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Characterizations of completely regular elements in semigroups

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Let S be an arbitrary semigroup, $a \in S$. The element a of S is said to be *completely regular* (c.r.) if there exists an element x in S such that

$$(1) \qquad axa = a \quad \text{and} \quad ax = xa$$

holds. Evidently, every c.r. element of S is regular, but not conversely. A semigroup S is called c.r. if all its elements are c.r.

We shall make use of the following well-known result:

Lemma. (CROISOT [2]) *An element a of a semigroup S is completely regular if and only if $a \in a^2Sa^2$.*

Our first result is stated in the following

Theorem 1. *An element a of a semigroup S is c.r. if and only if there exists an idempotent element e in S such that*

$$(2) \qquad B(a) = B(e)$$

holds.[1])

Proof. Let a be a c.r. element of a semigroup S. Then S has an element x with property (1). Hence it follows easily that $B(a)=aSa$. Let us introduce the notation $e=ax$. Then it is easy to see that $e^2=e$ and $B(e)=(ax)S(xa)\subseteq aSa$. On the other hand, we conclude that $B(a)=(axa)S(axa)=e(aSa)e\subseteq eSe$. Since $B(e)=eSe$, we obtain (2).

Conversely, suppose that for an element a of a semigroup S there exists an idempotent e such that condition (2) holds. This means that

$$(3) \qquad \{a\} \cup aS^1a = eSe.$$

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[1]) $B(a)$ denotes the principal bi-ideal of S generated by the element a of S. For other notations and terminology we refer to [1] and [5].

Hence it follows that $a=ese$, where $s\in S$. Thus we obtain the equations

(4) $$ea=a=ae.$$

Condition (3) implies also that $e=a$ or $e=ata$, where $t\in S^1$. In the latter case we get $ata^2=a=a^2ta$, whence $a=a^2(tat)a^2$, that is, the element a is c.r., indeed. This holds in the case $e=a$ too, whence Theorem 1 is proved.

Corollary 1. *Let $\mathcal{M}_n$ be the multiplicative semigroup of all $n\times n$ complex matrices. An element A of $\mathcal{M}_n$ has a group inverse (or Drazin 1-inverse) if and only if there is an idempotent matrix E in $\mathcal{M}_n$ such that $A\mathcal{M}_nA=E\mathcal{M}_nE$ holds.*

A matrix X is a *group inverse* of A if $AXA=A$, $XAX=X$, and $AX=XA$ (cf. Erdélyi [4]). This notion is a particular case of the Drazin pseudo-inverse (cf. Drazin [3]).

Corollary 1 follows at once from Theorem 1 because $\mathcal{M}_n$ is a regular semigroup.

Corollary 2. *A semigroup S is c.r. if and only if for every element a of S there exists an idempotent e_a in S such that $B(a)=B(e_a)$.*

It may be remarked that Theorem 1 and Corollary 2 remain true with quasi-ideal instead of bi-ideal.

Theorem 2. *An element a of a semigroup S is c.r. if and only if there exists an idempotent element e in S such that*

(5) $$Q(a)=Q(e)$$

holds.

Proof. Let a be a c.r. element of a semigroup S, i.e., there is an element x in S with property (1). Using the notation $e=ax$ we have $L(a)=Se$ and $R(a)=eS$. Thus it follows that $Q(a)=aS\cap Sa=L(a)\cap R(a)=eS\cap Se=Q(e)$.

Conversely, if to an element a of a semigroup S there is an idempotent e admitting the property (5), then

(6) $$aS^1\cap S^1a=eS\cap Se.$$

Hence $a=es=te$, where $s,t\in S$. Hence (4) follows. On the other hand, (6) implies $e\in aS^1\cap S^1a$, that is, $e=a$ or $e=au=va$ with $u,v\in S$. From the latter equations we get that $a^2u=a=va^2$, i.e., $a\in a^2Sa^2$. Therefore a is c.r. by the Lemma. Thus Theorem 2 is completely proved.

Corollary 1. *A semigroup S is c.r. if and only if for each element a of S there exists an idempotent e_a in S such that $Q(a)=Q(e_a)$.*

Corollary 2. *An element a of a semigroup S is c.r. if and only if the $\mathcal{H}$-class of a is a group.*

This follows from Theorem 2 and from a result by PONDĚLÍČEK [12].

Some further characterizations of c.r. elements and c.r. semigroups can be given, but the proofs are similar to the above proofs of Theorem 1 and Theorem 2, and we omit them.

Theorem 3. *For an element a of a semigroup S the following conditions are equivalent:*

(A) *a is completely regular.*
(B) $\exists e \in E(S)$ *such that* $B(a)=B(e)$.
(C) $\exists e \in E(S)$ *such that* $Q(a)=Q(e)$.
(D) $\exists e \in E(S)$ *such that* $B(a)=R(e)L(e)$.
(E) $\exists e \in E(S)$ *such that* $Q(a)=R(e)L(e)$.
(F) $\exists e \in E(S)$ *such that* $B(a)=L(e)\cap R(e)$.
(G) $\exists e \in E(S)$ *such that* $Q(a)=eS\cap Se$.
(H) $\exists e \in E(S)$ *such that* $B(a)\cap Q(a)=eSe$.
(I) $\exists e \in E(S)$ *such that* $B(a)\cap Q(a)=R(e)L(e)$.
(J) $\exists e \in E(S)$ *such that* $L(a)\cap R(a)=R(e)L(e)$.
(K) *The principal bi-ideal $B(a)$ is a monoid.*
(L) *The principal quasi-ideal $Q(a)$ is a monoid.*
(M) *The principal (m, n)-ideal generated by a is a monoid* $(m, n>0)$.
(N) $\exists e \in E(S)$ *such that* $\{a\}_{(m,n)}=\{e\}_{(m,n)}$, *where* $m, n>0$.

For the definition of (m, n)-ideals, see [6]. For earlier characterizations of completely regular semigroups as well as of completely regular elements, see [7], [8], [9], [10].

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