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C^*-algebras and derivation ranges

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Dedicated to P. R. Halmos

1. Introduction and Summary

Let T be an element of the algebra $\mathscr{B}(\mathfrak{H})$ of bounded linear operators on a complex Hilbert space $\mathfrak{H}$ and let $\delta_T(X)=TX-XT$ be the corresponding inner derivation. There are two natural closed subalgebras of $\mathscr{B}(\mathfrak{H})$ associated with T, namely the *inclusion algebra* $\mathscr{I}(T)$ of operators A for which the range $\mathscr{R}(\delta_A)$ of δ_A is contained in the norm closure $\mathscr{R}(\delta_T)^-$, and the *multiplier algebra* $\mathscr{M}(T)=$ $=\{Z\in\mathscr{B}(\mathfrak{H}):Z\mathscr{R}(\delta_T)+\mathscr{R}(\delta_T)Z\subseteqq\mathscr{R}(\delta_T)^-\}$. Most of the recent results [1, 3, 16, 19, 20, 21, 22] about the range of a derivation can be interpreted as assertions about these algebras or the two algebras that are defined similarly by replacing $\mathscr{R}(\delta_T)^-$ by $\mathscr{R}(\delta_T)$. In the finite dimensional case, $\mathscr{M}(T)=\{T\}'$ and $\mathscr{I}(T)=\{T\}''$ are the commutant and bicommutant of T.

In this paper we study the situation in which either (and, therefore, both) of these is a C^*-subalgebra of $\mathscr{B}(\mathfrak{H})$. The corresponding operators T, those for which $\mathscr{R}(\delta_T)^-=\mathscr{R}(\delta_{T^*})^-$ is a self-adjoint subspace of $\mathscr{B}(\mathfrak{H})$, are called *d-symmetric operators*. Any isometry or normal operator is d-symmetric and so is the image of a d-symmetric operator under an irreducible representation of the C^*-algebra $C^*(T)$ generated by T and the identity operator. However, if N is normal then $\mathscr{R}(\delta_N)$ is itself self-adjoint only if the spectrum of N has empty interior [11, Theorem 4.1].

If T is d-symmetric then $\mathscr{R}(\delta_T)^-$ is determined by the *T-central states* on $\mathscr{B}(\mathfrak{H})$, that is, linear functionals f with $f(I)=1=\|f\|$ and $f(TX)=f(XT)$ for all

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$X \in \mathscr{B}(\mathfrak{H})$. In fact, $\mathscr{R}(\delta_T)^-$ is even determined by those pure states of $\mathscr{B}(\mathfrak{H})$ whose restrictions to $\mathscr{I}(T)$ are multiplicative. These satisfy $f(AX)=f(XA)=f(A)f(X)$ for $X \in \mathscr{B}(\mathfrak{H})$ and $A \in \mathscr{I}(T)$ so that in particular $C^*(T)$ must have a character.

The C^*-algebra $C(T)$ of operators C for which $C\mathscr{B}(\mathfrak{H})+\mathscr{B}(\mathfrak{H})C$ is contained in $\mathscr{R}(\delta_T)^-$ plays a fundamental role here. For example, T is d-symmetric if and only if $T^*T-TT^* \in \mathscr{C}(T)$ and a d-symmetric operator has the Fuglede property: $TX-XT \in \mathscr{C}(T)$ for some operator X on $\mathfrak{H}$ only if $T^*X-XT^* \in \mathscr{C}(T)$. For a d-symmetric operator $\mathscr{C}(T)$ coincides with the commutator ideal of $\mathscr{I}(T)$. It is non-separable in general.

The inclusion algebra $\mathscr{I}(T)$ of a d-symmetric operator T is identified in the two extreme cases in which T has no reducing eigenvalues (complex numbers λ for which $\ker(T-\lambda I)$ reduces T) and in which T has a spanning set of orthonormal eigenvectors (T is a *diagonal operator*). In the first case $\mathscr{I}(T)=C^*(T)+\mathscr{C}(T)$, while in the second $\mathscr{I}(T)$ is the C^*-algebra generated by T and those projections onto eigenspaces corresponding to eigenvalues of finite multiplicity that are limit points of the spectrum of T.

Various criteria for d-symmetry are given in § 2 and the ideal $\mathscr{C}(T)$ is studied in § 3. We study the T-central states in § 4, present examples, counterexamples and information about special cases in § 5, and mention several questions we have been unable to resolve in the final section of the paper.

2. Conditions for d-symmetry

The proof of our first result was inspired by ROSENBLUM's proof [14] of the Fuglede theorem.

Theorem 2.1. *For T in $\mathscr{B}(\mathfrak{H})$ the following are equivalent:*

(a) *T is d-symmetric,* (b) $T^*T-TT^* \in \mathscr{C}(T)$,

(c) $T^*\mathscr{R}(\delta_T)+\mathscr{R}(\delta_T)T^* \subseteqq \mathscr{R}(\delta_T)^-$.

Proof. The equivalence of (b) and (c) is a consequence of the identities:

$$(T^*T-TT^*)X = T^*\delta_T(X)-\delta_T(T^*X), \quad X(T^*T-TT^*) = \delta_T(X)T^*-\delta_T(XT^*).$$

Since $T^*\delta_{T^*}(X)=\delta_{T^*}(T^*X)$ and $\delta_{T^*}(X)T^*=\delta_{T^*}(XT^*)$, (a) implies (c).

Now assume (c) holds. To prove that T is d-symmetric it suffices to show that $f(\mathscr{R}(\delta_{T^*}))=0$ for all f in $\mathscr{B}(\mathfrak{H})^*$ satisfying $f(\mathscr{R}(\delta_T))=0$. If $X \in \mathscr{B}(\mathfrak{H})$ then $f(TX)=f(XT)$ and

$$f(T^{*n}TX) = f(T^{*n}(TX-XT))+f(T^{*n}XT) = 0+f(T^{*n}XT) = f(TT^{*n}X)$$

since $T^*\mathscr{R}(\delta_T) \subseteqq \mathscr{R}(\delta_T)^-$ by (c). By induction $f(T^{*n}T^mX)=f(T^mT^{*n}X)$ for all non-negative integers n and m. From this one obtains

$$f(\exp(\alpha T+\beta T^*)X)=f(\exp(\alpha T)\exp(\beta T^*)X)=f(\exp(\beta T^*)\exp(\alpha T)X)$$

for all complex numbers α and β by imitating the standard proof of the identity $\exp(A+B)=\exp(A)\exp(B)$ (for commuting A and B) as given in [12, p. 397] for example. A similar argument, using $\mathscr{R}(\delta_T)T^*\subseteqq\mathscr{R}(\delta_T)^-$, gives

$$f(X\exp(\alpha T+\beta T^*))=f(X\exp(\alpha T)\exp(\beta T^*)=f(X\exp(\beta T^*)\exp(\alpha T)).$$

Since $f(TX)=f(XT)$, it follows by induction that $f(T^nX)=f(XT^n)$ for all n and hence $f(\exp(\alpha T)X)=f(X\exp(\alpha T))$ or $f(\exp(\alpha T)X\exp(-\alpha T))=f(X)$. These relations yield:

$$f(\exp(i\lambda T^*)X\exp(-i\lambda T))=f(\exp(i\bar{\lambda}T)\exp(i\lambda T^*)X\exp(-i\lambda T^*)\exp(-i\bar{\lambda}T))=$$
$$=f(\exp(i2\operatorname{Re}(\bar{\lambda}T))X\exp(-i2\operatorname{Re}(\bar{\lambda}T)))$$

for any complex λ. The right hand side of this equation is bounded, so by Liouville's theorem the entire function on the left hand side must be constant. In particular, the derivative vanishes at $\lambda=0$. This gives $f(T^*X-XT^*)=0$. □

Corollary 2.2. *Every normal operator is d-symmetric.*

Theorem 2.3. *Every isometry V is d-symmetric.*

Proof. If $Q=I-VV^*$ then $\delta_{V^*}(X)=\delta_V(-V^*XV^*)-QXV^*$ so it suffices to show that $QX\in\mathscr{R}(\delta_V)^-$ for all $X\in\mathscr{B}(\mathfrak{H})$. Let $T_n=\sum_{k=0}^{n-1}(k/n-1)V^kQXV^{*(k+1)}$ for $n=2,3,\ldots$. Then $\delta_V(T_n)-QX=-n^{-1}\sum_{k=1}^{n}V^kQXV^{*k}$. Since $(V^jQx,V^kQy)=0$ for $j\neq k$ and x,y in $\mathfrak{H}$,

$$\left\|\sum_{k=1}^{n}V^kQXV^{*k}x\right\|^2=\sum_{k=1}^{n}\|V^kQXV^{*k}x\|^2\leqq n\|QX\|^2\|x\|^2.$$

Hence $n^{-1}\|\sum_{k=1}^{n}V^kQXV^{*k}\|\leqq n^{-1/2}\|QX\|$ and $QX\in\mathscr{R}(\delta_V)^-$.

