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The value distribution of entire functions of order at most one

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§ 1. Introduction and results

Recently S. KIMURA [6] proved

Theorem A. *Let f be an entire function of order less than one and w_n a sequence such that $|w_n|\to\infty$ as $n\to\infty$. Suppose that all the roots of the equations $f(z)=w_n$ $(n=1, 2, \dots)$ lie in a half-plane (say $\mathrm{Re}\, z\geqq 0$). Then f is a polynomial of degree at most 2.*

We begin by improving Theorem A a little to

Theorem 1. *If f is an entire function whose growth is at most order one and minimal type, and w_n is a sequence such that $|w_n|\to\infty$ while all roots of $f(z)=w_n$ $(n=1, 2, \dots)$ lie in a half-plane, then f is a polynomial of degree at most 2.*

In this form the theorem is sharp. For any $d>0$ the function e^{dz} has type d and is bounded in $\mathrm{Re}\, z\leqq 0$ so that any sequence w_n such that $1<|w_n|\to\infty$ may be taken to satisfy the hypothesis in $\mathrm{Re}\, z\geqq 0$.

Theorem 1 has an application in the theory of iteration of entire functions (see e.g. FATOU [5] for proofs of the following results). The iterates f^n of an entire function f are defined by $f^1=f$, $f^{n+1}=f^n\circ f=f\circ f^n$ $(n=1, 2, \dots)$. If f is non-linear the set $\mathfrak{C}(f)$ of points in whose neighbourhood $\{f^n\}$ is a normal family, is a proper open subset of the plane. The complement $\mathfrak{F}(f)$ of $\mathfrak{C}(f)$ is a non-empty, unbounded, perfect set. $\mathfrak{F}(f)$ has the *invariance property:*

If $w\in\mathfrak{F}(f)$ and $f(z)=w$, then $z\in\mathfrak{F}(f)$ and $f(w)\in\mathfrak{F}(f)$.

In iteration theory the fixed points of f are important. A fixed point z of f of order k is a solution of $f^k(z)=z$. It is proved in [5] that every point of $\mathfrak{F}(f)$ is a limit point of fixed points of f.

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It may happen that a component of $\mathfrak{C}(f)$ contains a half-plane. Thus for $d>0$ the function

$$g(z) = d^{-1}(e^{dz}-1) \tag{1}$$

maps $H=\{z: \operatorname{Re} z<0\}$ into itself so that $\{g^n\}$ is normal in H.

Suppose that conversely g is a transcendental entire function and that $\mathfrak{C}(g)$ contains a half-plane, which we may take to be $\operatorname{Re} z<0$. Then $\mathfrak{F}(g)$ lies in $\operatorname{Re} z\geqq 0$ and if we take a sequence $w_n\in\mathfrak{F}(g)$ such that $|w_n|\to\infty$, all solutions of $f(z)=w_n$ lie in $\mathfrak{F}(g)$ by the invariance property, and hence in $\operatorname{Re} z\geqq 0$. Thus from Theorem 1 we have

Theorem 2. *If g is a transcendental entire function such that the domain of normality $\mathfrak{C}(g)$ of $\{g^n\}$ contains a half-plane, then the growth of g must be at least of order* 1, *positive type.*

Example (1) shows that this is sharp with respect to growth. Related problems have been discussed under more restrictive conditions by P. BHATTACHARYYA [4].

If $0\in\mathfrak{F}(g)$ then every solution z of $g(z)=0$ belongs to $\mathfrak{F}(g)$. The following Theorem 3a is thus a strengthening of theorem 2.

We introduce the notation

$$A(\theta,\delta) = \{z: |\arg z-\theta| < \delta\}. \tag{2}$$

Theorem 3a. *Suppose* (i) *g is a transcendental entire function whose growth is at most of order* 1, *minimal type,* (ii) *all the zeros of g lie in* $\operatorname{Re} z\geqq 0$.

