

Lemma 3. *Let T_1, T_1' be C_0 contractions on H_1, H_1' and let T_2, T_2' be unitary operators on H_2, H_2', respectively. If $T_1 \oplus T_2$ is a quasi-affine transform of $T_1' \oplus T_2'$, then T_1 is quasi-similar to T_1' and T_2 is unitarily equivalent to T_2'.*

Proof. Let $X: H_1 \oplus H_2 \to H_1' \oplus H_2'$ be a quasi-affinity such that $X(T_1 \oplus T_2) = (T_1' \oplus T_2')X$. For any $h \in H_1$, let $h_1 \oplus h_2 = X(h \oplus 0)$, where $h_1 \in H_1'$ and $h_2 \in H_2'$. Since T_1', being a C_0 contraction, is of class $C_{0\cdot}$, we have $(T_1'^n h_1) \oplus (T_2'^n h_2) = (T_1' \oplus T_2')^n X(h \oplus 0) = X(T_1 \oplus T_2)^n (h \oplus 0) = X(T_1^n h \oplus 0) \to 0$ as $n \to \infty$. Thus $T_2'^n h_2 \to 0$ as $n \to \infty$. Since T_2' is of class $C_{1\cdot}$, this implies that $h_2 = 0$, and hence that $X(h \oplus 0) \in H_1'$. Thus with respect to the decompositions $H_1 \oplus H_2$ and $H_1' \oplus H_2'$, X is triangulated as

$$X = \begin{bmatrix} X_1 & Z \\ 0 & X_2 \end{bmatrix}.$$

By considering the adjoint, a similar argument as above shows that $Z=0$. Hence we obtain quasi-affinities $X_1: H_1 \to H_1'$ and $X_2: H_2 \to H_2'$ such that $X_1 T_1 = T_1' X_1$ and $X_2 T_2 = T_2' X_2$, that is, $T_1 \prec T_1'$ and $T_2 \prec T_2'$. By the uniqueness of the Jordan model for C_0 contractions, we infer that T_1 is quasi-similar to T_1' (*cf.* [1], Theorem 1). On the other hand, that T_2 is unitarily equivalent to T_2' follows from [4], Lemma 4.1.

Lemma 4. *The operator* $S_\varphi \oplus M_E$ *on the space* $(H^2 \ominus \varphi H^2) \oplus L^2(E)$ *is cyclic.*

Proof. Let f be an essentially bounded function in $L^2(E)$, which is cyclic for M_E. If $E \neq C$, such is the identity function $1(e^{it}) = e^{it}$ on E. If $E = C$ then it is well known that the cyclic vectors for the bilateral shift are those functions $f \in L^2$ for which $|f| > 0$ a.e. and $\int \log |f| = -\infty$; we may assume that f is essentially bounded, for otherwise let $F = \{e^{it}: |f(e^{it})| \geqq 1\}$. Consider $\chi_{C \setminus F} f + \chi_F$. Let P be the (orthogonal) projection of H^2 onto $H^2 \ominus \varphi H^2$, and let 1 also denote the identity function in H^2. We want to show that $P(1) \oplus f$ is a cyclic vector for $S_\varphi \oplus M_E$.

Let $K = \bigvee_{n=0}^{\infty} (S_\varphi \oplus M_E)^n (P(1) \oplus f)$. For each $h \in H^2$, let $\{p_n\}$ be a sequence of polynomials such that $p_n \to \varphi h$ in L^2-norm. Since f is essentially bounded, we have $p_n f \to \varphi h f$, and hence $P(p_n) \oplus p_n f \to P(\varphi h) \oplus \varphi h f = 0 \oplus \varphi h f$. This shows that $0 \oplus \varphi h f$ is in K for any $h \in H^2$.

Now let g be an arbitrary function in $L^2(E)$. Since f is a cyclic vector for M_E, there exists a sequence of polynomials $\{q_n\}$ such that $q_n f \to \bar{\varphi} g$ in L^2-norm. Then $\varphi q_n f \to \varphi \bar{\varphi} g = g$. By what we proved before, we conclude that $0 \oplus g \in K$ for any $g \in L^2(E)$. On the other hand, since it is clear that $P(h) \oplus hf \in K$ for any $h \in H^2$, we have $P(h) \oplus 0 = (P(h) \oplus hf) - (0 \oplus hf) \in K$. Hence we obtain $(H^2 \ominus \varphi H^2) \oplus L^2(E) = K$, which completes the proof.