Remark. The proof of 2.3 shows that $Q\mathscr{B}(\mathfrak{H})\subseteqq\mathscr{R}(\delta_V)^-$. The closure cannot be deleted here, however, as $\mathscr{R}(\delta_V)$ contains no non-zero right ideal of $\mathscr{B}(\mathfrak{H})$ [21]. But $\mathscr{R}(\delta_V)$ *does* contain the left ideal of $\mathscr{B}(\mathfrak{H})$ generated by Q [18].

Let $\mathscr{K}=\mathscr{K}(\mathfrak{H})$ denote the compact operators on $\mathfrak{H}$. An operator T is *essentially d-symmetric* if it is d-symmetric in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}$, that is, if $[\nu(T),\nu(\mathscr{B}(\mathfrak{H}))]^-$ is a self-adjoint subspace of the Calkin algebra. (Here ν denotes the canonical homomorphism of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{B}(\mathfrak{H})/\mathscr{K}$.) We now determine the relationship between d-symmetric and essentially d-symmetric operators.

A closed subspace of $\mathscr{B}(\mathfrak{H})$ is self-adjoint if and only if its annihilator $\mathscr{F}$ is self-adjoint in the sense that $f\in\mathscr{F}$ implies $f^*\in\mathscr{F}$, where $f^*(X)=f(X^*)^*$. Now each

$f \in \mathscr{B}(\mathfrak{H})^*$ has a unique representation $f = f_0 + f_J$ where f_0 is a bounded linear functional on $\mathscr{B}(\mathfrak{H})$ that vanishes on $\mathscr{K}$ and f_J is induced by an operator J in the trace class by the formula $f_J(X) = \text{trace}(XJ)$ for X in $\mathscr{B}(\mathfrak{H})$. (See [9, 2.11.7 and 4.1.2].) Moreover, $f = f_0 + f_J$ is T-central for an operator T if and only if both f_0 and f_J are T-central, and f_J is T-central if and only if $TJ = JT$ [20, Theorem 3]. These facts give

Proposition 2.4. *An operator T on $\mathfrak{H}$ is d-symmetric if and only if*

(a) *T is essentially d-symmetric, and*

(b) *$TJ = JT$ for an operator J in the trace class implies $TJ^* = J^*T$.*

Corollary 2.5. (a) *An essentially normal operator T is d-symmetric if and only if $TJ = JT$ for an operator J in the trace class implies $TJ^* = J^*T$.*

(b) *An operator in the trace class is d-symmetric if and only if it is normal.*

Proof. Since the proof of Theorem 2.1 is valid in any C^*-algebra, any essentially normal operator is essentially d-symmetric. □

Corollary 2.6. *The following are equivalent for a d-symmetric operator T:*

(a) $\mathscr{K} \subseteqq \delta_T(\mathscr{K})^-$.

(b) $\mathscr{K} \subseteqq \mathscr{R}(\delta_T)^-$.

(c) *T has no reducing eigenvalues.*

Proof. If T has a reducing eigenvalue, then $(Sx, x) = 0$ for all S in $\mathscr{R}(\delta_T)^-$ and some non-zero $x \in H$ and $\mathscr{K}$ non $\subseteqq \mathscr{R}(\delta_T)^-$. Thus, (b) implies (c). If $\delta_T(\mathscr{K})$ is not dense in $\mathscr{K}$, then since $\mathscr{K}^*$ is the trace class operators, there is a non-zero J in the trace class such that f_J vanishes on $\delta_T(\mathscr{K})$, that is, $TJ = JT$. Since T is d-symmetric $TJ^* = J^*T$ by 2.4 and T commutes with a non-zero self-adjoint trace class operator. Therefore T has a finite dimensional reducing subspace $\mathscr{M}$. Clearly, any direct summand of a d-symmetric operator is d-symmetric so $T|\mathscr{M}$ is normal by 2.5(b). Hence T has a reducing eigenvalue and (c) implies (a). The remaining implication is obvious. □

Remarks. (a) If S and T are d-symmetric operators with disjoint spectra, then an easy application of Rosenblum's theorem [13] shows that $S \oplus T$ is d-symmetric.

(b) If λ is an eigenvalue of T but $\bar{\lambda}$ is not an eigenvalue of T^*, then $T \oplus \lambda I$ is not d-symmetric, where I is the identity on any non-zero Hilbert space. In particular, if U_+ denotes the unilateral shift and $|\lambda| < 1$, $U_+ \oplus \lambda I$ is not d-symmetric. However, if $|\lambda| \geqq 1$ then 2.4(b) and a calculation show that $U_+ \oplus \lambda I$ is d-symmetric.

(c) STAMPFLI [16] constructed a compact weighted shift K that commutes with no non-zero trace class operator and therefore $\mathscr{R}(\delta_K)^- = \mathscr{K}$. This operator K is then d-symmetric and quasinilpotent. As $K \oplus n^{-1} I$ is d-symmetric by (a) above and $K \oplus 0$ is not d-symmetric by (b), it follows that *the set of d-symmetric operators is not norm closed.* Stampfli has independently pointed out this same fact to us.

The proof of our next theorem requires non-separable versions of two known results. We now present these (slight) generalizations. Let $\mathscr{A}$ denote a unital separable C^*-algebra of operators in $\mathscr{B}(\mathfrak{H})$, where $\mathfrak{H}$ is *separable*. In [17] Voiculescu showed that if π is a representation of $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H}_\pi)$, where $\mathfrak{H}_\pi$ is separable and $\pi(\mathscr{A} \cap \mathscr{K}) = 0$, then there is a sequence of unitary transformations U_n of $\mathfrak{H} \oplus \mathfrak{H}_\pi$ onto $\mathfrak{H}$ such that $A - U_n(A \oplus \pi(A)) U_n^*$ is compact for all A in $\mathscr{A}$ and $\lim_n \|A - U_n(A \oplus \pi(A)) U_n^*\| = 0$ for all A in $\mathscr{A}$. (In symbols, $\mathrm{id} \underset{a}{\sim} \mathrm{id} \oplus \pi$). This fact was used in [2] to show that if f is a state on $\mathscr{A}$ that is zero on $\mathscr{A} \cap \mathscr{K}$ then f extends to a pure state on $\mathscr{B}(\mathfrak{H})$.

Proposition 2.7. *Let $\mathscr{A}$ denote a unital separable C^*-algebra of operators acting on a Hilbert space $\mathfrak{H}$ (of any dimension).*

(a) *If π is a representation of $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H}_\pi)$, where $\mathfrak{H}_\pi$ is separable and $\pi(\mathscr{A} \cap \mathscr{K}) = 0$, then* $\mathrm{id} \underset{a}{\sim} \mathrm{id} \oplus \pi$.

(b) *If f is a state on $\mathscr{A}$ such that f vanishes on $\mathscr{A} \cap \mathscr{K}$, then f extends to a pure state on $\mathscr{B}(\mathfrak{H})$.*

Proof. Choose a dense sequence $\{A_n\}$ of operators in $\mathscr{A}$ and select unit vectors in $\mathfrak{H}$ as follows. For each n choose an infinite orthonormal sequence $\{e_{nk}\}$ such that $\|A_n\|_e = \lim_k \|A_n e_{nk}\|$ and choose a sequence $\{x_{nj}\}$ such that $\|A_n\| = \lim_j \|A_n x_{nj}\|$. (Here, $\|A_n\|_e$ denotes the norm of $A_n + \mathscr{K}$ in $\mathscr{B}(\mathfrak{H})/\mathscr{K}$.) Write $\mathfrak{M}$ for the subspace of $\mathfrak{H}$ generated by $\{\mathscr{A} e_{nk}\} \cup \{\mathscr{A} x_{nj}\}$. Then the restriction map Φ induced by the projection P of $\mathfrak{H}$ onto M is an isometric isomorphism. Furthermore, an operator A in $\mathscr{A}$ is compact if and only if $\Phi(A)$ is compact. Now suppose π is a representation of $\mathscr{A}$ as in (a) above. Then $\pi' = \pi \circ \Phi^{-1}$ is a representation of $\Phi(\mathscr{A})$ which satisfies the hypotheses of Voiculescu's theorem. (If $\Phi(A)$ is compact, then $\pi'(\Phi(A)) = \pi(A) = 0$ since A is compact.) Hence $\mathrm{id}_{\Phi(A)} \underset{a}{\sim} \mathrm{id}_{\Phi(A)} \oplus \pi'$. Let Ψ denote the restriction of $\mathscr{A}$ to $P^\perp \mathfrak{H}$. Then

$$\mathrm{id}_{\mathscr{A}} = \Psi \oplus \Phi \underset{a}{\sim} \Psi \oplus (\mathrm{id}_{\Phi(\mathscr{A})} \oplus \pi') = \mathrm{id}_{\mathscr{A}} \oplus \pi$$

and (a) is established.