Then for any $\delta>0$ the set $\mathfrak{F}(g)\cap A(\pi,\delta)$ is unbounded.

Because of the importance of fixed points it is interesting that we can also prove

Theorem 3b. *If in* 3a (ii) *is replaced by the hypothesis that the first order fixed points lie in* $\operatorname{Re} z\geqq 0$, *the conclusion remains true.*

The example (1), for which all first order fixed points lie in $\operatorname{Re} z\geqq 0$, shows that 3b ceases to hold if the assumption of minimal type is dropped.

In the circumstances of Theorems 3a or 3b it follows that $A(\pi,\delta)$ must contain fixed points of some order of g. Can one be more explicit about the order of such fixed points? Let us take 3b and make the stronger hypothesis in (ii) that all the first order fixed points are real and positive. Our methods and results differ slightly according to the order of g. For order less than $\frac{1}{2}$ we have

Theorem 4a. *Suppose* (i) *g is transcendental entire of at most order $\frac{1}{2}$, minimal type, and*

(ii) *all but finitely many first order fixed points of g are real and positive.*

Then for any $\delta>0$, $A(\pi,\delta)$ contains infinitely many fixed points of order k for each $k\geqq 2$.

Indeed the fixed points of higher order, whose existence is shown in the theorem can be taken to be non-real. This is somewhat analogous to the result of the first author in [2] that if f is transcendental entire of order less than $\frac{1}{2}$ and l is a straight line, then not all solutions of $f^2(z)-z=0$ lie in l. Neither result includes the other but both show that second order fixed points tend to be scattered in their angular distribution.

If the order of g exceeds $\frac{1}{2}$ we have not been able to prove the existence of fixed points of order 2 in $A(\pi, \delta)$. However we can prove

Theorem 4b. *If in Theorem* 4a (i) *is replaced by the assumption that the order of g is strictly positive, but at most order* 1 *minimal type, then for any* θ, δ *subject to* $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, $\delta>0$, *we have that* $A(\theta, \delta)$ *contains infinitely many fixed points of order k for each* $k\geqq 3$.

Thus in particular if g is at most of order 1 minimal type and all first order fixed points are real and positive, f has fixed points of every order greater than 2 in $A(\pi, \delta)$, however small $\delta>0$ is taken.

The arguments used in this discussion can also be applied to show that functions of certain classes are not expressible as iterates of entire functions. An example is furnished by

Theorem 5. *Suppose the transcendental function* F *is such that*

(i) $\limsup_{r\to\infty} \{\log\log\log M(F, r)\}/\log r<1$,

(ii) *all first order fixed points of* F *lie in* $\operatorname{Re} z\geqq 0$, *and*

(iii) F *is bounded in* $A(\pi, \delta)$ *for some* $\delta>0$.

Then F *is not expressible as* $f^k, k\geqq 2$, *for any entire* f.

In (ii) we may replace fixed points by zeros without affecting the validity of the theorem. The function e^{e^z} has all its fixed points in $\operatorname{Re} z\geqq 0$ and shows that we cannot allow equality in (i).

§ 2. Proof of Theorem 1

We may assume $f(0)\neq 0$ (otherwise consider $f(z-\delta)$ for a suitable positive constant δ).

We shall use the following results about functions of minimal type whose zeros lie in a half-plane. They may be found e.g. in the proof of theorem 1 of [8], where the additional hypothesis $f(-r)=O(r^k)$ of that theorem is not used until after these facts have been derived.