Proof of Theorem 3. Let $T = U \oplus T'$ be the decomposition of T into the direct sum of its unitary part U and its c.n.u. part T'. Since T' is also a weak contraction, we may consider its C_0 part T_0 and C_{11} part T_1. It was proved in [16] that T' is quasi-similar to $T_0 \oplus T_1$. Hence T is quasi-similar to $T_0 \oplus T_1 \oplus U$. By Theorem 1, Lemma 3 and the multiplicity theory of normal operators [5], we conclude that T is quasi-similar to a uniquely determined Jordan operator (2).

If T is quasi-similar to (2), then T^* is quasi-similar to

$$S_{\varphi_1^\sim} \oplus \ldots \oplus S_{\varphi_m^\sim} \oplus \ldots \oplus M_{E_1^\sim} \oplus \ldots \oplus M_{E_n^\sim} \oplus \ldots,$$

where $\varphi_j^\sim(\lambda) = \overline{\varphi_j(\bar{\lambda})}$ $(j=1, 2, \ldots)$ and $E_k^\sim = \{e^{it}: e^{-it} \in E_k\}$ $(k=1, 2, \ldots)$. Hence it is clear that to show that $\mu_T = \mu_{T^*} = \max\{m, n\}$, we have only to show that $\mu_T = \max\{m, n\}$. For convenience, we assume that $n \leqq m$. Let d_{T_0} and d_{T_1} denote the defect indices of T_0 and T_1, respectively. By Theorem 1 and Lemma 1 we have $m = \mu_{T_0} \leqq \mu_{T_0 \oplus T_1 \oplus U} = \mu_T$. If $m = \infty$ then we have already had the result. Hence we may assume that $m<\infty$. Since (2) is unitarily equivalent to

$$(S_{\varphi_1}\oplus M_{E_1})\oplus\ldots\oplus(S_{\varphi_n}\oplus M_{E_n})\oplus M_{E_{n+1}}\oplus\ldots\oplus M_{E_m},$$

using Lemmas 1 and 4 we have $\mu_T\leqq\underbrace{1+\ldots+1}_{n}+\underbrace{1+\ldots+1}_{m-n}=m$. Thus $\mu_T=m=$ $=\max\{m,n\}$. The case $m<n$ is similarly proved.

Now we assume that T is c.n.u., that is, $T=T_0\oplus T_1$. We show that the Jordan model (2) is a weak contraction. Indeed, it is enough to show that $S\equiv S_{\varphi_1}\oplus\ldots\oplus S_{\varphi_m}\oplus\ldots$ is weak. But S is the Jordan model of T_0, which is a weak C_0 contraction, so the assertion follows from the results of §8 of [2]. By Theorems 1 and 2 we have $m=\mu_{T_0}\leqq d_{T_0}$ and $n=\mu_{T_1}\leqq d_{T_1}$. Since $d_{T_0}\leqq d_T$ and $d_{T_1}\leqq d_T$ (cf. [9] p. 302) we obtain $\max\{m,n\}\leqq d_T=d_{T^*}$, completing the proof.

We make some remarks to conclude this section.

By Theorem 3 and Lemma 3 we infer that for weak contractions T_1, T_2, if T_1 is a quasi-affine transform of T_2 then T_1 and T_2 are quasi-similar to each other.

For c.n.u. weak contractions the unitary part of the Jordan model has an absolutely continuous spectrum.

If T is a c.n.u. weak contraction with finite defect indices, then in the Jordan model of T we have $E_k=\{e^{it}\colon \operatorname{rank}\Delta(e^{it})\geqq k\}$ $(k=1,2,\ldots,n)$, where $\Delta(e^{it})=$ $=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$ and $\Theta(\lambda)$ denotes the characteristic function of T. Indeed, since the characteristic function $\Theta_1(\lambda)$ of the C_{11} part T_1 is the purely contractive part of the outer factor $\Theta_e(\lambda)$ of $\Theta(\lambda)$, if $\Delta_1(e^{it})=[I-\Theta_1(e^{it})^*\Theta_1(e^{it})]^{1/2}$ then $\operatorname{rank}\Delta(e^{it})=\operatorname{rank}\Delta_1(e^{it})$ a.e.. Thus the assertion follows from [9] Theorem VI. 6.1. In particular, $E_1=\{e^{it}\colon \Theta(e^{it})$ is not isometric$\}$ and $n=\operatorname{ess\,sup}\operatorname{rank}\Delta(e^{it})$.

3. Multiplicity-free operators. A C_0 contraction T is called multiplicity-free if $\mu_T=1$, or equivalently, T has a cyclic vector. Some of the equivalent conditions for multiplicity-free C_0 contractions are gathered in the next theorem (*cf.* [10] and [13]).

