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Unary algebras with regular endomorphism monoids

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The pair (A,f) where A is a non-void set and f is a unary operation will be briefly called a *unar*. For simplicity we often write A instead of (A,f). Let f^0 be the identity transformation and $f^n=ff^{n-1}$ for every $n\geqq 1$. We define a relation $\sim$ on the unar A as follows:

$$a \sim b \underset{\text{def}}{\Leftrightarrow} f^m(a) = f^n(b) \quad \text{for some} \quad m, n \geqq 0.$$

This relation turns out to be an equivalence relation, the classes of which are called *components*. A unar consisting of a single component is termed *connected*. An element a of a unar is *cyclic* if $f^n(a)=a$ for some $n\geqq 1$. A unar is called a *cycle of length n* if it consists of the distinct elements $a, f(a), \ldots, f^{n-1}(a)$ with $f^n(a)=a$. The term *loop* stands for a cycle of length 1. The set

$$a^{\triangle} \underset{\text{def}}{=} \{f^n(a) \mid n = 0, 1, 2, \ldots\}$$

is called the *upper cone* of the element a. If $f(x)=a$ then the element x is called a *parent* of a. A connected unar which is not a cycle but in which every element has a unique parent is said to be a *line*. A connected unar A is called a *cycle*, a *loop* or a *line with short tails* if A contains a cycle, resp. a loop or a line C such that $f(x)\in C$ for every $x\in A$. We agree on denoting the cardinality of a set A by $|A|$. If $X\subseteq A$, set $f(X) \underset{\text{def}}{=} \{f(x)|x\in X\}$.

The mapping φ of the unar A into the unar B is called a *homomorphism* if $\varphi(f(x))=f(\varphi(x))$ for all $x\in A$. In particular, if $A=B$ then we obtain the definition of an *endomorphism* of A. The set of all endomorphism of A forms a monoid which is denoted by End A. The set of all *automorphisms* (i.e. bijective endomorphisms) of A forms a group denoted by Aut A.

If m and n are positive integers or ∞ then the symbol $m|n$ means that either $n=\infty$ or $m, n\neq\infty$ and m divides n.

In the present paper the following results are established:

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Theorem 1. *The endomorphism monoid of a unar is regular if and only if each component of the unar is either a cycle with short tails or a line with short tails and for any components K, L and M the following conditions are satisfied:*

(1) *if* $|f(L)|\,\big|\,|f(K)|$, $|f(M)|\,\big|\,|f(L)|$ *and* $L \neq M$ *then* $|f(K)|=|f(L)|$;

(2) *if* $|f(L)|\,\big|\,|f(K)|$, $K \neq f(K)$ *and* $L \neq f(L)$ *then* $|f(K)|=|f(L)|$;

(3) *if* $|f(L)|\,\big|\,|f(K)|$ *and* $|L \setminus f(L)| \geqq 2$ *then* $K=f(K)$ *or* $K=L$.

Theorem 2. *The endomorphism monoid of a unar is an inverse semigroup if and only if every element in the unar has at most two parents, each of its components is either a cycle with short tails or a line with short tails and beyond conditions* (1)—(3), *the following are also fulfilled for any components K, L and M:*

(4) *if* $|f(L)|\,\big|\,|f(K)|$ *and* $|f(M)|\,\big|\,|f(K)|$ *then* $K=L$ *or* $K=M$ *or* $L=M$;

(5) *if* $K \neq L$ *and* $|f(L)|\,\big|\,|f(K)|$ *then* $|f(L)|=1$ *and* $|f(K)|>1$, *and if, in addition,* $L \neq f(L)$ *then* $K=f(K)$.

Theorem 3. *The endomorphism monoid of a unar is a group if and only if each of its components is either a cycle or a line and for arbitrary components K and L the relation* $|L|\,\big|\,|K|$ *implies* $K=L$.

In the proof of these theorems we need some lemmas. The first one characterizes inverse semigroups, while the others concern the unar (A,f).

Lemma 1. (cf. [1] Theorem 1.17) *The following conditions on a semigroup S are equivalent:*

(i) *S is regular and any two idempotents of S commute with each other;*

(ii) *S is an inverse semigroup (i.e., every element of S has a unique inverse).*

Lemma 2. (cf. [2] Theorem 2.4) *In a connected unar A the following conditions are equivalent:*

(i) *A is either a cycle or a line;*

(ii) *f is bijective;*

(iii) *the endomorphisms of A are the elements of the set* $\{f^k\colon k=0, \pm 1, \pm 2, \ldots\}$.