Lemma 1. *Let f be a transcendental entire function of at most order one and minimal (i.e. zero) exponential type. Suppose $f(0)\neq 0$ and that all zeros a_n of f lie in the right half plane* $\operatorname{Re} z\geqq 0$. *Then there are constants A and c such that*

$$f(z) = Ae^{cz}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}, \tag{3}$$

where $a_n=r_n e^{i\theta_n}$ *is such that*

$$\lambda = \operatorname{Re}\sum_{n=1}^{\infty} a_n^{-1} = \sum_{n=1}^{\infty}(\cos\theta_n)/r_n \tag{4}$$

is convergent and

$$\lambda+\operatorname{Re} c = 0. \tag{5}$$

Further, for any fixed k

$$|f(-r)|/r^k \to \infty \quad \text{as} \quad r\to\infty. \tag{6}$$

Proof of Theorem 1. We may suppose $w_1=0$ (for otherwise consider $f(z)-w_1$) and suppose first that f is transcendental entire of at most order one, minimal type and that all solutions of $f(z)=w_n$ lie in H: $\operatorname{Re} z\geqq 0$. In particular the zeros $a_n=r_n e^{i\theta_n}$ lie in H, so by Lemma 1

$$\frac{f'(z)}{f(z)} = c+\sum_{n=1}^{\infty}\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right).$$

Using (4) and (5) this yields

$$\operatorname{Re}\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\operatorname{Re}\frac{1}{z-a_n}. \tag{7}$$

If $\operatorname{Re} z<0$ and $\operatorname{Re} a\geqq 0$ we have $\operatorname{Re}\frac{1}{z-a}<0$, while if $z=\varrho e^{i\varphi}$ then for fixed φ

$$|z|\operatorname{Re}\frac{1}{z-a}\to\cos\varphi \quad \text{as } \varrho\to\infty.$$

Thus by (7), if δ is a fixed number such that $0<\delta<\frac{\pi}{2}$,

$$|z|\operatorname{Re}\frac{f'(z)}{f(z)}\to-\infty \quad \text{as} \quad z\to\infty \quad \text{in} \quad A(\pi,\delta).$$

Take a fixed constant $K>2\pi/\delta$. Then there is a constant r_0 such that

$$|z|\left|\frac{f'}{f}\right| > K \quad \text{for} \quad z\in A(\pi,\delta), \quad |z|>r_0. \tag{8}$$

Next choose a member of the given sequence w_n so that $|f(z)|<|w_n|$ for $|z|\leqq r_0$. By (6) there is a largest r_n such that $|f(-r_n)|=|w_n|$. There is a component G of $\{z: |f(z)|>|w_n|\}$ which contains $\{z: z=-r<-r_n\}$ and this component is bounded by a level curve $\Gamma\colon |f(z)|=|w_n|$ which passes through $z=-r_n$. Γ cannot close in $\operatorname{Re} z<0$ for there are no zeros of f in this region.

If Γ meets neither of the lines $\arg z=\pi\pm\delta$, then G lies entirely in the angle $A(\pi, \delta)$. Let $r\theta(r)$ be the length of that segment γ_r of $|z|=r$ which lies in G and contains $z=-r$. By the arguments used in the proof of the Denjoy—Carleman—Ahlfors theorem in [9, pp. 310—311] it follows that for all sufficiently large $r(>r_1$ say) the maximum modulus function $M(f, r)$ of f satisfies

$$\log\log M(f, r) > \log\log \operatorname*{Max}_{\gamma_r} |f(z)| > \pi \int_{r_1}^{r} \frac{dt}{t\theta(t)}+C,$$

for a suitable constant C. Since $\theta(r)<2\delta$ this implies that f has order at least $\pi/2\delta>1$, which is impossible.