Now suppose that f is a state on $\mathscr{A}$ that is zero on $\mathscr{A} \cap \mathscr{K}$. Then $f' = f \circ \Phi^{-1}$ is a state on $\Phi(\mathscr{A})$ that is zero on $\Phi(\mathscr{A}) \cap \mathscr{K}(\mathfrak{M})$ and so by [2] there is a pure state g' on $\mathscr{B}(\mathfrak{M})$ which extends f'. Define g'' on $\mathscr{B}(P^\perp \mathfrak{H}) \oplus \mathscr{B}(\mathfrak{M})$ by $g''(X \oplus Y) = g'(Y)$. Then g'' is a pure state on $\mathscr{B}(P^\perp \mathfrak{H}) \oplus \mathscr{B}(\mathfrak{M})$ and if $A \in \mathscr{A}$, $A = A_1 \oplus \Phi(A)$

and $g''(A)=g'(\Phi(A))=f'(\Phi(A))=f(A)$. Thus g'' is a pure state that extends f, so we may choose a pure state on $\mathscr{B}(\mathfrak{H})$ that extends g''. □

Recall that a representation $\pi:\mathscr{A}\to\mathscr{B}(\mathfrak{H}_\pi)$ of a C^*-algebra $\mathscr{A}$ into the operators on the Hilbert space $\mathfrak{H}_\pi$ is called *cyclic* if there is a vector x in $\mathfrak{H}_\pi$ such that $\pi(\mathscr{A})x$ is dense in $\mathfrak{H}_\pi$.

Theorem 2.8. *If T is a d-symmetric operator on a Hilbert space $\mathfrak{H}$ and $\pi:C^*(T)\to\mathscr{B}(\mathfrak{H}_\pi)$ is a cyclic representation such that either*
(a) $\pi(C^*(T)\cap\mathscr{K})=0$, *or* (b) $\pi(C^*(T))$ *is irreducible, then $\pi(T)$ is d-symmetric.*

Proof. Assume that $\pi(C^*(T)\cap\mathscr{K})=0$. Then by Proposition 2.7(a) there is a unitary transformation U mapping $\mathfrak{H}\oplus\mathfrak{H}_\pi$ onto $\mathfrak{H}$ such that $T-U(T\oplus\pi(T))U^*$ is compact. Since T is d-symmetric, $U(T\oplus\pi(T))U^*$ is essentially d-symmetric and so $T\oplus\pi(T)$ is essentially d-symmetric. Therefore $\pi(T)$ is essentially d-symmetric. Let f denote a $\pi(T)$-central bounded linear functional on $\mathscr{B}(\mathfrak{H}_\pi)$. We must show that f^* is $\pi(T)$-central. Write $f=f_0+f_w$ where f_0 vanishes on $\mathscr{K}(\mathfrak{H}_\pi)$ and f_w is ultraweakly continuous (that is, induced by a trace class operator.) Then f_0 and f_w are $\pi(T)$-central [20] and f_0^* is $\pi(T)$-central because $\pi(T)$ is essentially d-symmetric. We need only show that f_w^* is $\pi(T)$-central. Fix a cyclic vector x for $\pi(C^*(T))$ and define a state ω on $C^*(T)$ by $\omega(A)=(\pi(A)x,x)$. Since $C^*(T)$ is separable and ω vanishes on $\mathscr{A}\cap\mathscr{K}$, there is a pure state ϱ on $\mathscr{B}(\mathfrak{H})$ that extends ω by Proposition 2.7(b). It follows (as in [9, 2.10.2]) that there is a Hilbert space $\mathfrak{H}'$ containing $\mathfrak{H}_\pi$ and an irreducible representation π' of $\mathscr{B}(\mathfrak{H})$ in $\mathscr{B}(\mathfrak{H}')$ such that the projection P of $\mathfrak{H}'$ onto $\mathfrak{H}_\pi$ reduces $\pi'(C^*(T))$ and $P\pi'(A)|\mathfrak{H}_\pi=\pi(A)$ for all A in $C^*(T)$. Define a linear functional g on $\mathscr{B}(\mathfrak{H})$ by $g(X)=f_w(P\pi'(X)P|\mathfrak{H}_\pi)$.) Then

$$g(TX)=f_w(P\pi'(TX)P|\mathfrak{H}_\pi)=f_w(\pi(T)P\pi'(X)P|\mathfrak{H}_\pi)=$$
$$=f_w((P\pi'(X)P|\mathfrak{H}_\pi)\pi(T))=f_w(P\pi'(XT)P|\mathfrak{H}_\pi)=g(XT),$$

since f_w is $\pi(T)$-central. Thus g is T-central; so, since T is d-symmetric, g is T^*-central. Therefore, for all X in $\mathscr{B}(\mathfrak{H})$ we have

$$f_w((P\pi'(X)P|\mathfrak{H}_\pi)\pi(T)^*)=f_w(\pi(T)^*P\pi'(X)P|\mathfrak{H}_\pi).$$

Since π' is irreducible and f_w is ultraweakly continuous, f_w is $\pi(T)^*$-central.

Now suppose that π is irreducible. By the first part of the proof, we may assume that π is not zero on $C^*(T)\cap\mathscr{K}$. Then $\pi_0=\pi|C^*(T)\cap\mathscr{K}(\mathfrak{H})$ is irreducible [9, 2.11.3] and by [5, 1.4.4] there is a subspace $\mathfrak{M}$ of $\mathfrak{H}$ such that π_0 is unitarily equivalent to the restriction to $\mathfrak{M}$ of the identity representation of $C^*(T)\cap\mathscr{K}$. Since $C^*(T)\cap\mathscr{K}$ is irreducible on $\mathfrak{M}$, $\mathfrak{M}$ must reduce T. A similar argument shows that $\pi(C^*(T))$ is unitarily equivalent to $C^*(T)|\mathfrak{M}$. Thus $\pi(T)$ is unitarily equivalent to a direct summand of T and $\pi(T)$ is d-symmetric. □

Remarks. (a) The operator $T\oplus\pi(T)$ in the proof of Theorem 2.8 need not be d-symmetric. Indeed, let K denote the compact d-symmetric operator in Remark (c) following 2.6 and define π on $C^*(K)=\mathcal{K}(\mathfrak{H})+\mathbf{C}I$ by $\pi(K_1+\lambda I)=\lambda$. Then $K\oplus\pi(K)=K\oplus 0$ is not a d-symmetric operator.

(b) If T is d-symmetric and π is an irreducible representation of $C^*(T)$ then $\mathfrak{H}_\pi$ is either infinite dimensional or one dimensional. For $\pi(T)$ is d-symmetric by 2.8 and if $\mathfrak{H}_\pi$ has dimension $n<\infty$, then $\pi(T)$ is normal (2.5(b)), and irreducible, hence $n=1$. Thus, *if T is essentially n-normal and d-symmetric, then T is essentially normal.*

(c) We shall show (3.6) that if T is d-symmetric, then $C^*(T)$ has a character. Hence $C^*(\pi(T))$ has a character for every irreducible representation π of $C^*(T)$.

3. The inclusion and multiplier algebras

As noted prior to Proposition 2.4 d-symmetry of an operator is equivalent to the condition that the annihilator of its derivation range be a self-adjoint subspace of $\mathcal{B}(\mathfrak{H})^*$. We now show that the annihilator is actually determined by the states it contains.

Let $E(T)$ denote the set of all T-central states on $\mathcal{B}(\mathfrak{H})$; that is, the set of states f on $\mathcal{B}(\mathfrak{H})$ such that $f(TX)=f(XT)$ for all X in $\mathcal{B}(\mathfrak{H})$.

Theorem 3.1. *If T is a d-symmetric operator, then $\mathcal{R}(\delta_T)^-=\bigcap\{\ker(f):f\in E(T)\}$.*

Proof. Fix $f=f^*$ in the annihilator of $\mathcal{R}(\delta_T)$. Then there are unique positive linear functionals f^+ and f^- on $\mathcal{B}(\mathfrak{H})$ such that $f=f^+-f^-$ and $\|f\|=\|f^+\|+\|f^-\|$ [9, 12.3.4]. To prove the theorem, it suffices to show that f^+ and f^- are T-central. To do this we use an argument due to Effros and Hahn [10, p. 24].