Lemma 3. (cf. [2] Lemma 2.8) *If C is a cycle of length n in A and* $a \in C$ *then for every endomorphism* φ *the element* $a\varphi$ *is contained in a cycle of length p where p divides n.*

Lemma 4. (cf. [2] Lemma 2.11) *If* $a, b \in A$ *belong to the same component K,* $\varphi \in \operatorname{End} A$ *and* $a\varphi$ *belongs to the component L then* $b\varphi$ *also belongs to L.*

The following lemma is easily verified.

Lemma 5. *The set* $f(A)$ *is a subalgebra in A which is invariant with respect to every endomorphism in* End *A*.

Lemma 6. *If* End *A is regular then* $f \in \operatorname{Aut} f(A)$.

Proof. Since $f \in \operatorname{End} A$, we have $f\Phi f = f$ for some $\Phi \in \operatorname{End} A$. If $x \in f(A)$, i.e., $x = f(y)$ for some $y \in A$ then we have

$$f\Phi(x) = f\Phi f(y) = f(y) = x \quad \text{and} \quad \Phi f(x) = f\Phi(x) = x,$$

which completes the proof.

Lemma 7. *If K is a component in A then* End K *can be embedded in* End A. *If* End A *is a regular or an inverse semigroup or a group then* End K *has the same property.*

Proof. If $\varphi \in \operatorname{End} K$ then put

$$\Phi(\varphi)(x) = \begin{cases} \varphi(x) & \text{if } x \in K \\ x & \text{otherwise} \end{cases}$$

for every x in A. It is easy to see that Φ embeds End K in End A. If there exists $(\Phi(\varphi))^{-1} \in \operatorname{End} A$ then K is invariant with respect to $(\Phi(\varphi))^{-1}$ and, consequently, the restriction of $(\Phi(\varphi))^{-1}$ to K can be chosen as φ^{-1}. Hence, End K is a group provided End A is a group. Assume now that the monoid End A is regular. Then $\Phi(\varphi)\Psi\Phi(\varphi) = \Phi(\varphi)$ for some $\Psi \in \operatorname{End} A$. If $\Psi(K) \subseteq K$ then the regularity of End K follows. In the opposite case we have $\Psi(a) \notin K$ for some $a \in K$. Then Lemma 4 implies that

$$\Phi(\varphi)\Psi\Phi(\varphi)(a) = \Phi(\varphi)\Psi(\varphi(a)) = \Psi(\varphi(a)) \notin K,$$

in contrary to the fact that $\Phi(\varphi)(a) = \varphi(a) \in K$. Finally, it remains to note that the rest follows from Lemma 1 since $\varepsilon^2 = \varepsilon$ implies $(\Phi(\varepsilon))^2 = \Phi(\varepsilon)$.

Lemma 8. *Let K and L be cycles with short tails or lines with short tails such that $|f(L)|\,\big|\,|f(K)|$. Let $a \in K$ and $b \in L$. Then there exists a homomorphism $\varphi\colon K \to L$ such that $\varphi(a) = b$ and $\varphi(x) \in f(L)$ for every $x \neq a$.*

Proof. If $f^m(a) = f^n(a)$ and $m > n$ then $|f(K)|\,\big|\,(m-n)$ and, moreover, $n \geqq 1$ provided $a \notin f(K)$. Since $|f(L)|\,\big|\,(m-n)$, we have $f^m(b) = f^n(b)$. Thus there exists a homomorphism $\varphi\colon a^\Delta \to b^\Delta$. If $|f(K)| < \infty$ then $\varphi(f(K)) = f(L)$. If $|f(K)| = \infty$, i.e. $f(K)$ is a line, then φ can be naturally continued to a homomorphism $\varphi\colon (a \cup f(K)) \to L$ such that again $\varphi(f(K)) = f(L)$. If $x \in K \setminus f(K)$ then $\varphi(f(x))$ is defined and there exists a unique element $x' \in f(L)$ such that $f(x') = \varphi(f(x))$. Choosing $\varphi(x) = x'$ we obtain the required homomorphism.

The proof of Theorem 1. Let A be a unar and End A a regular monoid. If A is connected then, by Lemma 6, we have $f \in \operatorname{Aut} f(A)$. Then Lemma 2 implies $f(A)$ to be a cycle or a line. In view of Lemma 7 the components of A have the required structure. Now suppose the components K, L and M satisfy the assumptions of property (1). Owing to Lemma 8, there exist homomorphisms $\varphi\colon K \to L$

