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On Jordan models of C_0-contractions

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In [7] the following theorem was proved:

Let T be a contraction of class C_0 on a separable Hilbert space. Then there exists an (up to constant factors of modulus 1) unique sequence $\{m_i\}_1^\infty$ of inner scalar functions such that:

1) $m_{i+1}|m_i$, *i.e.* m_{i+1} *divides* m_i, *for each* i,

2) T *is quasisimilar to* $S(m_1)\oplus S(m_2)\oplus\dots$ *(the "Jordan model" of* T*).*

In [1] and [2] it was proved that if T has finite defect indices $\delta_T=\delta_{T^*}=n$ then, for $i=1, 2, \dots, n$, m_i is equal to the $(n-i+1)$-th invariant factor of the characteristic function of T.

At the end of [8] the problem was raised what is the relation of the functions m_i to the characteristic function of T in the general case. We are going to give an answer to this question.

The main result (Theorem 1) can be also deduced from [9], Corollary 3.4. The methods of the proof, however, are quite different.

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We shall use the notations introduced in [2], [4], [6] and [8]. By H^∞ we mean the Banach algebra of bounded holomorphic functions on the disc $|\lambda|<1$. If $u, v\in H^\infty$ then $u\wedge v$ means the largest common inner divisor of u and v. For $1\leqq n\leqq\infty$ we define, as in [8], $\mathcal{M}(n)$ as the set of $n\times n$ matrices $A=(a_{ij})$ over H^∞ for which

$$\sum_i \Big|\sum_j \xi_j a_{ij}(\lambda)\Big|^2 \leqq K^2 \sum_j |\xi_j|^2 \quad (K\geqq 0)$$

holds for $|\lambda|<1$ and for every square-summable sequence of complex numbers, i.e. whose values $A(\lambda)$ ($|\lambda|<1$) are operators on the complex euclidean n-space E_n, bounded by the constant K.

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For $A\in\mathcal{M}(n)$ and a natural number r, $r\leqq n$, we define $\mathcal{D}_r(A)$ as the largest common inner divisor of all minors of A of order r. The invariant factors $\mathcal{E}_r(A)$ of A are then defined by

$$\mathcal{E}_1(A)=\mathcal{D}_1(A) \quad \text{and} \quad \mathcal{E}_r(A)=\mathcal{D}_r(A)/\mathcal{D}_{r-1}(A) \quad \text{for} \quad r\geqq 2$$

(we put $\mathcal{E}_r(A)=0$ if $\mathcal{D}_r(A)=0$). A matrix $A\in\mathcal{M}(n)$ is called inner (inner from both sides) if A is isometry (unitary) valued almost everywhere on the unit circle.

For $A\in\mathcal{M}(n)$ inner we define the operator $S(A)$ on the Hilbert space $\mathfrak{H}(A)=H^2(E_n)\ominus AH^2(E_n)$ by $S(A)u=P_{\mathfrak{H}(A)}(\lambda u)$. If T is an operator on $\mathfrak{H}$ and T' is an operator on $\mathfrak{H}'$ we write $T\overset{i}{\prec}T'$ if there exists an injective operator $X:\mathfrak{H}\to\mathfrak{H}'$ such that $XT=T'X$. If X can be chosen such that $\overline{X\mathfrak{H}}=\mathfrak{H}'$ we write $T\prec T'$. T and T' are called quasisimilar ($T\sim T'$) if $T\prec T'$ and $T'\prec T$. We write $T\overset{u}{\sim}T'$ if T and T' are unitarily equivalent.

Lemma 1. *Let the matrix $A\in\mathcal{M}(n)$ $(1\leqq n\leqq\infty)$ be inner from both sides and have a scalar multiple $\psi\in H^\infty$, ψ inner. Let $\Omega\in\mathcal{M}(n)$ be such that $\Omega A=A\Omega=\psi I_n$. Then $\mathcal{E}_k(\Omega)|\psi$ for $k=1,2,\dots$. Let $\psi_k=\psi|\mathcal{E}_k(\Omega)$. Then for every natural number $r\leqq n$ there exist matrices Δ, $\Lambda\in\mathcal{M}(n)$ with a common scalar multiple $h\in H^\infty$, $h\wedge\psi=1$. and a matrix $B_r\in\mathcal{M}(n')$ $(n'+r=n)$ inner from both sides and having the scalar multiple $\psi_r\cdot h$ such that $\Delta A=B\Lambda$, where $B=\operatorname{diag}[\psi_1,\dots,\psi_r,B_r]\in\mathcal{M}(n)$.*