Since f is self-adjoint, the set $\{A\in\mathcal{B}(\mathfrak{K}): f(AX)=f(XA)$ for all X in $\mathcal{B}(\mathfrak{H})\}$ is a C^*-algebra containing T. Fix a unitary owerator U in $C^*(T)$ and write $g_1(X)=f^+(U^*XU)$, $g_2(X)=f^-(U^*XU)$ for X in $\mathcal{B}(\mathfrak{H})$. Then g_1 and g_2 are positive linear functionals with $g_1(X)-g_2(X)=f(U^*XU)=f(X)$ and $\|g_1\|+\|g_2\|=$ $=g_1(I)+g_2(I)=\|f\|$. So, by the uniqueness of the decomposition of f, $f^+=g_1$ and $f^-=g_2$. Hence f^+ and f^- are U-central for every unitary U in $C^*(T)$. Since the unitaries in $C^*(T)$ span $C^*(T)$, f^+ and f^- are T-central. □

Remark. The proof of Theorem 3.1 shows that an operator T has a T-central state if and only if the commutator subspace $[C^*(T),\mathcal{B}(\mathfrak{H})]$ is not norm dense in $\mathcal{B}(\mathfrak{H})$. (See [6].) This is equivalent to the non-density of $\mathcal{R}(\delta_T)+\mathcal{R}(\delta_{T^*})$ or to the condition that 0 belong to the closure of the numerical range of every commutator $TX-XT$ [19]. The mere existence of a central state, however, does not imply d-symmetry as the example $T=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ shows.

Corollary 3.2. *If T is a d-symmetric operator, then:*

(a) $\mathscr{R}(\delta_T)^-$ *is an hereditary subspace of* $\mathscr{B}(\mathfrak{H})$*: that is, if* $0 \leqq X \leqq Y$ *and* $Y \in \mathscr{R}(\delta_T)^-$*, then* $X \in \mathscr{R}(\delta_T)^-$.

(b) $\mathscr{R}(\delta_A) \subseteqq \mathscr{R}(\delta_T)^-$ *for all A in* $C^*(T)$.

(c) $\mathscr{C}(T)$ *is the linear span of the positive elements in* $\mathscr{R}(\delta_T)^-$ *and* $\mathscr{C}(T)$ *is hereditary in* $\mathscr{B}(\mathfrak{H})$.

(d) $\mathscr{K}(\mathfrak{H}) \subseteqq \mathscr{C}(T)$ *if and only if T has no reducing eigenvalues.*

Proof. Parts (a) and (b) are clear from 3.1. We prove part (c). If C is a positive operator in $\mathscr{R}(\delta_T)^-$, then $f(C)=0$ for each T-central state f, and so, $|f(XC^{1/2})|^2 \leqq \leqq f(XX^*)\, f((C^{1/2})^2)=0$. Similarly, $f(C^{1/2}X)=0$. Hence, by Theorem 3.1, $C^{1/2} \in \mathscr{C}(T)$ and so $C = C^{1/2}C^{1/2} \in \mathscr{C}(T)$. On the other hand, if $C \in \mathscr{C}(T)$ is self-adjoint with spectral measure $E(\cdot)$, then $C = CE([0, \infty)) + E((-\infty, 0))C$ is a linear combination of positive operators in $\mathscr{R}(\delta_T)^-$. $\mathscr{C}(T)$ is a hereditary subspace of $\mathscr{B}(\mathfrak{H})$ by (a). Part (d) follows from (c) and 2.6. □

We now study the sets $\mathscr{C}(T)$, $\mathscr{I}(T)$, and $\mathscr{M}(T)$ in more detail.

Theorem 3.3: *If T is a d-symmetric operator, then:*

(a) $\mathscr{C}(T)$, $\mathscr{I}(T)$, *and* $\mathscr{M}(T)$ *are C^*-algebras.*

(b) $\mathscr{C}(T)$ *is a norm closed two-sided ideal in* $\mathscr{M}(T)$ *which is properly contained in* $\mathscr{I}(T)$. *Furthermore,* $\mathscr{I}(T) \subseteqq \mathscr{I}(T) + \{T\}' \subseteqq \mathscr{M}(T)$.

(c) $\mathscr{I}(T)/\mathscr{C}(T)$ *is contained in the center of* $\mathscr{M}(T)/\mathscr{C}(T)$.

(d) $\mathscr{M}(T) = \{Z \in \mathscr{B}(\mathfrak{H}) : [Z, \mathscr{I}(T)] \subseteqq \mathscr{C}(T)\} = \{Z \in \mathscr{B}(\mathfrak{H}) : [Z, T] \in \mathscr{C}(T)\} = = \{Z \in \mathscr{B}(\mathfrak{H}) : [Z, T] \in \mathscr{I}(T)\}$.

(e) $\mathscr{C}(T) = \mathscr{I}(T) \cap \mathscr{R}(\delta_T)^- = \mathscr{M}(T) \cap \mathscr{R}(\delta_T)^-$.

Proof. As $\mathscr{R}(\delta_T)^-$ is self-adjoint it is clear that $\mathscr{M}(T)$ and $\mathscr{C}(T)$ are C^*-algebras. It is also clear that $\mathscr{C}(T) \subseteqq \mathscr{I}(T)$ and that $\{T\}' \subseteqq \mathscr{M}(T)$.

If $A \in \mathscr{I}(T)$ then $A\delta_T(X) = \delta_T(AX) + \delta_A(TX) - T\delta_A(X)$ is in $\mathscr{R}(\delta_T)^-$. Hence $A\mathscr{R}(\delta_T) \subseteqq \mathscr{R}(\delta_T)^-$. Similarly $\mathscr{R}(\delta_T)A \subseteqq \mathscr{R}(\delta_T)^-$ so that $\mathscr{I}(T) \subseteqq \mathscr{M}(T)$. Therefore if $A_1, A_2 \in \mathscr{I}(T)$ then $A_1A_2X - XA_1A_2 = A_1(A_2X - XA_2) + (A_1X - XA_1)A_2 \in \in \mathscr{R}(\delta_T)^-$ and $A_1A_2 \in \mathscr{I}(T)$. Hence $\mathscr{I}(T)$ is a norm closed subalgebra of $\mathscr{B}(\mathfrak{H})$. Since $\mathscr{R}(\delta_T)^-$ is self-adjoint, $\mathscr{I}(T)$ is a C^*-algebra.

If $Z \in \mathscr{M}(T)$, $C \in \mathscr{C}(T)$, and X is any operator then

$$X(CZ) = (XC)Z \in \mathscr{R}(\delta_T)^- Z \subseteqq \mathscr{R}(\delta_T)^- \quad \text{and} \quad (CZ)X = C(ZX) \in \mathscr{R}(\delta_T)^-.$$

Hence $\mathscr{C}(T)$ is right ideal of $\mathscr{M}(T)$. Since $\mathscr{C}(T)$ is a C^*-algebra, it is a norm closed two sided ideal of $\mathscr{M}(T)$. Also, $I \notin \mathscr{C}(T)$ because $\mathscr{R}(\delta_T)^- \neq \mathscr{B}(\mathfrak{H})$ [16, Theorem 1], so $\mathscr{C}(T)$ is properly contained in $\mathscr{I}(T)$. This proves (a) and (b).

If $Z \in \mathscr{M}(T)$, $A \in \mathscr{I}(T)$, and X is any operator, then

$$\delta_Z(A)X = Z\delta_A(X) - \delta_A(ZX) \in \mathscr{R}(\delta_T)^- \quad \text{and} \quad X\delta_Z(A) = \delta_A(X)Z - \delta_A(XZ) \in \mathscr{R}(\delta_T)^-.$$

Hence $\delta_Z(A)\in\mathscr{C}(T)$. This proves (c). It also shows that

$$\mathscr{M}(T)\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,\mathscr{I}(T)]\subseteqq\mathscr{C}(T)\}\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,T]\in\mathscr{C}(T)\}\subseteqq$$
$$\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,T]\in\mathscr{I}(T)\}.$$

Before showing the reverse inclusions, we establish (e). Suppose that $A\in\mathscr{M}(T)\cap\cap\mathscr{R}(\delta_T)^-$. Then $AA^*+A^*A\in A\mathscr{R}(\delta_T)^-+\mathscr{R}(\delta_T)^-A\subseteqq\mathscr{R}(\delta_T)^-$. Hence both AA^* A^*A belong to $\mathscr{C}(T)$ as $\mathscr{C}(T)$ is hereditary. By considering the polar decompositions of A and A^* one gets $A\in\mathscr{C}(T)$. Thus $\mathscr{M}(T)\cap\mathscr{R}(\delta_T)^-\subseteqq\mathscr{C}(T)$ and the inclusions $\mathscr{C}(T)\subseteqq\mathscr{I}(T)\cap\mathscr{R}(\delta_T)^-\subseteqq\mathscr{M}(T)\cap\mathscr{R}(\delta_T)^-$ are trivial.