and $\psi: L \to M$. For every $x \in A$ put

$$\Phi(x) = \begin{cases} \varphi(x) & \text{if } x \in K \\ \psi(x) & \text{if } x \in L \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\Phi \in \text{End } A$. Since End A is regular, we have $\Phi\Psi\Phi = \Phi$ for some $\Psi \in \text{End } A$. If $x \in K$ then $\Phi(x) \in L$. Since $\Phi\Psi\Phi(x) = \Phi(x)$, we conclude $\Psi\Phi(x) \in K$. Hence it follows by Lemma 4 that $\Psi(L) \subseteqq K$. Consequently, $|f(K)| \,|\, |f(L)|$ by Lemma 3 and therefore $|f(K)| = |f(L)|$. Let us assume now that the components K and L fulfil the assumptions of property (2). Choose $a \in K \setminus f(K)$, $b \in L \setminus f(L)$ and, making use of Lemma 8, let $\varphi: K \to L$ and $\psi: L \to L$ be homomorphisms satisfying $\varphi(a) = b$ and $\psi(L) \subseteqq f(L)$, respectively. We define $\Phi \in \text{End } A$ as above and select $\Psi \in \text{End } A$ such that $\Phi\Psi\Phi = \Phi$. If $|f(K)| \neq |f(L)|$ then $\Psi(L) \cap K = \emptyset$ by Lemmas 3 and 4. Consequently, $\Phi\Psi\Phi(a) \neq b = \Phi(a)$ which is a contradiction. Finally, let K and L satisfy the assumptions of property (3). Choose $b, c \in L \setminus f(L)$ such that $b \neq c$. Suppose there exists $a \in K \setminus f(K)$. By Lemma 8, we can find homomorphisms $\varphi: K \to L$, $\psi: L \to L$ such that $\varphi(a) = b, \psi(b) = c$ and $\varphi(x), \psi(y) \in f(L)$ provided $x \neq a$ and $y \neq b$. Define Φ as above and choose Ψ such that $\Phi\Psi\Phi = \Phi$. If $K \neq L$ then, by Lemma 4, we have $\Psi(L) \cap K = \emptyset$ or $\Psi(L) \cap L = \emptyset$. In the first case we obtain that $\Phi\Psi\Phi(a) = \Phi\Psi(b) \neq b = \Phi(a)$ while in the second case we have $\Phi\Psi\Phi(b) = \Phi\Psi(c) \neq c = \Phi(b)$. But, of course, both cases are impossible. Thus the necessity of the conditions of Theorem 1 is proved.

Conversely, suppose now that the unar A satisfies these conditions and $\Phi \in \text{End } A$. For every component L consider the set of components

$$L^{\triangle} = \{K \mid \Phi(K) \subseteqq L\}.$$

We establish that the following statement is valid:

If $L^{\triangle} \neq \emptyset$ then there exists a component L^0 and a homomorphism $\psi_L: L \to L^0$ such that $\Phi\psi_L(x) = x$ for every $x \in \text{Im } \Phi \cap L$.

In fact, taking into consideration Lemma 4, denote by M the component containing $\Phi(L)$. By the structure of the components of A we have $f(L) \subseteqq \text{Im } \Phi$. Suppose first that $\text{Im } \Phi \cap L = f(L)$. If $M = L$ then choose an element $a \in f(L)$ and, putting $L^0 = L$, choose an element $b \in f(L^0)$ with $\Phi(b) = a$. Applying Lemma 8 we can find a homomorphism $\psi_L: L \to L^0$ with $\psi_L(a) = b$. If $x \in \text{Im } \Phi \cap L$ and $x = f^k(a)$ for some k then

$$\Phi\psi_L(x) = f^k \Phi\psi_L(a) = f^k \Phi(b) = f^k(a) = x.$$

If there exists no such k then $f(L)$ is a line. Therefore $f^k(x) = a$ for some k whence we have

$$f^k(\Phi\psi_L(x)) = \Phi\psi_L(a) = \Phi(b) = a = f^k(x).$$

Since $\Phi(L)=f(L)$, it follows that $\Phi\psi_L(x)=x$. If $M\neq L$ then, by Lemma 4, we can see that $\Phi(K)\subseteq L$ for a component $K\neq L$. Lemma 3 and property (1) imply that $|f(K)|=|f(L)|$. Then we can set $L^0=K$ and literally repeat the foregoing argument. Assume now that $\operatorname{Im}\Phi\cap L\neq f(L)$. If there exists a component K in $L^\vartriangle$ such that $K\neq L$ and $K\neq f(K)$ then, by property (3) and Lemma 3, we obtain that $L\setminus f(L)$ consists of a single element, say a. Then $a=\Phi(b)$ for some $b\in A$ and we can choose L^0 to be the component containing b. It is easy to see that $b\notin f(L^0)$. Due to property (2), $|f(L^0)|=|f(L)|$ which allows us to apply the above reasoning again. It remains to treat the case when $\operatorname{Im}\Phi\cap L\neq f(L)$ and $K=f(K)$ for each $K\in L^\vartriangle\setminus\{L\}$. Then $L\in L^\vartriangle$. There is no difficulty in verifying that Φ induces an automorphism, say φ, on $f(L)$. Let $\psi_L\colon f(L)\to f(L)$ be the inverse of this automorphism. For every $x\in(\operatorname{Im}\Phi\cap L)\setminus f(L)$, choose and fix an $x'\in L$ with $\Phi(x')=x$ and set $\psi_L(x)=x'$. Then ψ_L maps $\operatorname{Im}\Phi\cap L$ into L and