Thus there is a level curve $\Gamma\colon |f(z)|=|w_n|$, which starts at $z=-r_n$ and runs to either $\arg z=\pi+\delta$ or $\pi-\delta$. Moreover Γ lies in $|z|\geqq r_0$ so that the inequality (8) holds on Γ. But $w=f(z)$ maps Γ onto $|w|=|w_n|$ and as z traverses Γ, w traverses $|w|=|w_n|$ without change of direction. Further, we have

$$\frac{dw}{w}=\frac{dz}{z}\,\frac{zf'(z)}{f(z)},$$

whence, if $w=|w_n|e^{i\varphi}$ and $z=re^{i\theta}\in\Gamma$ we have

$$i\,d\varphi = \left(\frac{dr}{r}+i\,d\theta\right)\frac{zf'(z)}{f(z)}, \tag{9}$$

so that by (8)

$$|d\varphi| \geqq |d\theta|\left|\frac{zf'(z)}{f(z)}\right| > K|d\theta|.$$

The image of Γ is therefore an arc of $|w|=|w_n|$ whose angular measure is at least $K\delta>2\pi$. Thus Γ, and in particular $A(\pi, \delta)$ must contain a root z of $f(z)=w_n$, against the hypothesis of the theorem.

We conclude that f cannot therefore be transcendental. If f is a polynomial its degree can clearly not exceed two.

§ 3. Proof of Theorem 3

Suppose g is a transcendental entire function of growth at most order 1, minimal type and is such that

$$A(\pi, \delta)\cap \mathfrak{F}(g) \tag{10}$$

is bounded for some $\delta>0$. Without loss of generality we may assume that the set in (10) is empty — it is only necessary to shift the origin and consider the iteration of $g(z+a)-a$ for sufficiently large negative a.

Whether the zeros of $g(z)$ or the fixed points (i.e. the zeros of $g(z)-z$) are in $\operatorname{Re} z \geqq 0$ it follows from Lemma 1 that for any k

$$g(-r)/r^k \to \infty \quad \text{as} \quad r \to \infty. \tag{11}$$

Since $A = A(\pi, \delta)$ does not meet $\mathfrak{F}$, A belongs to an unbounded component G of the set $\mathfrak{C}(g)$ of normality of g^n. Indeed by [3] G is simply-connected. The boundary ∂G belongs to $\mathfrak{F}$ and is a continuum in the complex sphere. By the invariance property of $\mathfrak{F}$, $g(z)$ omits all the values of ∂G for $z \in A$.

If $M = \pi/(2\delta)$ the transformation

$$u = (1+t)/(1-t), \quad z = -u^{\frac{1}{M}} \tag{T}$$

maps $|t|<1$ onto A, so that the function

$$w = h(t) = g\left\{-\left(\frac{1+t}{1-t}\right)^{\frac{1}{M}}\right\}$$

is regular in $|t|<1$ and omits the values $w \in \partial G$.

By a result of J. E. Littlewood [7]

$$M(h, \varrho) = O\{(1-\varrho)^{-2}\} \quad \text{as} \quad \varrho \to 1^-.$$

If $z = re^{i\theta} \in A$, and $|\theta - \pi| < \delta/2$, then in (T)

$$\begin{aligned} t &= (1 - e^{Mi(\pi-\theta)} r^{-M})/(1 + e^{Mi(\pi-\theta)} r^{-M}) \\ &\sim 1 - 2e^{Mi(\pi-\theta)} r^{-M} \quad \text{as} \quad r \to \infty. \end{aligned}$$

Since $|M(\pi-\theta)| < \pi/4$ we have $1-|t| > r^{-M}$ for large r. Thus as $z = re^{i\theta} \to \infty$ in $|\theta - \pi| < \frac{1}{2}\delta$ we have

$$|g(z)| = |h(t)| < M(h, 1-r^{-M}) = O(r^{2M}).$$

But this conflicts with (11). The result follows.

§ 4. Preliminaries to the proof of Theorems 4a and 4b

Throughout this section assume that g is an entire function such that

(i) g is transcendental and of at most order one, minimal type,

(ii) all but finitely many fixed points of first order of g are real and positive.