To finish the proof of (d), suppose $\delta_T(Z)\in\mathscr{I}(T)$ and X is an operator. Then $\delta_T(Z)\in\mathscr{C}(T)$ by (e) and so

$$Z\delta_T(X)=\delta_T(ZX)-\delta_T(Z)X\in\mathscr{R}(\delta_T)^-\quad\text{and}\quad\delta_T(X)Z=\delta_T(XZ)-X\delta_T(Z)\in\mathscr{R}(\delta_T)^-.$$

Hence $Z\in\mathscr{M}(T)$. □

The following is a version of the Fuglede theorem for d-symmetric operators.

Corollary 3.4. *Let T be d-symmetric and let $X\in\mathscr{B}(\mathfrak{H})$. If $TX-XT\in\mathscr{C}(T)$ then $TX^*-X^*T\in\mathscr{C}(T)$.*

Example. Let K denote an irreducible compact operator that does not commute with any trace class operator (as in remark (c) following 2.6, for example.) Then $\mathscr{C}(K)=\mathscr{R}(\delta_K)^-=\mathscr{K}$ so that $\mathscr{M}(K)=\mathscr{B}(\mathfrak{H})$ and $\mathscr{I}(K)=\mathscr{K}+\mathbf{C}I$ by [7, Theorem 2.9].

We now show that $\mathscr{C}(T)$ is the commutator ideal of $\mathscr{I}(T)$ if T is a d-symmetric operator.

Recall [4, § 3.3] that the *commutator ideal* Comm $(\mathscr{A})$ of a C^*-algebra $\mathscr{A}$ is the smallest closed two-sided ideal of $\mathscr{A}$ containing all of the commutators $A_1A_2-A_2A_1$ for A_1, A_2 in $\mathscr{A}$. Comm $(\mathscr{A})$ is also the smallest closed ideal $\mathscr{C}$ such that $\mathscr{A}/\mathscr{C}$ is commutative and, furthermore, $\operatorname{Comm}\mathscr{A}=\bigcap\ker(\varphi)$, where the intersection is taken over all the characters (non-zero complex homomorphisms) of $\mathscr{A}$. If T is an operator, then $\operatorname{Comm}(C^*(T))=\operatorname{Comm}C^*(T)$ is the ideal generated by T^*T-TT^*.

We make use of the fact that *if f is a state on a C^*-algebra $\mathscr{B}$ whose restriction φ to a C^*-subalgebra $\mathscr{A}$ is a character, then f is $\mathscr{A}$-multiplicative on $\mathscr{B}$* in the sense that $f(XA)=f(X)f(A)=f(AX)$ for all X in $\mathscr{B}$ and all A in $\mathscr{A}$. Indeed, $A-f(A)I$ belongs to the left kernel of f because $\varphi\big((A-\varphi(A))^*(A-\varphi(A))\big)=0$.

Theorem 3.5. *If T is a d-symmetric operator, then:*

(a) $\mathscr{C}(T)=\operatorname{Comm}\big(\mathscr{I}(T)\big)$.

(b) $\operatorname{Comm}C^*(T)=C^*(T)\cap\mathscr{R}(\delta_T)^-=C^*(T)\cap\mathscr{C}(T)$.

(c) *The map α of $C^*(T)/\operatorname{Comm}C^*(T)$ into $\mathscr{I}(T)/\mathscr{C}(T)$ given by $\alpha\big(A+\operatorname{Comm}C^*(T)\big)=A+\mathscr{C}(T)$ is an isomorphism.*

Proof. Let φ be a character on $\mathcal{I}(T)$ and let f be any extension of φ to a state on $\mathcal{B}(\mathfrak{H})$. Then f is T-central by the remark preceding the statement of the theorem, hence $\varphi(\mathcal{C}(T))=f(\mathcal{C}(T))=0$ as $\mathcal{C}(T)\subseteq\mathcal{R}(\delta_T)^-$. Thus $\mathcal{C}(T)\subseteq\text{Comm}\,\mathcal{I}(T)$. The reverse inclusion is clear from Theorem 3.3(c).

The same remark shows that any character of $C^*(T)$ vanishes on $C^*(T)\cap\cap\mathcal{R}(\delta_T)^-$ so that $C^*(T)\cap\mathcal{R}(\delta_T)^-\subseteq\text{Comm}\,C^*(T)\subseteq C^*(T)\cap\mathcal{C}(T)\subseteq C^*(T)\cap\cap\mathcal{R}(\delta_T)^-$ by the first part of the argument. This proves (b) and (c) is then clear. □

Corollary 3.6. *If T is a d-symmetric operator, then $C^*(T)$ has a character.*

Proof. Since $I\notin\mathcal{C}(T)$ (3.3(b)), $\text{Comm}\,C^*(T)\neq C^*(T)$ by 3.5(b).

Remark. Note that $C^*(T)$ may have only one character, however. For example, this is the case for the compact operator considered in the example following 3.4.

We now derive additional results about the inclusion and multiplier algebras under the additional hypothesis that T has no reducing eigenvalues. Then $\mathcal{R}(\delta_T)^-\supseteq\mathcal{K}$ by 2.6 and so $\mathcal{C}(T)\supseteq\mathcal{K}$.

Theorem 3.7. *If T is a d-symmetric operator that has no reducing eigenvalues, then*

(a) $\mathcal{I}(T)=C^*(T)+\mathcal{C}(T)$, (b) *The center of* $\mathcal{M}(T)/\mathcal{C}(T)$ is $\mathcal{I}(T)/\mathcal{C}(T)$.

Proof. Suppose there is an operator S in $\mathcal{I}(T)$ such that $S\notin C^*(T)+\mathcal{C}(T)$. Then the commutative C^*-algebra $(C^*(S,T)+\mathcal{C}(T))/\mathcal{C}(T)$ properly contains $(C^*(T)+\mathcal{C}(T))/\mathcal{C}(T)$ and so by the Stone—Weierstrass theorem, there are distinct characters φ_1 and φ_2 on $C^*(S,T)+\mathcal{C}(T)$ that vanish on $\mathcal{C}(T)$ and agree on $C^*(T)+\mathcal{C}(T)$. Hence there are one-dimensional representations π_1 and π_2 of $C^*(S,T)+\mathcal{K}$ such that $\pi_1(S)\neq\pi_2(S)$, π_1 and π_2 agree on $C^*(T)+\mathcal{K}$, and π_1 and π_2 vanish on $\mathcal{K}$ (since $\mathcal{K}\subseteq\mathcal{C}(T)$). Let π denote the direct sum of $\aleph_0$ copies of π_1 and π_2. By Proposition 2.7(a) id is unitarily equivalent to $\text{id}\oplus\pi$ modulo the compacts and it follows that there are infinite dimensional projections P_1 and P_2 on $\mathfrak{H}$ such that $P_iA-\varphi_i(A)P_i$ and $AP_i-\varphi_i(A)P_i$ are compact for $i=1,2$ and all A in $C^*(S,T)+\mathcal{K}$. Choose orthonormal bases $\{e_n\}$ and $\{f_n\}$ for $P_1\mathfrak{H}$ and $P_2\mathfrak{H}$, respectively and define W in $\mathcal{B}(\mathfrak{H})$ by $We_n=f_n$, $n=1,2,\ldots$, and $Wx=0$ for x in $(P_1\mathfrak{H})^\perp$. If $X\in\mathcal{B}(\mathfrak{H})$, then for $n=1,2,\ldots$, $((TX-XT)e_n,f_n)=((P_2TX-XTP_1)e_n,f_n)=(\varphi_2(T)-\varphi_1(T))(Xe_n,f_n)+(Ke_n,f_n)$, where K is a compact operator. Since $\varphi_1(T)=\varphi_2(T)$ and $\|Ke_n\|\to 0$, $\|W-(TX-XT)\|\geqq 1$ and $W\notin\mathcal{R}(\delta_T)^-$. On the other hand, $SW-WS=SP_2W-WP_1S=(\varphi_2(S)-\varphi_1(S))W+K$, where K is a compact operator. Since $\varphi_1(S)\neq\varphi_2(S)$ and $S\in\mathcal{I}(T)$, $W\in\mathcal{R}(\delta_S)+\mathcal{K}\subseteq\mathcal{R}(\delta_T)^-$, a contradiction. This proves part (a) of the theorem.