$$\psi_L(f(x))=\psi_L f\Phi(x')=\psi_L\Phi(f(x'))=\psi_L\varphi(f(x'))=f(x')=f\psi_L(x).$$

Just as above, we extend ψ_L to a homomorphism of L into L for which we will use the same notation ψ_L and set $L^0=L$.

Returning to the proof of the theorem, put

$$\Psi(x)=\begin{cases}\psi_L(x) & \text{if } x\in L \text{ with } L^\vartriangle\neq\emptyset\\ x & \text{otherwise.}\end{cases}$$

Obviously, $\Psi\in\operatorname{End}A$. Moreover, we have $L^\vartriangle\neq\emptyset$ provided L is a component containing $\Phi(x)$ for some $x\in A$. Hence, utilizing the property of the homomorphism ψ_L we conclude that $\Phi\Psi\Phi(x)=\Phi\psi_L\Phi(x)=\Phi(x)$ which proves the regularity of the monoid $\operatorname{End}A$.

The proof of Theorem 2. Let A be a unar and $\operatorname{End}A$ an inverse monoid. Suppose a, b, c are distinct elements in A and $f(a)=f(b)=f(c)$. Denote by K the component containing these elements. By Theorem 1, $f(K)$ is a cycle or a line. Therefore, for example, $a, b\notin f(K)$. The transformations ε and δ defined by

$$\varepsilon(x)=\begin{cases}b & \text{if } x=a\\ x & \text{otherwise}\end{cases}\quad\text{and}\quad\delta(x)=\begin{cases}a & \text{if } x=b\\ x & \text{otherwise,}\end{cases}$$

respectively, turn out to be endomorphisms of K. Here $\varepsilon^2=\varepsilon$, $\delta^2=\delta$,

$$\varepsilon\delta(a)=\varepsilon(a)=b\quad\text{and}\quad\delta\varepsilon(a)=\delta(b)=a.$$

Since the idempotents in an inverse semigroup commute with each other by Lemma 1, this contradicts Lemma 7. Thus, every element of A has at most two parents. The validity of conditions (1)—(3) is implied by Theorem 1. Assume now that the distinct components K, L and M satisfy the assumptions of property (4). Owing to Lemma 8, there exist homomorphisms $\varphi\colon K\to L$ and $\psi\colon K\to M$. The transformations Φ and Ψ where

$$\Phi(x)=\begin{cases}\varphi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases} \quad\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases}$$

are easily shown to be endomorphisms of A. Here $\Phi^2=\Phi$, $\Psi^2=\Psi$. Still, if $x\in K$, we have

$$\Phi\Psi(x)=\Phi(\psi(x))=\psi(x)\in M \quad\text{and}\quad \Psi\Phi(x)=\Psi(\varphi(x))=\varphi(x)\in L,$$

which, by Lemma 1, fails to hold in the inverse monoid End A. If K and L are distinct components with $|f(L)|\,\big|\,|f(K)|$ and $|f(L)|\geqq 2$ then select elements $a\in f(K)$ and $b, c\in f(L)$ such that $b\neq c$. Lemma 8 implies the existence of homomorphisms $\varphi\colon K\to L$ and $\psi\colon K\to L$ such that $\varphi(a)=b$ and $\psi(a)=c$. Furthermore, we define endomorphisms Φ and Ψ by setting

$$\Phi(x)=\begin{cases}\varphi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases} \quad\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{if } x\in K\\ x & \text{if } x\notin K.\end{cases}$$

Then $\Phi^2=\Phi$, $\Psi^2=\Psi$ and $\Phi\Psi(a)=c\neq b=\Psi\Phi(a)$. If $f(K)=\{v\}$ and $f(L)=\{w\}$ then $\Phi^2=\Phi$, $\Psi^2=\Psi$ and $\Phi\Psi(v)=v\neq w=\Psi\Phi(v)$, where

$$\Phi(x)=\begin{cases}v & \text{if } x\in K\cup L\\ x & \text{if } x\notin K\cup L\end{cases} \quad\text{and}\quad \Psi(x)=\begin{cases}w & \text{if } x\in K\\ x & \text{if } x\notin K.\end{cases}$$