Theorem 4. *Let T be a C_0 contraction on a separable Hilbert space. Then the following conditions are equivalent to each other:*

(i) *T is multiplicity-free;*

(ii) *T is quasi-similar to S_φ for some inner function φ;*

(iii) *$\{T\}'$ is commutative.*

Here $\{T\}'$ denotes the commutant of T.

We generalize this to the following

Theorem 5. *Let T be a c.n.u. weak contraction on a separable Hilbert space. Let T_0 and T_1 denote the C_0 and C_{11} part of T, respectively. Then the following conditions are equivalent to each other:*

(i) *T admits a cyclic vector*;
(ii) T_0 *and* T_1 *admit cyclic vectors*;
(iii) *T is quasi-similar to* $S_\varphi \oplus M_E$ *for some inner function* φ *and some measurable subset E of C (here* φ *may be constant and E may have measure zero)*;
(iv) $\{T\}'$ *is commutative*;
(v) $\{T_0\}'$ *and* $\{T_1\}'$ *are commutative.*

This theorem suggests the following

Definition. A c.n.u. weak contraction T is called *multiplicity-free* if it satisfies the equivalent conditions (i)—(v) in Theorem 5.

Note that CLARK [3] also defined multiplicity-free operators among operators of class $[C_0. \cup C_1.] \cap [C._0 \cup C._1]$. It is clear that our definition is consistent with his.

Proof of Theorem 5. The equivalence of (i), (ii) and (iii) follows from Theorems 1, 2 and 3. The implication (i)$\Rightarrow$(iv) and the equivalence (ii)$\Leftrightarrow$(v) were proved by SZ.-NAGY and FOIAŞ (*cf.* [11], [12] or [7], [13]). Thus to complete the proof we have only to show that (iv) implies one of the other conditions. Let us prove the implication (iv)$\Rightarrow$(iii). Let $S \oplus M$ denote the Jordan model of T, where $S = S_{\varphi_1} \oplus S_{\varphi_2} \oplus \dots$ and $M = M_{E_1} \oplus M_{E_2} \oplus \dots$, and let X, Y be two quasi-affinities such that

$$TX = X(S \oplus M) \quad \text{and} \quad (S \oplus M)Y = YT.$$

Then, from (iv) it follows that the relation

$$(XAY)(XBY) = (XBY)(XAY)$$

holds whenever $A, B \in \{S \oplus M\}'$ and hence

$$A(YX)B = B(YX)A.$$

Now by Lemma 3 it follows that $YX = Z \oplus V$ where $Z \in \{S\}'$, $V \in \{M\}'$ and we have

$$AZB = BZA, \quad A'VB' = B'VA' \tag{3}$$

for any $A, B \in \{S\}'$, $A', B' \in \{M\}'$. Taking $B = I$, $B' = I$ in (3), it follows that $Z \in \{S\}''$ and $V \in \{M\}''$ such that, again by (3), we infer that $\{S\}'$ and $\{M\}'$ are commutative. From the implication (v)$\Rightarrow$(ii) it follows that $S = S_{\varphi_1}$ and $M = M_{E_1}$ and (iii) follows.

We remark that conditions (i)—(v) in Theorem 5 are equivalent to the corresponding conditions for T^*. (This follows from Theorem 3 that $\mu_T = \mu_{T^*}$.) Also note that if the defect indices of T are finite, then these conditions are equivalent to:

(vi) The minors of order $d_{T_0} - 1$ of the matrix of $\Theta_{*i}(\lambda)$ have no common inner divisor, and rank $\Delta(e^{it}) \leqq 1$ a.e. (*cf.* [9], pp. 267 and 271). In particular, we have

Corollary 1. *If T is a c.n.u. contraction with scalar-valued characteristic function $\varphi(\lambda)\not\equiv 0$, then T is cyclic and $\{T\}'$ is commutative.*

Proof. T is certainly a c.n.u. weak contraction which satisfies condition (vi). The assertion follows from the remark we made above.

Part of the previous result was obtained earlier by Sz.-Nagy and Foiaş [14].

Corollary 2. *Let T be a c.n.u. multiplicity-free weak contraction on H. If K is an invariant subspace for T such that $T|K$ is also a weak contraction, then $T|K$ is multiplicity-free.*

Proof. Since T is multiplicity-free, we have $\mu_{T^*}=1$, so that $\mu_{(T|K)^*}=1$. Therefore, if $T|K$ is a weak contraction, it follows that it is multiplicity-free.

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