Then we have

$$g(z) - z = p(z) e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}},$$

where p is a polynomial of degree say $d \geqq 0$, and $a_n > 0$. Applying lemma 1 to $\{g(z)-z\}/p(z)$ we see that $\sum a_n^{-1}$ converges and in fact

$$g(z)-z = p(z)\exp(i\gamma z)\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right),$$

where γ is real. If $\gamma \neq 0$ then

$$\operatorname{Max}|g(\pm iy)| > \exp|\gamma y|$$

so $\gamma=0$ since g has minimal type. Thus

$$g(z) = z+h(z), \quad h(z) = p(z)Q(z) = p(z)\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right). \tag{12}$$

Lemma 2. *If g satisfies* (i), (ii) *then there is some $r_0>0$ such that $|g(-r)|$ is increasing for $r>r_0$, so that $w=g(-r)$, $r>r_0$ describes a simple curve Γ. Γ approaches infinity in a limiting direction* $\arg w=\alpha$.

For, let δ satisfy $0<\delta<\frac{\pi}{2}$. From (12) it follows that as $z\to\infty$ in $A(\pi,\delta)$ we have $|h(z)/z|\to\infty$ and

$$\left|\frac{zh'}{h}\right| = \left|\frac{zp'}{p}+\frac{zQ'}{Q}\right| = \left|d+o(1)+\frac{zQ'}{Q}\right| \to \infty,$$

(c.f. (7) and (8) in theorem 1). Thus $|h'(z)|\to\infty$ and

$$\frac{zg'}{g} = \frac{zh'}{h}\frac{(1+1/h')}{(1+z/h)} \to \infty \quad \text{as} \quad z\to\infty \quad \text{in} \quad A(\pi,\delta). \tag{13}$$

In particular

$$g'(-r)/g(-r) = \frac{-1}{r}\left\{d+o(1)+\sum_{n=1}^{\infty}\frac{r}{r+a_n}\right\} \quad \{1+o(1)\} \tag{14}$$

as $r\to\infty$, and if $g(-r)=\mathrm{Re}^{i\varphi}$ we have

$$\frac{dR}{R}+i\,d\varphi = \frac{g'(-r)}{g(-r)}(-dr). \tag{15}$$

By (14) the argument of (15) approaches zero as $r\to\infty$, so that $\frac{dR}{R}>0$ for large r.

Clearly $|h(-r)|\to\infty$ faster than any power of r and $\arg h(-r)$ tends to a constant value, namely the argument of the leading coefficient of $p(z)$. Hence $\arg g(-r)$ approaches the same limit. The lemma is proved.

Lemma 3. *If g satisfies* (i), (ii) *then, given any real θ_0, δ, σ such that* $0<\delta<\frac{\pi}{2}$, $0<\sigma\leqq\pi$, *there exist a constant R_1 and two branches ψ and χ of $z=g^{-1}(w)$ regular in*

$$S = A(\theta_0, \sigma)\cap\{|w| > R_1\},$$

such that the values of ψ, χ satisfy $\pi-\delta<\arg\psi<\pi$ and $\pi<\arg\chi<\pi+\delta$, respectively. For any $k>0$ we have

$$\mathrm{Max}\,\{|\psi(w)|, |\chi(w)|\} = O\big(|w|^{\frac{1}{k}}\big) \quad as \quad w\to\infty \quad in \quad S. \tag{16}$$

Proof. As w traverses Γ from $w_0=g(-r_0)$ to ∞ the branch of $z=g^{-1}(w)$ such that $r_0=g^{-1}(w_0)$ has a regular continuation and the values of z are all real and negative $(<-r_0)$.

For $r_1>r_0$ put $R=|g(-r_1)|$ and consider the level-curve $\lambda=|g(z)|=R$ which passes through $z=-r_1$. Along λ we have as in (9)

$$i\,d\varphi = (zg'/g)\left\{i\,d\theta+\frac{dr}{r}\right\},$$

where $z=re^{i\theta}\in\lambda$, $g(z)=\mathrm{R}e^{i\varphi}$.