Proof. According to Theorem 1 of [8] there exist matrices $M,N\in\mathcal{M}(n)$ with the respective scalar multiples φ_1 and φ_2, such that $\varphi_1\cdot\varphi_2\wedge\psi=1$, $M\Omega=\Omega'N$ where Ω' is a matrix of the form $\Omega'=\operatorname{diag}[\mathcal{E}_1(\Omega),\dots,\mathcal{E}_r(\Omega),\Omega'_r]$ with $\Omega'_r\in\mathcal{M}(n')$ $(n'+r=n)$ and $\mathcal{E}_1(\Omega)|\mathcal{E}_2(\Omega)|\dots|\mathcal{E}_r(\Omega)|\Omega'_r$.

Because M and N also have the scalar multiple $\varphi=\varphi_1\cdot\varphi_2$ there exist matrices M^a, $N^a\in\mathcal{M}(n)$ such that $MM^a=M^aM=NN^a=N^aN=\varphi\cdot I_n$. Set $A'=NAM^a$. Then we have $\Omega'A'=\Omega'NAM^a=M\Omega AM^a=\varphi\psi\cdot I_n$ and $\varphi A'\Omega'=A'\Omega'NN^a=NAM^aM\Omega N^a=\varphi^2\psi\cdot I_n$, hence $A'\Omega'=\varphi\psi\cdot I_n$. A' is necessarily of the form $\operatorname{diag}[\varphi\psi_1,\dots,\varphi\psi_r,A_r]$, where $A_r\in\mathcal{M}(n)$ and $A_r\dfrac{\Omega'_r}{E_r(\Omega)}=\dfrac{\Omega'_r}{E_r(\Omega)}A_r=\varphi\psi_r\cdot I_n$. Let $\varphi=\varphi_i\cdot\varphi_e$ and $A_r=A_{ri}A_{re}$ be the canonical inner-outer factorizations of φ and A_r, respectively. Set $B=\operatorname{diag}[\psi_1,\dots,\psi_r,A_{ri}]$, $\Delta=\varphi N$, $\Lambda=\operatorname{diag}[\underbrace{\varphi,\dots,\varphi}_{r\text{ times}},A_{re}]M$. Then we have $B\Lambda=\operatorname{diag}[\varphi\psi_1,\dots,\varphi\psi_r,A_r]\cdot M=A'M=NAM^aM=\varphi NA=\Delta A$. By [4] (V. 6. 4) A_{re} has the scalar multiple φ_e so the matrices Δ and Λ have the scalar multiple $h=\varphi^2$. On the other hand, A_{ri} has the scalar multiple $\varphi_i\cdot\psi_r$. So A_{ri} is inner from both sides by [4] (V. 6.2), and B, Δ and Λ satisfy all the conditions required.

Lemma 2. *Let the matrices $A,B\in\mathcal{M}(n)$ $(1\leqq n\leqq\infty)$ be inner from both sides and have scalar multiples ψ and ψh, respectively, where $\psi\wedge h=1$ and ψ is* *inner. Let* Δ *and* $\Lambda \in \mathcal{M}(n)$ *be matrices with the scalar multiple h, i.e.* $\Delta\Delta^a = \Delta^a\Delta = \Lambda\Lambda^a = \Lambda^a\Lambda = hI_n$ *for some* $\Delta^a, \Lambda^a \in \mathcal{M}(n)$. *Suppose* $\Delta A = B\Lambda$. *Then:*

1) *the operator* $X\colon \mathfrak{H}(A) \to \mathfrak{H}(B)$ *defined by* $X = P_{\mathfrak{H}(B)}\Delta|\mathfrak{H}(B)$ *is injective and satisfies* $XS(A) = S(B)X$;
2) *the operator* $Y\colon \mathfrak{H}(B) \to \mathfrak{H}(A)$ *defined by* $Y = P_{\mathfrak{H}(A)}\Delta^a|\mathfrak{H}(B)$ *satisfies* $YS(B) = S(A)Y$ *and we have* $h(S(B)|\mathfrak{N}) = 0$, *where* $\mathfrak{N} = \ker Y$.

Proof. The proof of (1) is the same as in [1] or [6], and we repeat it only to be seen how does case 2 differ from case 1.