Now suppose that $Z+\mathcal{I}(T)$ is in the center of $\mathcal{M}(T)/\mathcal{C}(T)$ but $Z\notin\mathcal{I}(T)=C^*(T)+\mathcal{C}(T)$. Then by the argument given in the first part of the proof, there are characters φ_1 and φ_2 on $C^*(Z, T)+\mathcal{K}$ such that $\varphi_1(Z)\neq\varphi_2(Z)$, φ_1 and φ_2 vanish on $\mathcal{K}$, and φ_1 and φ_2 agree on $C^*(T)+\mathcal{K}$. Also, there are orthogonal infinite dimensional projections P_1 and P_2 on $\mathfrak{H}$ such that $P_iA-\varphi_i(A)P_i$ and $AP_i-\varphi_i(A)P_i$ are compact for all A in $C^*(Z, T)+\mathcal{K}$ and $i=1, 2$. If $X\in\mathcal{B}(\mathfrak{H})$, then $(P_1+P_2)(TX-XT)(P_1+P_2)=(\varphi_1(T)-\varphi_2(T))(P_1XP_2-P_2XP_1)+K=K$, for some compact operator K, since $\varphi_1(T)=\varphi_2(T)$. Thus $(P_1+P_2)\mathcal{R}(\delta_T)^-(P_1+P_2)=$ $=\mathcal{K}((P_1+P_2)\mathfrak{H})$. Let W denote a partial isometry of $P_1\mathfrak{H}$ onto $P_2\mathfrak{H}$ as in the first part of the proof. Then $W\delta_T(X)-\delta_T(WX)=-\delta_T(W)X=-(TP_2W-WP_1T)X=$ $=(\varphi_1(T)-\varphi_2(T))WX+K=K$, for some compact operator K, since $\varphi_1(T)=$ $=\varphi_2(T)$ and so $W\mathcal{R}(\delta_T)\subseteqq\mathcal{R}(\delta_T)+\mathcal{K}\subseteqq\mathcal{R}(\delta_T)^-$. A similar argument shows that $\mathcal{R}(\delta_T)W\subseteqq\mathcal{R}(\delta_T)^-$ so that $W\in\mathcal{M}(T)$. Hence, $ZW-WZ\in\mathcal{C}(T)\subseteqq\mathcal{R}(\delta_T)^-$. But $P_2(ZW-WZ)P_1=(\varphi_2(Z)-\varphi_1(Z))W+K$ for some compact operator K and since $\varphi_2(Z)\neq\varphi_1(Z)$, $P_2(ZW-WZ)P_1$ is not compact, a contradiction. □

Corollary 3.8. *If T is a d-symmetric operator that has no reducing eigenvalues, then* $C^*(T)/\text{Comm}\, C^*(T)\cong\mathcal{J}(T)/\mathcal{C}(T)$.

In the concluding result of this section we show that $\mathcal{C}(T)$ can be quite large.

Theorem 3.9. *Suppose $\mathfrak{H}$ is separable and that T is a d-symmetric operator with no reducing eigenvalues. If T is not essentially normal, or if T is essentially normal with uncountable spectrum, then $\mathcal{C}(T)$ contains a C^*-algebra that is spatially isomorphic to $\mathcal{B}(\mathfrak{H})\oplus\mathcal{K}(\mathfrak{H})$.*

Proof. It is enough to show that $\mathcal{R}(\delta_T)^-$ contains a projection P of infinite rank. For then, since P is positive, $P\in\mathcal{C}(T)$ by 3.2(c) and $P\mathcal{B}(\mathfrak{H})P+P^{\perp}\mathcal{K}(\mathfrak{H})P^{\perp}$ is the desired subalgebra of $\mathcal{C}(T)$.

If $\mathcal{R}(\delta_T)^-$ fails to contain a projection of infinite rank, then $\mathcal{C}(T)\subseteqq\mathcal{K}$ by 3.2(c) and spectral theory. Hence T is essentially normal. Since $\mathcal{K}\subseteqq\mathcal{R}(\delta_T)^-$, Remark 1 of [22] implies that the spectrum of T is countable. □

4. The T-central states

The set $E(T)$ of all T-central states on $\mathcal{B}(\mathfrak{H})$ is convex and weak*-compact. We begin this section by examining the extreme points of $E(T)$. Recall that a state f on a C^*-algebra $\mathcal{B}$ is $\mathcal{A}$-multiplicative if $f(AX)=f(A)f(X)=f(XA)$ for all X in $\mathcal{B}$ and all A in $\mathcal{A}$ and that the extreme points in the set of all states on $\mathcal{B}$ are also called pure states.

Theorem 4.1. *If T is a d-symmetric operator and f is an extreme point of $E(T)$, then f is $\mathcal{J}(T)$-multiplicative on $\mathcal{B}(\mathfrak{H})$ and f is a pure state on $\mathcal{B}(\mathfrak{H})$.*

Proof. Fix a self-adjoint element A in $\mathscr{I}(T)$ with $0<\varepsilon<A<I-\varepsilon$, for some $\varepsilon>0$. Define f_1 and f_2 on $\mathscr{B}(\mathfrak{H})$ by $f_1(X)=f(A)^{-1}f(XA)$ and $f_2(X)=f(I-A)^{-1}f(X(I-A))$. Then $f(XA)=f(XA^{1/2}A^{1/2})=f(A^{1/2}XA^{1/2})$ as $A^{1/2}\in\mathscr{I}(T)$. Hence f_1 and (similarly) f_2 are states on $\mathscr{B}(\mathfrak{H})$. Since $TA-AT\in\mathscr{C}(T)$ by 3.3(c),

$$f(A)f_1(XT)=f(XTA)=f(X(TA-AT))+f(XAT)=0+f(XAT)=$$
$$=f(TXA)=f(A)f_1(TX).$$

Thus, f_1 and (similarly) f_2 are T-central. Since $f=f(A)f_1+f(I-A)f_2$ is an extreme point of $E(T)$, $f=f_1$ and so f is A-multiplicative. Since $\mathscr{I}(T)$ is the linear span of operators of this form, the first assertion is proved.

Now suppose that there are states f_1 and f_2 on $\mathscr{B}(\mathfrak{H})$ and $0<\alpha<1$ such that $f=\alpha f_1+(1-\alpha)f_2$, where f is an extreme point of $E(T)$. Since f is multiplicative on $\mathscr{I}(T)$ by the first part of the proof, f is a pure state on $\mathscr{I}(T)$ and so f, f_1, and f_2 agree on $\mathscr{I}(T)$. In particular, each f_i is multiplicative on $C^*(T)$. It follows (see our remark preceding 3.5) that each f_i is T-central. Hence, $f=f_1=f_2$. □

Corollary 4.2. *If T is a d-symmetric operator, then each character on $C^*(T)$ extends to a character on $\mathscr{I}(T)$.*

Proof. Fix a character φ on $C^*(T)$ and let f be a pure state on $\mathscr{B}(\mathfrak{H})$ that extends φ. Then f is T-multiplicative since it extends φ and, therefore, f is an extreme point of $E(T)$. Hence f is multiplicative on $\mathscr{I}(T)$ by the theorem. □

Remark. It follows from 4.1 that if T is d-symmetric then $\mathscr{R}(\delta_T)^-$ is the intersection of the kernels of the T-multiplicative states on $\mathscr{B}(\mathfrak{H})$. Also, $\mathscr{I}(T)$ is the set of operators A such that every extreme point of $E(T)$ is A-multiplicative.

It is natural at this point to ask: Which states on $C^*(T)$ extend to T-central states on $\mathscr{B}(\mathfrak{H})$? The answer is what one might expect.

Theorem 4.3. *If T is a d-symmetric operator, then:*

(a) *A state f on $C^*(T)$ extends to a T-central state on $\mathscr{B}(\mathfrak{H})$ if and only if $f(\mathrm{Comm}\, C^*(T))=0$.*

(b) *A state g on $\mathscr{I}(T)$ extends to a T-central state on $\mathscr{B}(\mathfrak{H})$ if and only if $g(\mathscr{C}(T))=0$.*

Proof. Since $\mathscr{C}(T)\subseteqq\mathscr{R}(\delta_T)^-$, each T-central state on $\mathscr{B}(\mathfrak{H})$ vanishes on $\mathscr{C}(T)$ and so on $\mathrm{Comm}\, C^*(T)$ (by 3.5(b)) so that the conditions $f(\mathrm{Comm}\, C^*(T))=g(\mathscr{C}(T))=0$ are necessary. Now suppose g is a state on $\mathscr{I}(T)$ such that $g(\mathscr{C}(T))=0$. Then g may be viewed as a state on the commutative C^*-algebra $\mathscr{I}(T)/\mathscr{C}(T)$. Hence, g is the weak*-limit of a net of convex combinations of characters on $\mathscr{I}(T)$. Each of the characters appearing in these convex combinations has an extension to a pure state on $\mathscr{B}(\mathfrak{H})$ which is T-multiplicative by 4.1. By taking the same convex combinations of the extended states, we obtain a net of T-central states that has a subnet that converges to a T-central extension of g. The proof of sufficiency in part (a) is the same. $\square$

5. Examples

In this section we consider the C^*-algebras $\mathscr{C}(T)$, $\mathscr{I}(T)$, and $\mathscr{M}(T)$ for special d-symmetric operators.