This contradicts Lemma 1 as above. If $|f(L)|=1$ and assume $L\neq f(L)$ and $K\neq f(K)$ then, by property (3), $L=\{b, w\}$ where $f(b)=f(w)=w$. Putting

$$\Phi(x)=\begin{cases}w & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases} \quad\text{and}\quad \Psi(x)=\begin{cases}w & \text{if } x\in f(K)\\ b & \text{if } x\in K\setminus f(K)\\ x & \text{if } x\notin K,\end{cases}$$

we can see that $\Phi, \Psi\in$ End A, $\Phi^2=\Phi$ and $\Psi^2=\Psi$. However, for every $x\in K\setminus f(K)$ we have

$$\Phi\Psi(x)=\Phi(b)=b \quad\text{and}\quad \Psi\Phi(x)=\Psi(w)=w,$$

which is impossible. Thus we have proved the necessity of the conditions of Theorem 2.

Assume now that these conditions are satisfied in the unar A. In consequence of Theorem 1, End A is a regular monoid. Let $\Phi, \Psi\in$ End A such that $\Phi^2=\Phi$ and $\Psi^2=\Psi$. By Lemma 1, we have only to show that $\Phi\Psi=\Psi\Phi$. Let x be an arbitrary element in A and K the component containing x. Denote by L and M the components containing $\Phi(x)$ and $\Psi(x)$, respectively. By Lemma 4, $\Phi(K)\subseteq L$ and $\Psi(K)\subseteq M$. By virtue of Lemma 3 and property (4) we have $K=L$, $K=M$

or $L=M$. If $K=L=M$ then both Φ and Ψ induce idempotent endomorphisms on $f(K)$. Thus Lemma 2 implies that $\Phi(z)=\Psi(z)=z$ for every $z\in f(K)$, i.e. $\Phi\Psi(x)= =x=\Psi\Phi(x)$ provided $x\in f(K)$. Otherwise, if $x\notin f(K)$ then, since $f(x)$ has at most two parents, we obtain $\Phi(x)=x$ or $\Phi(x)=x'$ where $x'\in f(K)$ and $f(x')=f(x)$. A similar statement holds for Ψ, too. If $\Phi(x)=\Psi(x)=x$ then $\Phi\Psi(x)=x=\Psi\Phi(x)$. If $\Phi(x)=x'$ or $\Psi(x)=x'$ then we have $\Phi\Psi(x)=x'=\Psi\Phi(x)$. Suppose now $L=M$ but $K\neq L$. Then property (5) implies that $|f(L)|=1$ and either $K=f(K)$ or $L=f(L)$. Hence we have $\Phi(z)=\Psi(z)=w$ for every $z\in K$ where w denotes the single element in $f(L)$. Moreover, $\Phi(w)=\Phi^2(z)=\Phi(z)=w$. Analogously, $\Psi(w)=w$. Thus

$$\Phi\Psi(x) = \Phi(w) = w = \Psi(w) = \Psi\Phi(x).$$

Finally, consider the case when $K=L$ but $L\neq M$. Property (5) implies that $|f(M)|=1$ and either $K=f(K)$ or $M=f(M)$. Denoting by w the single element of $f(M)$, we conclude as above that $\Psi(z)=w$ for every $z\in K$. In addition, properties (4) and (5) imply $\Phi(M)\subseteq M$ by Lemma 3. Thus

$$\Phi\Psi(x) = \Phi(w) = w = \Psi\Phi(x).$$

The case when $K=M$ but $M\neq L$ is handled similarly. Therefore $\Phi\Psi=\Psi\Phi$ which completes the proof.

The proof of Theorem 3. Let A be a unar and End A a group. If $|A|=1$ then the conditions of Theorem 3 are trivially fulfilled. Let $|A|\neq 1$. Lemmas 2 and 7 imply each component to be a cycle or a line. If we have distinct components K and L with $|L|\,\big|\,|K|$ then, according to property (5) in Theorem 2, $|L|=1$. If w is the single element in L then, defining Φ by $\Phi(x)=w$ for every $x\in A$, we have $\Phi\in\text{End}\,A=\text{Aut}\,A$. Consequently, $A=L$, contradicting our assumption. The proof of the necessity of the conditions of Theorem 3 is complete. In the case when these conditions are satisfied it is not difficult to show by Lemma 3 that every endomorphism induces an endomorphism on each component. To complete the proof it remains only to make use of Lemma 2.

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