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Extensions of Lomonosov's invariant subspace theorem

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1. Introduction

The famous invariant subspace theorem of V. LOMONOSOV [9] includes the assertion that each algebra of operators on a Banach space which commutes with a nonzero compact operator has a nontrivial invariant subspace. That is, if K is a compact operator other than 0, and if $AK=KA$ for all A in some algebra $\mathcal{A}$, then $\mathcal{A}$ has an invariant subspace. In [10] it was shown that this could be generalized, in the case where K is injective and $\mathcal{A}$ is uniformly closed, to the same conclusion under the assumption that $\mathcal{A}K\subset K\mathcal{A}$ (in the sense that $A\in\mathcal{A}$ implies that $AK=KA_1$ for some $A_1\in\mathcal{A}$). In [12] it was shown that the hypothesis that K be injective is not needed.

In the present note we prove that $\mathcal{A}$ uniformly closed and $\mathcal{A}K_1\subset K_2\mathcal{A}$, for K_1 and K_2 compact and nonzero, implies $\mathcal{A}$ has an invariant subspace (Theorem 3) and the commutant of $\mathcal{A}$ has an invariant subspace (Theorem 4). In fact, we obtain results slightly more general than this. The proofs presented are considerably simpler than those in [10] and [12].

Our work is merely a perturbation of LOMONOSOV's [9]; it relies on the following lemma.

Lomonosov's Lemma. ([9], [13, p. 156], [11]) *If $\mathcal{A}$ is an algebra of bounded operators on a Banach space which has no nontrivial invariant subspace, and if K is any nonzero compact operator, then there is a vector $x\neq 0$ and an A in $\mathcal{A}$ such that $AKx=x$.*

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2. Preliminary results: An operator equation and operator ranges

We need to consider maps which may be nonlinear, but which are bounded in a certain sense.

Definition. A function S taking a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$ is a *bounded map* if there is a constant M such that $\|Sx\| \leqq M\|x\|$ for all $x \in \mathfrak{X}$; a *bounded operator* is a bounded map which is linear.

Note that a nonlinear bounded map need not be continuous.

The next lemma is implicit in [10]. We are grateful to Ivan Kupka for providing a suggestion which led to the simpler proof given below.

Lemma 1. *Suppose that S is a bounded map taking $\mathfrak{X}$ into itself, K is a bounded linear operator on $\mathfrak{Y}$ with spectral radius $r(K)$, and T is a bounded linear operator taking $\mathfrak{X}$ into $\mathfrak{Y}$. If $T=KTS$, if $\varepsilon>0$, and if $\|Sx\| \leqq (r(K)+\varepsilon)^{-1}\|x\|$ for all $x \in \mathfrak{X}$ then $T=0$.*

Proof. Fix $x \in \mathfrak{X}$. For each positive integer n, $Tx=K^nTS^nx$ (just keep applying K and S on the left and the right, respectively). Thus, for all n,

$$\|Tx\| \leqq \|K^n\|\|T\|r((K)+\varepsilon)^{-n}\|x\|.$$

Given any $\delta>0$, $\|K^n\|^{1/n}<r(K)+\delta$ for n sufficiently large. For sufficiently large n, then,

$$\|Tx\| \leqq (r(K)+\delta)^n\|T\|(r(K)+\varepsilon)^{-n}\|x\|.$$

If $\delta<\varepsilon$, then $\left\{\left(\frac{r(K)+\delta}{r(K)+\varepsilon}\right)^n\right\} \to 0$ as $n\to\infty$, so $Tx=0$.

Recall that a *Riesz operator* is an operator with spectral properties like those of a compact operator; i.e., a Riesz operator is a noninvertible operator whose nonzero spectrum consists of eigenvalues of finite multiplicity with no limit points other than 0.

Definition. The operator K is *decomposable at* 0 if for each $\varepsilon>0$ there is an invariant subspace $\mathfrak{M}\neq\{0\}$ of K which has an invariant complement and is such that the spectral radius of the restriction of K to $\mathfrak{M}$ is less than ε.

Theorem 1. *If $T=KTS$, where S is a bounded map on $\mathfrak{X}$, K is a bounded operator on $\mathfrak{Y}$ and T is a bounded operator taking $\mathfrak{X}$ into $\mathfrak{Y}$, then*

(i) *K quasinilpotent implies $T=0$;*

(ii) *K a Riesz operator implies T has finite rank;*

(iii) *K decomposable at 0 implies the range of T is not dense.*

Proof of (i): For ε sufficiently small and positive, $\|Sx\| \leqq \varepsilon^{-1}\|x\|$, so the result follows immediately from Lemma 1.

