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A remark on Gehér's theorem

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By SMIRNOV's theorem [3] every metrizable space X can be homeomorphically embedded into a Hilbert space. In [2] GEHÉR proved that for every metric on X this embedding can be chosen to be uniformly continuous. The aim of this note is to give a short and simple proof of the Gehér's result.

Theorem. *Every metric space (X, d) can be embedded into a Hilbert space by a uniformly continuous homeomorphism.*

Proof. By the Bing Metrization Theorem [1] the space X has a σ-discrete base $\mathscr{B}$. Let $\mathscr{B}=\{U_{(s,n)}\}_{(s,n)\in S\times N}$ where $U_{(s,n)}\cap U_{(s',n)}=\emptyset$ for every $s, s'\in S$, $s\neq s'$ and $n\in N$ (natural numbers). We may assume that every element of $\mathscr{B}$ has a diameter less than 1.

Denote by H the Hilbert space with $S\times N$ as the index set.

We show that the function $f\colon X\to H$ (well)-defined by

$$f(x)=\{2^{-n/2}(d(x, X-U_{(s,n)}))\}_{(s,n)\in S\times N}$$

is the embedding we were to construct.

The function f is uniformly continuous — for every two points $x, y\in X$ we have

$$\|f(x)-f(y)\|^2=\sum_{(s,n)\in S\times N}\frac{1}{2^n}[d(x, X-U_{(s,n)})-d(y, X-U_{(s,n)})]^2\leqq$$

$$\leqq\sum_{n\in N}\frac{1}{2^n}[d(x,y)]^2=[d(x,y)]^2.$$

On the other hand, for every open set U in X and every point $x\in U$ there is a pair $(s,n)\in S\times N$ such that $x\in U_{(s,n)}\subset U$. Hence, if $y\in X-U_{(s,n)}$, then

$$\|f(x)-f(y)\|^2\geqq\frac{1}{2^n}\{d(x, X-U_{(s,n)}\}^2$$

which proves that f is one-to-one and f^{-1} is continuous.

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References

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