We have $\Delta AH^2(E_n) = B\Lambda H^2(E_n) \subset BH^2(E_n)$ so $P_{\mathfrak{H}(B)}\Delta P_{\mathfrak{H}(A)} = P_{\mathfrak{H}(B)}\Delta$. Hence $XS(A) = P_{\mathfrak{H}(B)}\Delta P_{\mathfrak{H}(A)}U_+|\mathfrak{H}(A) = P_{\mathfrak{H}(B)}\Delta U_+|\mathfrak{H}(A) = P_{\mathfrak{H}(B)}U_+\Delta|\mathfrak{H}(A) = P_{\mathfrak{H}(B)}U_+ P_{\mathfrak{H}(B)}\Delta|\mathfrak{H}(A) = S(B)X$ (where $U_+\colon H^2(E_n) \to H^2(E_n)$ is defined by $U_+u = \lambda u$). Let $u \in \mathfrak{H}(A)$ and $Xu = 0$, i.e. $\Delta u \in BH^2(E_n)$. As A and B are analytic functions inner from both sides, the corresponding multiplication operators on $L^2(E_n)$ are unitary. Set $f = A^{-1}u$. Then $B\Lambda f = \Delta Af = \Delta u \in BH^2(E_n)$; hence $\Lambda f \in H^2(E_n)$. On the other hand, we have $Af = u \in H^2(E_n)$. Hence $hf = \Lambda^a\Lambda f \in \Lambda^a H^2(E_n) \subset H^2(E_n)$ and $\psi f = A^a Af = A^a u \in A^a H^2(E_n) \subset H^2(E_n)$ (this is the place where the proof does not work in case 2 because we get only $\psi hf = B^a Bf \in H^2(E_n)$ then). Since $h \wedge \psi = 1$, $hf \in H^2(E_n)$ and $\psi f \in H^2(E_n)$ imply $f \in H^2(E_n)$ by the Lemma of [3]. So $u = Af \in AH^2(E_n)$. Since, on the other hand, $u \in H(A) = H^2(E_n) \ominus AH^2(E_n)$ we conclude that $u = 0$. Thus X is an injective operator.

As for 2, note that $hA\Lambda^a = \Delta^a\Delta A\Lambda^a = \Delta^a B\Lambda\Lambda^a = h\Delta^a B$ and hence $A\Lambda^a = \Lambda^a B$. We prove as above that $YS(B) = S(A)Y$. From this it follows that the subspace $\mathfrak{N} = \ker Y$ is $S(B)$-invariant. Let $u \in N$, i.e. $\Delta^a u \in AH^2(E_n)$. Then $hu = \Delta\Delta^a u \in \Delta AH^2(E_n) = B\Lambda H^2(E_n) \subset BH^2(E_n)$. We have $h(S(B)) = uP_{\mathfrak{H}(B)}(hu) \in P_{\mathfrak{H}(B)}BH^2(E_n) = 0$. Hence $h(S(B)|\mathfrak{N}) = 0$.

Now we are able to prove our main theorem:

Theorem 1. *Let T be an operator of class* C_0 *acting on a separable Hilbert space. Let* Θ *be the characteristic function of T and let* Ω *be a contractive analytic function such that* $\Theta\Omega = \Omega\Theta = \psi I_n$, *where* $\psi \in H^\infty$ *is inner and n is the defect index of T (such an* Ω *exists by* [4], VI. 5.1). *Let* $S(m_1) \oplus S(m_2) \oplus \cdots$ *be the Jordan model of T. Then* $m_r = \psi/\mathcal{E}_r(\Omega)$ *for every natural number* $r \leqq n$ *(if n is finite then in this notation* $m_i = 1$ *for* $i > n$*).*

Proof. Let r be an integer, $r \leqq n$. By Lemma 1 there exist matrices $\Delta, \Lambda, \Theta' \in \mathcal{M}(n)$ such that $\Delta\Theta = \Theta'\Lambda$, Δ and Λ have a scalar multiple h, $h \wedge \psi = 1$ and $\Theta' = \operatorname{diag}[\psi_1, \ldots, \psi_r, \Theta'_r]$, where $\psi_i = \psi/\mathcal{E}_i(\Omega)$ $(i = 1, \ldots, r)$, and Θ'_r is inner from both sides and has the scalar multiple $h\psi_r$.