Proof of (ii): Choose ε sufficiently small so that $\|Sx\| \leq (2\varepsilon)^{-1}\|x\|$ for all x. Then the Riesz functional calculus yields an idempotent P which commutes with K such that the spectral radius of PK is less than ε. From $T=KTS$ it follows that $PT=PKTS=(PK)(PT)S$, so Lemma 1 implies that $PT=0$. Hence $T=(1-P)T$, and the range of T is contained in the range of the finite-rank operator $1-P$.

Proof of (iii): Begin as in (ii) above; get P by the assumption of decomposability at 0. Then $T=(1-P)T$, and the range of T is contained in the range of $1-P$ and thus is not dense.

Halmos and Douglas showed (see [4]) that if A and B are operators on Hilbert space, and if the range of A is contained in the range of B, then $A=BS$ for some operator S. This result is false, in general, on Banach spaces (cf. [5]), unless B is injective. We note that the result is true in general if we do not require S to be linear.

Lemma 2. *Let A be a bounded operator taking $\mathfrak{X}$ into $\mathfrak{Y}$ and B a bounded operator taking $\mathfrak{Z}$ into $\mathfrak{Y}$. If the range of A is contained in the range of B, then there is a bounded mapping S from $\mathfrak{X}$ into $\mathfrak{Z}$ such that $A=BS$.*

Proof. Let $\ker B=\{z\in\mathfrak{Z}: Bz=0\}$. Define $\hat{B}: (\mathfrak{Z}/\ker B)\to\mathfrak{Y}$ by

$$\hat{B}(z+\ker B) = Bz;$$

then $\hat{B}$ is an injective bounded operator. Now $\hat{B}^{-1}A: \mathfrak{Y}\to\mathfrak{Z}/\ker B$ is trivially seen to be a closed operator, so the closed graph theorem implies that $\hat{B}^{-1}A=\hat{S}$ for some bounded operator $\hat{S}: \mathfrak{X}\to\mathfrak{Z}/\ker B$. Then $A=\hat{B}\hat{S}$. Define the map $S: \mathfrak{X}\to\mathfrak{Z}$ by letting, for each $x\in\mathfrak{X}$, Sx be any element in $\hat{S}x$ of norm at most $\|\hat{S}x\|+\|x\|$; the definition of the norm on a quotient space implies that such an Sx exists. Then $\|Sx\|\leq(\|\hat{S}\|+1)\|x\|$. Also $A=BS$, for if $x\in\mathfrak{X}$, then $Ax=\hat{B}\hat{S}x=Bz$ for any $z\in\hat{S}x$. Since Sx is such a z, $Ax=BSx$, and the lemma is proven.

Definition. A linear manifold $\mathfrak{M}$ in a Banach space $\mathfrak{X}$ is an *operator range* if there is a Banach space $\mathfrak{Y}$ and a bounded operator $T: \mathfrak{Y}\to\mathfrak{X}$ such that $T(\mathfrak{Y})=\mathfrak{M}$.

A comprehensive treatment of operator ranges in Hilbert space is given in [6]. Grabiner [7] contains some results about operator ranges in Banach spaces, including part (i) of the next theorem (with a proof different from ours).

Theorem 2. *If $\mathfrak{M}$ is an operator range in $\mathfrak{Y}$, and if K is a bounded operator on $\mathfrak{Y}$ such that $\mathfrak{M}\subset K\mathfrak{M}$, then*

(i) ([7]) *K quasinilpotent implies $\mathfrak{M}=\{0\}$;*

(ii) *K a Riesz operator implies $\mathfrak{M}$ is finite-dimensional;*

(iii) *K decomposable at 0 implies $\mathfrak{M}$ is not dense.*

Proof. Suppose that $T: \mathfrak{X} \to \mathfrak{Y}$ and $T(\mathfrak{X}) \doteq \mathfrak{M}$. Then the range of T is contained in the range of KT, so Lemma 2 implies that $T = KTS$ for some bounded map S. Now parts (i), (ii) and (iii) of this theorem follow from the corresponding parts of Theorem 1.