By (13) for z of sufficiently large modulus in $A(\pi,\delta)$ we have for any given $K>4\pi/\delta$ that $|zg'(z)/g(z)|>K$. Thus if R and hence r are sufficiently large we have $|d\varphi|>K|d\theta|$, $|d\varphi|>K|dr|/r$. As z leaves $-r_1$ on λ and travels in a given direction to $re^{i\theta}$ the corresponding φ changes monotonely so that

$$K|\theta-\pi| = \left|\int K d\theta\right| \leqq \int K|d\theta| \leqq \int |d\varphi| = \left|\int d\varphi\right| = \Delta\varphi,$$

and similarly $K|\log(r/r_1)|\leqq\Delta\varphi$. As $w=g(z)$ traverses $|w|=R$, increasing from $\arg g(-r)$ by 4π, z traverses λ in one direction with θ changing by at most $4\pi/K<\delta$, while r satisfies

$$r_1\exp(-4\pi/K) < r < r_1\exp(4\pi/K). \tag{17}$$

Thus if r_1 is large enough the value of z remains in $A(\pi,\delta)$ and by (13) $g'(z)\neq 0$ on λ so the value of z gives a regular continuation of $g^{-1}(w)$ from $g(-r_1)$ in Γ round $|w|=R$ through an angle of 4π. The values of z lie in $A(\pi,\delta)$ but do not meet the negative real axis except at $z=-r_1$, since $g(-r)$ is increasing. Since Γ can be taken to lie in any sector $|\arg w-\alpha|<\varepsilon$, $\varepsilon>0$, it follows that we can derive from these values of $g^{-1}(w)$ a branch ψ which satisfies the statements of lemma 3, including either $\pi-\delta<\arg\psi<\pi$ or $\pi<\arg\psi<\pi+\delta$.

If in the above construction we begin by proceeding along λ in the opposite direction from that chosen originally we construct the other branch χ of g^{-1}.

For $re^{i\theta}=\psi(\mathrm{Re}^{i\varphi})$ we have by

$$|r| = |\psi(\mathrm{Re}^{i\varphi})| < r_1 \exp(4\pi/K)$$

and from $R=|g(-r_1)|>r_1^{2k}$ for large r_1 the estimate (16) follows.

We shall also need

Lemma 4 (PÓLYA [10]). *Let e, f, h be entire functions which satisfy $e=f\circ h$, $h(0)=0$. Then there is a positive constant c independent of e, f, h such that*

$$M(e, r) > M\left[g, cM\left(h, \frac{r}{2}\right)\right]. \tag{18}$$

The condition $h(0)=0$ can be dropped if (18) is to hold only for all sufficiently large r.

§ 5. Proofs of Theorems 4a and 4b

Theorem 4a. *Suppose g satisfies the hypotheses of the theorem. The first of these implies that the minimum modulus of g is large ($>R_n$) on a sequence of circles $|z|=R_n\to\infty$. The R_n may be chosen so that there is at least one zero of g in each $R_n<|z|<R_{n+1}$. Since $|g(-r)|/r\to\infty$ as $r\to\infty$ each of the simply-connected slit annuli*

$$A_n = \{z\colon R_n < |z| < R_{n+1},\ |\arg z| < \pi\}, \quad n = 1, 2, \ldots$$

contains a zero of g and has the property that

$$|g(z)| > |z| \quad \textit{on the boundary} \quad \partial A_n. \tag{19}$$

Denote by φ a branch of $z=g^{-1}(w)$ which is regular in $A(0, \pi)$ for sufficiently large w, with values in $\pi>\arg z>\pi-\delta$, δ being the fixed number, $0<\delta<\dfrac{\pi}{2}$ chosen in § 4. Such a φ exists by lemma 3.