I. We prove first $m_r|\psi_r$.

By Lemma 2 the operator $X=P_{\mathfrak{H}(\Theta')}\Delta|\mathfrak{H}(\Theta)$ is injective and $XS(\Theta)=$ $=S(\Theta')X$, i.e. $S(\Theta)\overset{i}{\prec} S(\Theta')$. In the same time $S(\Theta)\overset{u}{\sim} T\sim S(M)$, where $M=\operatorname{diag}[m_1, m_2, \ldots]$, $S(M)=S(m_1)\oplus S(m_2)\oplus\cdots$ and $\mathfrak{H}(M)=\mathfrak{H}(m_1)\oplus\mathfrak{H}(m_2)\oplus\cdots$. Hence $S(M)\overset{i}{\prec} S(\Theta')$. Let $Z\colon \mathfrak{H}(M)\to\mathfrak{H}(\Theta')$ be an injective operator such that $ZS(M)=S(\Theta')Z$. Put $\varphi=h\psi_r$, $\mathfrak{M}=\overline{\varphi(S(M))\mathfrak{H}(M)}$ and $\mathfrak{M}'=\overline{\varphi(S(\Theta'))\mathfrak{H}(\Theta')}$. We proceed as in [5]. We have $Z\mathfrak{M}\subset\overline{Z\varphi(S(M))\mathfrak{H}(M)}=\overline{\varphi(S(\Theta'))Z\mathfrak{H}(M)}\subset$ $\subset\overline{\varphi(S(\Theta'))\mathfrak{H}(\Theta')}=\mathfrak{M}'$ and $\mathfrak{M}$ and $\mathfrak{M}'$ are obviously subspaces invariant to $S(M)$ and $S(\Theta')$, respectively. Hence $S(M)|\mathfrak{M}\overset{i}{\prec} S(\Theta')|\mathfrak{M}'$. But $S(\Theta')|\mathfrak{M}'$ is unitarily equivalent to the operator $S(\psi_1/(\psi_1\wedge\varphi))\oplus\ldots\oplus S(\psi_{r-1}/(\psi_{r-1}\wedge\varphi))=S(\psi_1/\psi_r)\oplus\ldots$ $\ldots\oplus S(\psi_{r-1}/\psi_r)=S(\mathscr{E}_r(\Omega)/\mathscr{E}_1(\Omega))\oplus\ldots\oplus S(\mathscr{E}_r(\Omega)/\mathscr{E}_{r-1}(\Omega))$ (see [6]). In the same way $S(M)|\mathfrak{M}$ is unitarily equivalent to the operator

$$S(m_1/(m_1\wedge\varphi))\oplus S(m_2/(m_2\wedge\varphi))\oplus\cdots.$$

From Proposition 2 of [5] it follows that $S(M)|\mathfrak{M}\in C_0(r-1)$; hence $m_r|\varphi$. Since $m_r\wedge h=1$ it is necessarily $m_r|\psi_r$.

II. Next we prove $\psi_r|m_r$.

By Lemma 2 there exists an operator $Y\colon \mathfrak{H}(\Theta')\to\mathfrak{H}(\Theta)$ such that $YS(\Theta')=$ $=S(\Theta)Y$ and $h(S(\Theta')|\mathfrak{N})=0$, where $\mathfrak{N}=\ker Y$. Let Y_1 be the restriction of Y to the subspace $M=\mathfrak{H}(\psi_1)\oplus\ldots\oplus\mathfrak{H}(\psi_r)\oplus\{0\}$ of the Hilbert space $\mathfrak{H}(\Theta')$. Denote $M'=\operatorname{diag}[\psi_1, \ldots, \psi_r, I_{n'}]$ $(n'+r=n)$, $S(M')=S(\Theta')|\mathfrak{M}$ and $\mathfrak{N}_1=\mathfrak{N}\cap\mathfrak{M}$. Obviously, $Y_1S(M')=S(\Theta)Y_1$ and $h(S(M')|\mathfrak{N}_1)=0$. On the other hand, $\psi(S(M')|\mathfrak{N}_1)=\psi(S(M'))|\mathfrak{N}_1=0$. As $\psi\wedge h=1$, the minimal function of $S(M')|\mathfrak{N}_1$ is 1, i.e. $\mathfrak{N}_1=\{0\}$. So Y_1 is an injective operator and $S(M')\overset{i}{\prec} S(\Theta)$. Now we have $S(\psi_1)\oplus\ldots\oplus S(\psi_r)\overset{u}{\sim} S(M')\overset{i}{\prec} S(\Theta)\sim S(m_1)\oplus S(m_2)\oplus\ldots$. It follows as in I (or [6]) that $\psi_r|m_r$. Together with I this gives $m_r=\psi_r$, thus finishing the proof.