I. *Normal operators without eigenvalues.* Let N denote a normal operator without eigenvalues. Then the spectrum σ of N is uncountable, N is d-symmetric, and $\mathscr{K} \subseteqq \mathscr{C}(N)$ (2.2 and 3.2(d)). Hence $\mathscr{C}(N)$ is nonseparable by 3.9. Also, $\mathscr{I}(N) = C^*(N) + \mathscr{C}(N)$ by 3.7(a) and if $C(\sigma)$ denotes the continuous functions on σ, then $C(\sigma) \cong \mathscr{I}(N)/\mathscr{C}(N)$ is the center of $\mathscr{M}(N)/\mathscr{C}(N)$ by 3.7(b). Further, $\mathscr{M}(N)$ contains the von Neumann algebra $\{N\}'$ by 3.3(b).

Recall [15, 4.4.19] that there is a norm one projection $\mathscr{P}$ of $\mathscr{B}(\mathfrak{H})$ onto $\{N\}'$ such that $\mathscr{P}(AXB) = A\mathscr{P}(X)B$ for A and B in $\{N\}'$ and X in $\mathscr{B}(\mathfrak{H})$. Thus, $\mathscr{P}(\mathscr{R}(\delta_N)^-) = 0$ and so if $A \in \{N\}'$ and $X \in \mathscr{R}(\delta_N)^-$, $\|A\| = \|\mathscr{P}(A+X)\| \leqq \|A+X\|$ and $\{N\}' + \mathscr{C}(N)$ is an orthogonal direct sum in $\mathscr{M}(N)$. However, $\{N\}' + \mathscr{C}(N) \neq \mathscr{M}(N)$. Otherwise the center of $\mathscr{M}(N)/\mathscr{C}(N)$ would be isomorphic to $\{N\}'$. This is not the case. In fact, as noted above, the center of $\mathscr{M}(N)/\mathscr{C}(N)$ is isomorphic to $C^*(N)$.

II. *Diagonal operators.* In this example and the next all operators will be assumed to be acting on separable Hilbert space. An operator D is *diagonal* if there is a sequence $\{E_n\}$ of orthogonal projections such that $\Sigma E_n = I$ and a bounded sequence $\{d_n\}$ of distinct complex numbers such that $D = \Sigma d_n E_n$.

Proposition 5.1. *The following are equivalent for a d-symmetric operator T:*

(a) *T is a diagonal operator*, (b) *$\mathscr{I}(T)$ is commutative.* (c) $\mathscr{C}(T) = 0$.

(d) $\mathscr{M}(T) = \{T\}'$.

Proof. Since $\mathscr{C}(T) = \mathrm{Comm}\,\mathscr{I}(T)$ by 3.5(a), (b) and (c) are equivalent. By 2.1(b) and 3.2(c), the condition $\mathscr{C}(T) = 0$ is equivalent to the conditions that T be normal and $\mathscr{R}(\delta_T)^-$ contain no nonzero positive operator. Therefore (c) and (a) are equivalent by [22]. Finally (c) and 3.3(d) imply (d), and if (d) holds, then $\{T\}'$ is self-adjoint and T is normal. Hence $\mathscr{C}(T) = \mathscr{M}(T) \cap \mathscr{R}(\delta_T)^- = \{T\}' \cap \mathscr{R}(\delta_T)^- = 0$. (This latter intersection is 0 for any normal operator T as shown in example 1. See also [1]).

Thus, for a diagonal operator D, $\mathscr{C}(D)$ and $\mathscr{M}(D)$ are easily described. The C^*-algebra $\mathscr{I}(D)$ is more complicated. Before describing it we need a preliminary result.

Lemma 5.2. *If* $D=\Sigma d_n E_n$ *is a diagonal operator, then* $C^*(D)\subseteqq\mathscr{I}(D)\subseteqq\subseteqq C^*(D, E_1, E_2, \ldots)$.

Proof. The first inclusion is trivial. Also $\mathscr{C}(D)=0$ as D is a diagonal operator, hence $\{D\}'=\mathscr{M}(D)=\mathscr{I}(D)'$ by 3.3(d). Therefore $\mathscr{I}(D)\subseteqq(D)''$. To finish the proof, fix a diagonal operator $D'=\Sigma a_n E_n$ in $\{D\}''$ that is not in $C^*(D, E_1, E_2, \ldots)$. We must show that $D'\notin\mathscr{I}(D)$. Choose a sequence $\{e_n\}$ of unit vectors in $\mathfrak{H}$ such that $E_n e_n=e_n$ for each n, and let ω_n denote the vector state induced by e_n (so that $\omega_n(X)=(Xe_n, e_n)$). Then each ω_n is a character on $\mathscr{A}=C^*(D, D', E_1, \ldots)$ and if φ is a character on $\mathscr{A}$ then either $\varphi(E_n)=1$ for some unique integer n and $\varphi=\omega_n$, or else $\varphi(E_n)=0$ for all n and $\varphi=\lim_n \omega_{\sigma(n)}$ is the weak*-limit of a subsequence of the ω_n's induced by an injective map σ of the natural numbers $\mathbf{N}$ into $\mathbf{N}$. Since $C^*(D, E_1, \ldots)$ is a proper C^*-subalgebra of $\mathscr{A}$, there are distinct characters φ and ψ on $\mathscr{A}$ that agree on $C^*(D, E_1, E_2, \ldots)$ by the Stone—Weierstrass theorem. If $\varphi(E_n)=1$ for some n, then $\psi(E_n)=1$ and $\varphi=\psi=\omega_n$ because $E_n\in C^*(D, E_1, \ldots)$. Hence, $\varphi(E_n)=\psi(E_n)=0$ for all n and $\varphi=\lim_n \omega_{\sigma(n)}$, $\psi=\lim_n \omega_{\tau(n)}$ are weak*-limits of disjoint subsequences of $\{\omega_n\}$ induced by injective maps σ and τ of $\mathbf{N}$ into disjoint subsets of $\mathbf{N}$. Write

$$\alpha = \varphi(D') = \lim_n (D'e_{\sigma(n)}, e_{\sigma(n)}) = \lim_n a_{\sigma(n)},$$

$$\beta = \psi(D') = \lim_n (D'e_{\tau(n)}, e_{\tau(n)}) = \lim_n a_{\tau(n)}$$

and

$$\gamma = \varphi(D) = \psi(D) = \lim_n d_{\sigma(n)} = \lim_n d_{\tau(n)}$$

so that $\alpha\neq\beta$. Define an operator W by $We_{\sigma(n)}=e_{\tau(n)}$ for $n=1, 2, \ldots$ and $Wx=0$ if $x\in\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots\}^{\perp}$. Then if X is any operator, $\|D', W]-[D, X]\|\geqq\geqq\lim_n |([D', W]e_{\sigma(n)}, e_{\tau(n)})-([D, X]e_{\sigma(n)}, e_{\tau(n)})| = \lim_n |a_{\tau(n)}-a_{\sigma(n)})(We_{\sigma(n)}, e_{\tau(n)})-$ $-(d_{\tau(n)}-d_{\sigma(n)})(Xe_{\sigma(n)}, e_{\tau(n)})|=|\beta-\alpha|$. Thus $[D', W]\notin\mathscr{R}(\delta_D)^-$ and $D'\notin\mathscr{I}(D)$. □

Theorem 5.3. *Suppose* $D=\Sigma d_n E_n$ *is a diagonal operator and write*

$\Lambda=\{n\in\mathbf{N}: E_n$ has finite rank and d_n is not an isolated point of the spectrum of $D\}$.

Then $\mathscr{I}(D)=C^*(D, \{E_n\}_{n\in\Lambda})$.