For any fixed $l=2, 3, \ldots,$ the $(l-1)$-th iterate $\varphi^{l-1}(w)$ is defined in $A(0, \pi)$ for sufficiently large w, with values in $\pi>\arg z>\pi-\delta$. For sufficiently large n then φ^{l-1} maps A_n univalently onto a simply-connected domain D_n in $\pi>\arg z>$ $>\pi-\delta$. For $z\in\partial D_n$ we have $g^{l-1}(z)\in\partial A_n$. Now since $|g(z)|>|z|$ for large $|z|$, $z\in A(\pi, \delta)$, it follows from $z\in\partial D_n$ that $|g^{l-1}(z)|>|z|$ and from $g^{l-1}(z)\in\partial A_n$ and (19) that at

$$|g^l(z)| = \left|g\left(g^{l-1}(z)\right)\right| > |g^{l-1}(z)| > |z|,$$

at least for large n.

By Rouché's theorem $g^l(z)-z$ and $g^l(z)$ have equal numbers of zeros in D_n and $0\in g(A_n)=g^l(D_n)$. Thus the region $\pi>\arg z>\pi-\delta$ and a fortiori $A(\pi, \delta)$ contains an infinity of solutions of $g^l(z)-z=0$.

Theorem 4b. *Suppose g has order ϱ, $0<\varrho\leqq 1$, and is at most of order one, minimal type, while all but finitely many first order fixed points are positive. Suppose also that $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$ and that σ, $0<\sigma<\pi/2$ is so small that $\frac{\pi}{2}<\theta\pm\sigma\leqq\frac{3\pi}{2}$. Let ψ and χ be the two branches of g^{-1} whose existence is asserted in Lemma 3, in the case $\theta_0=\theta$. Then $\psi=\chi$ has no solution in $A(\theta,\sigma)\cap\{|w|>R_1\}$.*

Suppose g has only finitely many fixed points of order k in $A(\theta,\sigma)$. Then

$$F=(g^{k-1}-\psi)/(\chi-\psi)$$

is regular and different from 0, 1, ∞ for large z in $A(\theta,\sigma)$. By applying Schottky's theorem to F in $A(\theta,\sigma)$ (or in a slightly smaller sector within $A(\theta,\sigma)$ and with origin shifted so that $F\neq 0, 1, \infty$ in this sector) we find

$$F(z)=O\{\exp(C|z|^{\pi/\sigma})\} \tag{20}$$

for some constant C as $z\to\infty$ in $A(\theta,\sigma')$, $\sigma'<\sigma$. From (16) the same estimate follows for $|g^{k-1}(z)|$ with perhaps a different C.

Now there exists δ_1 such that $0<\delta_1<\frac{\pi}{2}$ and $A(\theta,\sigma')\subset A(\pi,\delta_1)$. Thus $|g(re^{i\theta})|\to\infty$ as $r\to\infty$ and $|zg'/g|>K>2\pi/\sigma'$ for large $|z|$, $z\in A(\theta,\sigma')$. As in the proof of theorem 1 there is for large R a level curve $\Gamma(R)$: $|g(z)|=R$, which passes through $z=re^{i\theta}$, say. Such a curve cannot close in $A(\theta,\delta)$ for arbitrarily large R, since $|g(z)|\to\infty$ in $A(\theta,\delta)$ and $A(\theta,\delta)$ contains only finitely many zeros of g. As in theorem 1 Γ must run to the boundary of $A(\theta,\sigma')$ in at least one direction. If γ is an arc of Γ which goes from $re^{i\theta}$ to $\partial A(\theta,\sigma')$, then from $|zg'/g|>K$ it follows that the image of γ under $w=g(z)$ is the whole of $|w|=R$.