Remark 1. We return now to the case of n finite. Then we can take $\psi=\det\Theta_T$ (in an arbitrary choice of orthonormal bases of the defect spaces of T), $\Omega=\Theta_T^{A}$ (the adjoint matrix of Θ_T). The theorem above gives $T\sim S(\psi_1)\oplus\cdots\oplus S(\psi_n)$ with $\psi_{i+1}|\psi_i$ where $\psi_i=\det\Theta_T/\mathscr{E}_i(\Theta_T^{A})$. In the same time by [2] it holds $T\sim S(\mathscr{E}_n)\oplus\cdots$ $\cdots\oplus S(\mathscr{E}_1)$ with $\mathscr{E}_i|\mathscr{E}_{i+1}$ where $\mathscr{E}_i=\mathscr{E}_i(\Theta_T)$. From the unicity of the Jordan model of T it follows $\psi_i=\mathscr{E}_{n-i+1}$ $(i=1, \ldots, n)$, i.e.

$$\det\Theta_T=\mathscr{E}_i(\Theta_T^{A})\cdot\mathscr{E}_{n-i+1}(\Theta_T),\quad i=1, \ldots, n.$$

We shall prove this relation directly, by using the following well-known fact (see e.g. [10]):

Proposition. *Let M be an $n\times n$ matrix (n finite) over the complex numbers, M^A its adjoint matrix. Let M_1 be an $r\times r$ submatrix of M formed by the rows $i_1, \ldots, i_r$ $(1\leqq i_1<\ldots<i_r\leqq n)$ and the columns $j_1, \ldots, j_r$ $(1\leqq j_1<\ldots<j_r\leqq n)$. Let M_2 be the $(n-r)\times(n-r)$ submatrix of M^A obtained by leaving out of M^A its $i_1, \ldots, i_r$-th rows and $j_1, \ldots, j_r$-th columns. Then*

$$\det M_2 = (\det M)^{n-r-1}\cdot\det M_1\cdot(-1)^c \quad \textit{where} \quad c=\sum_{k=1}^{r}(i_k+j_k).$$

Now let N be an $n\times n$ matrix over H^∞. From the Proposition easily follows $\mathscr{D}_{n-r}(N^A)=(\det N)^{n-r-1}\mathscr{D}_r(N)$ $(r=1, \ldots, n)$ and $\mathscr{E}_{n-r+1}(N^A)=\mathscr{D}_{n-r+1}(N^A)/\mathscr{D}_{n-r}(N^A)=$ $=\det N\cdot\mathscr{D}_{r-1}(N)/\mathscr{D}_r(N)=\det N/\mathscr{E}_r(N)$, $r=1, \ldots, n$; whence (1).

Remark 2. Let $A=\operatorname{diag}[\varphi, \psi, \varphi, \psi, \ldots]$, $B=\operatorname{diag}[\varphi\psi, \varphi\psi, \ldots]=\varphi\psi I_\infty$, where $\varphi, \psi\in H^\infty$ are inner, $\varphi, \psi\neq 1$, $\varphi\wedge\psi=1$. Then A and B are matrices inner from both sides with a scalar multiple $\varphi\psi$. Obviously, $A^a=\operatorname{diag}[\psi, \varphi, \psi, \varphi, \ldots]$, $B^a=\operatorname{diag}[1, 1, \ldots]=I_\infty$. According to Theorem 1 we have $S(A)\sim\bigoplus_1^\infty S(\varphi\psi)=S(B)$. An easy computation shows on the other hand that A and B are not quasiequivalent. This situation cannot happen in the case of finite matrices (see [1]).

In the same manner the matrix $A^a=\operatorname{diag}[\psi, \varphi, \psi, \varphi, \ldots]$ is not quasiequivalent to the diagonal matrix formed by its invariant factors $\operatorname{diag}[\mathscr{E}_1(A^a), \mathscr{E}_2(A^a), \ldots]=$ $=B^a=I_\infty$.

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