Proof. First note that if d_n is an isolated point of the spectrum, then $E_n\in C^*(D)$ by the Gelfand theory. Now fix an eigenvalue d_n of D that is a limit point of the spectrum of D. By 5.2 it suffices to show that $E_n\in\mathscr{I}(D)$ if and only if E_n has finite rank. Suppose that E_n has infinite rank and choose an orthonormal basis

$\{e_1, e_2, \dots\}$ for E_n. Since d_n is a limit point, there are projections E_{n_j} and unit vectors f_j such that $d_{n_j} \to d_n$ and $E_{n_j} f_j = f_j$. Define W in $\mathscr{B}(\mathfrak{H})$ by $We_j = f_j$ and $Wx = 0$ if $x \in (E_n \mathfrak{H})^\perp$. Then if X is any operator,

$$\|[E_n, W] - [D, X]\| \geqq \lim_j |([E_n, W]e_j, f_j) - ([D, X]e_j, f_j)| =$$

$$= \lim_j |(f_j, f_j) - 0 - (d_n - d_{n_j})(Xe_j, f_j)| = 1.$$

Thus, $[E_n, W] \notin \mathscr{R}(\delta_D)^-$ and $E_n \notin \mathscr{I}(D)$. Now suppose E_n has finite rank. Fix vectors x in $E_n \mathfrak{H}$ and y in $E_m \mathfrak{H}$, where $n \neq m$. Let $x \otimes y$ denote the rank one operator given by $x \otimes y(z) = (z, y)x$. Then $[D, x \otimes y] = Dx \otimes y - x \otimes D^* y = (d_n - d_m) x \otimes y$. Since $d_m \neq d_n$, $x \otimes y \in \mathscr{R}(\delta_D)^-$. If $z \in (E_n \mathfrak{H})^\perp$, then $z = \sum_{n \neq m} a_m y_m$, where $E_m y_m = y_m$, and $\Sigma |a_m|^2 = \|z\|^2$. Thus, $x \otimes z = \Sigma \bar{a}_m (x \otimes y_m)$ is in $\mathscr{R}(\delta_D)^-$ for all $x \in E_n \mathfrak{H}$ and $z \in (E_n \mathfrak{H})^\perp$. It follows that $E_n X E_n^\perp \in \mathscr{R}(\delta_D)^-$ for all X in $\mathscr{B}(\mathfrak{H})$ and so since $\mathscr{R}(\delta_D)^-$ is self-adjoint $\mathscr{R}(\delta_{E_n}) = \mathscr{R}(\delta_{E_n})^- = E_n \mathscr{B}(\mathfrak{H}) E_n^\perp + E_n^\perp \mathscr{B}(\mathfrak{H}) E_n \subseteqq \mathscr{R}(\delta_D)^-$. Thus, $E_n \in \mathscr{I}(D)$. □

Corollary 5.4. *If* $D = \Sigma d_n E_n$, *where each* E_n *has infinite rank, then* $\mathscr{I}(D) = C^*(D)$.

Corollary 5.5. *If* $D = \Sigma d_n E_n$, *where each* E_n *has rank one and* $\{d_n\}$ *is an enumeration of the rationals between* 0 *and* 1, *then:*

(a) $E_n \notin C^*(D)$, $n = 1, 2, \dots$, (b) $\mathscr{I}(D) = C^*(D, E_1, E_2, \dots)$.

Remarks. (a) Let D be the diagonal operator defined in 5.5. Then $\{D\}' = \{D\}'' = \mathscr{M}(D)$ (5.2(d)) so that $\mathscr{M}(D)$ is commutative, $\mathscr{C}(D) = 0$ by 5.2(c) and $C^*(D) = C^*(D) + \mathscr{C}(D) \neq \mathscr{I}(D)$ by 5.5. Thus, if the condition T has no reducing eigenvalues is omitted, Theorems 3.7 and 3.9 and Corollary 3.8 are no longer true.

(b) Let $\{E_n\}$ denote a sequence of orthogonal rank one projections with $\Sigma E_n = I$ and write

$$D = \sum_{n=1}^{\infty} n^{-1} E_n, \quad D' = \sum_{n=2}^{\infty} n^{-1} E_n.$$

Then $C^*(D) = C^*(E_1, E_2, \dots) \neq C^*(D') = C^*(E_2, E_3, \dots)$. However, by Theorem 5.1, $\mathscr{I}(D) = \mathscr{I}(D') = C^*(D)$ and by (the last part of) the proof of 5.4, $\mathscr{R}(\delta_D)^- = \mathscr{R}(\delta_{D'})^-$. Thus, in general, neither the includion algebra nor the derivation range determines $C^*(T)$.

(c) If T is an essentially normal d-symmetric operator with countable spectrum, then $\mathscr{C}(T) \subseteqq \mathscr{K}$; and $\mathscr{C}(T) = 0$ if and only if T is normal. Indeed, since the spectrum is countable, there is a non-zero representation π of $\mathscr{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}_\pi$ such that $\ker \pi = \mathscr{K}(\mathfrak{H})$ and $\pi(T)$ is a diagonal operator on $\mathfrak{H}_\pi$ [22].

Then $\mathscr{C}(\pi(T))=0$ by 5.2(c), and by 3.2(c) we have $\pi(\mathscr{C}(T))\subseteq\mathscr{C}(\pi(T))$. Hence $\mathscr{C}(T)\subseteq\ker(\pi)=\mathscr{K}$. The spectral theorem and 5.1(c) imply that $\mathscr{C}(T)=0$ if and only if T is normal.

III. *Pure isometries.* Let V denote a pure isometry (that is, an isometry with no unitary direct summand). Then V is d-symmetric by 2.3 and has no reducing eigenvalues; hence, $\mathscr{K}(\mathfrak{H})\subseteq\mathscr{C}(V)$ by 3.2(d) and so $\mathscr{I}(V)=C^*(V)+\mathscr{C}(V)$ and the center of $\mathscr{M}(V)/\mathscr{C}(V)$ is $\mathscr{I}(V)/\mathscr{C}(V)\cong C^*(V)/\mathrm{Comm}\, C^*(V)\cong C$ (unit circle), the continuous functions on the unit circle by 3.5, 3.7 and [8, Theorem 3]. Also, by 3.9 $\mathscr{C}(V)$ contains a subalgebra that is spatially isomorphic to $\mathscr{B}(\mathfrak{H})\oplus\mathscr{K}(\mathfrak{H})$. We now show that $\mathscr{M}(V)/\mathscr{C}(V)$ is also large.

Proposition 5.6. *An operator Z is in $\mathscr{M}(V)$ if and only if $V^*ZV-Z\in\mathscr{C}(V)$. Hence $\mathscr{M}(V)\supseteq\mathscr{C}(V)\oplus\mathscr{T}_V$, where $\mathscr{T}_V=\{X\in\mathscr{B}(\mathfrak{H}): V^*XV=X\}$ is the set of Toeplitz operators associated with V. Thus $\mathscr{M}(V)/\mathscr{C}(V)$ is non-separable.*

Proof. If $Z\in\mathscr{M}(V)$ then $V^*ZV-Z=V^*(ZV-VZ)\in\mathscr{C}(V)$ by 3.3(d). Conversely, suppose that $V^*ZV-Z\in\mathscr{C}(V)$. Then $Z(VX-XV)=\delta_V(ZX)+ +(I-VV^*)ZVX+V(V^*ZV-Z)X$ belongs to $\mathscr{R}(\delta_V)^-$ for any operator X because $I-VV^*=[V^*V]$, $\in\mathscr{C}(V)$. Also, $V^*Z^*V-Z^*\in\mathscr{C}(V)$ so that $Z^*\mathscr{R}(\delta_{V^*})^-= =Z^*\mathscr{R}(\delta_V)^-\subseteq\mathscr{R}(\delta_V)^-$ and this implies $\mathscr{R}(\delta_V)Z\subseteq\mathscr{R}(\delta_V)^-$ on taking adjoints. Thus Z is a two-sided multiplier of $\mathscr{R}(\delta_V)^-$ and therefore $Z\in\mathscr{M}(V)$.

That the subspaces $\mathscr{C}(V)$ and $\mathscr{T}_V$ have trivial intersection and are in fact "orthogonal" follows from the existence of a norm one projection of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{T}_V$ that vanishes on $\mathscr{R}(\delta_V)$. (See [21]). □

6. Some open problems

(a) If T is a d-symmetric operator, must $C^*(T)$ be a postliminaire or GCR C^*-algebra [9, Paragraphe IV]? If T is d-symmetric and $C^*(T)$ is GCR, then by direct integral theory and Theorem 2.8 T would be a direct integral of irreducible d-symmetric operators. Which irreducible operators are d-symmetric? We do not know when a direct integral of d-symmetric operators is d-symmetric.

(b) The example in Remark (c) following Corollary 2.6 raises the question: Is the set $\{T+K: T\ d\text{-symmetric},\ K \text{ compact}\}$ norm-closed?

(c) It follows from Proposition 5.2 that there does not exist a normal operator N such that $\mathscr{R}(\delta_N)^-=\mathscr{R}(\delta_K)^-$, where K is the compact d-symmetric operator of Remark (c) following Corollary 2.6. If V is the simple unilateral shift does there exist a normal operator N such that $\mathscr{I}(N)=\mathscr{I}(V)$? If N is normal must $\mathscr{I}(N)= =\mathscr{I}(A)$ for some self-adjoint operator A?

(d) Is there a property of δ_T as an element of the Banach algebra $\mathscr{B}(\mathscr{B}(\mathfrak{H}))$, which characterizes when T is d-symmetric?

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