For large R we have that if t is the point on $|t|=R$ where $|g^{k-2}(t)|=$ $=M(g^{k-2},R)$ then for some $z\in\gamma$, $g(z)=t$

$$M(g^{k-2},R)=|g^{k-1}(z)|. \tag{21}$$

Now in $A(\theta,\sigma')\subset A(\pi,\delta_1)$, $|g(z)|/|z|^N\to\infty$ as $|z|\to\infty$, for any N. Take $N>2\pi/(\varrho\sigma)$, where ϱ is the order of g. Then for large R we have from (21)

$$\operatorname*{Max}_{\substack{|z|=r\\ z\in A(\theta,\sigma')}} |g^{k-1}(z)|>M(g^{k-2},r^N). \tag{22}$$

Since $k-2\geqq 1$ the right hand side is (for large r) at least

$$M(g,r^N)>\exp(r^{N\varrho})>\exp(r^{2\pi/\sigma})$$

for some arbitrarily large r. Thus we have a contradiction between (22) and the estimate for g^{k-1} from (20). Hence g must in fact have an infinity of fixed points in $A(\theta,\sigma)$.

§ 6. Proof of Theorem 5

Suppose F satisfies the hypotheses of Theorem 5 and that there exist an entire function f and an integer $k \geqq 2$ such that $F = f^k$. Since F is bounded on the path γ which consists of the negative axis running to $-\infty$, it follows that one of $\gamma, f(\gamma), \ldots, f^{k-1}(\gamma)$ is an unbounded path on which f is bounded. From this it follows that the lower order of f is positive.

From lemma 4 and the fact that the lower order of f is positive we easily obtain a contradiction to hypothesis (i) of the theorem, provided $k \geqq 3$.

It remains to prove the theorem for $k=2$. From hypothesis (i), $F=f^2$ and the fact that the lower order of f is positive it follows from Lemma 4 (as is proved in [1, Satz 12]) that the order of f is less than one.

Now $f(z)=z+g(z)$ where the zeros of g are fixed points of f and hence of F. Thus the zeros of g lie in $\operatorname{Re} z \geqq 0$ and the order of z is less than 1. By lemma 1 we have

$$\frac{|f(-r)|}{r^2} \quad \text{and} \quad \frac{|g(-r)|}{r^2} \to \infty \quad \text{as} \quad r \to \infty, \tag{23}$$

while

$$|z| \left| \frac{g'}{g} \right| > K > 2\pi/\delta \quad \text{in} \quad |z| > r_0, \quad |\arg z - \pi| < \delta. \tag{24}$$

For a large $R\,(>M(g, r_0))$ there is a level curve Γ: $|g(z)|=R$ passing through $z=-r$ such that $|g(-r)|=R>r^2$. Just as in the proof of theorem 1 it follows that Γ must run to at least one of $\arg z = \pi+\delta$ or $\pi-\delta$, say the former, and that the image under $w=g(z)$ of this arc must cover $|w|=R$ with angular measure at least $K\delta > 2\pi$. Let γ denote the arc of Γ between $-r$ and a point z' chosen so that the image $g(\gamma)$ covers exactly the angular length $K\delta$ of $|w|=R$. As in the proof of Lemma 3 (17) it follows that for all $z_1 = r_1 e^{i\theta_1} \in \gamma$ we have $|\log(r_1/r)| < \delta$.

The arc γ is mapped by $f(z)=z+g(z)$ onto a (not necessarily closed) curve in such a way that the image of z_1 is $z_1 + \mathrm{Re}^{i\varphi_1}$ where $|z_1| < re^{\delta}$, $R > r^2$, and φ_1 increases by $K\delta > 2\pi$ as z_1 describes γ. Thus $f(\gamma)$ certainly cuts the negative real axis, say in a point $w' = f(z'')$, $z'' \in \gamma$. Then

$$|F(z'')| = |f(f(z''))| = |f(w')| > |w'|^2 > (R - re^{\delta})^2 > \frac{1}{4} r^4$$

if R and hence r are sufficiently large. Thus $A(\pi, \delta)$ contains points z'' of arbitrarily large modulus for which

$$|F(z'')| > \frac{1}{4} e^{-4\delta} |z''|^4,$$

which contradicts (iii). This completes the proof.

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