3. Invariant subspaces for certain operator algebras

If $\mathscr{A}$ is an algebra of operators contained in the commutant of a compact operator K, then the closure of $\mathscr{A}$ in any of the standard operator topologies is also contained in the commutant of K. Thus no closure assumption on such an $\mathscr{A}$ will be helpful in obtaining invariant subspaces. In the case where $\mathscr{A}$ merely intertwines a compact operator some closure assumption is essential (cf. remark (iii), p. 118 of [10]). For certain applications discussed below, however, we need to include cases where $\mathscr{A}$ is not closed even in the norm topology. It turns out to be sufficient that $\mathscr{A}$ be an operator range, in the sense that there is a bounded linear operator taking some Banach space into the space of operators such that the range of T is $\mathscr{A}$. (If $\mathscr{A}$ is uniformly closed it is an operator range; it is the range of the injection of $\mathscr{A}$ into the space of operators.)

Theorem 3. *If $\mathscr{A}$ is an algebra of operators and $\mathscr{A}$ is an operator range, and if there exist a nonzero compact operator K_1 and an operator K_2 which is decomposable at 0 such that $\mathscr{A}K_1 \subset K_2\mathscr{A}$, then $\mathscr{A}$ has a nontrivial invariant subspace.*

Proof. If $\mathscr{A}$ had no invariant subspaces, then Lomonosov's Lemma would imply that $A_0K_1x = x$ for some $A_0 \in \mathscr{A}$ and some $x \neq 0$. Now $\mathscr{A} = S\mathfrak{Y}$ for some Banach space $\mathfrak{Y}$. Define $Ty = (Sy)(x)$ for each $y \in \mathfrak{Y}$. Then the range of $\mathfrak{T}$ is $\mathscr{A}x = \{Ax: A \in \mathscr{A}\}$, so $\mathscr{A}x$ is an operator range. If $\mathscr{A}x = \{0\}$ then the one-dimensional space spanned by x is invariant under $\mathscr{A}$. If $\mathscr{A}x \neq \{0\}$ then $\mathscr{A}x$ is an operator range invariant under $\mathscr{A}$. For $A \in \mathscr{A}$,

$$Ax = AA_0K_1x = K_2A_2x \quad \text{for some} \quad A_2 \in \mathscr{A}.$$

Hence $\mathscr{A}x \subset K_2\mathscr{A}x$. Thus part (iii) of Theorem 2 implies $\mathscr{A}x$ is not dense, so its closure is a proper invariant subspace for $\mathscr{A}$.

Remark. If K_2 is compact then the linear manifold $\mathscr{A}x$ occurring in the proof of Theorem 3 is finite-dimensional. This does not prove, however, the obviously false assertion that the hypotheses of Theorem 3 and the additional requirement that K_2 be compact yield a finite-dimensional invariant subspace for $\mathscr{A}$. We get the finite-dimensional subspace $\mathscr{A}x$ via Lomonosov's Lemma, on the assumption that we have no invariant subspaces at all.

On the other hand, if $\mathscr{A}$ is any algebra of operators with a finite-dimensional invariant subspace $\mathfrak{M}$, then $\mathfrak{M}$ could arise from Theorem 3. For let $\mathscr{A}_0$ be the set of all operators leaving $\mathfrak{M}$ invariant and let P denote an idempotent with range $\mathfrak{M}$. Then $\mathcal{A}_0 P \subset P\mathcal{A}_0$, so Theorem 3 applies to $\mathcal{A}_0$ (with $K_1 = K_2 = P$). An invariant subspace for $\mathcal{A}_0$ is also invariant under its subalgebra $\mathcal{A}$. In particular, the answer to question 1 of [12] is "no"; $\mathcal{A}_0$ is a counter-example.

Theorem 4. *If $\mathcal{A}$ is an algebra of operators which is an operator range, if there exist compact operators K_1 and K_2 different from 0 such that $\mathcal{A}K_1 \subset K_2\mathcal{A}$, and if $\mathcal{A}$ contains an operator which is not a multiple of the identity, then the commutant of $\mathcal{A}$ has a nontrivial invariant subspace.*

Proof. If the commutant of $\mathcal{A}$ had no invariant subspace then Lomonosov's Lemma would imply that there exists a B commuting with $\mathcal{A}$ and an $x \neq 0$ such that $BK_1 x = x$. For A in $\mathcal{A}$, then,

$$Ax = ABK_1 x = BAK_1 x = (BK_2) A_1 x$$

for some $A_1 \in \mathcal{A}$. Thus the linear manifold $\mathcal{A}x$ satisfies $\mathcal{A}x \subset (BK_2)(\mathcal{A}x)$. Part (ii) of Theorem 2 above implies that $\mathcal{A}x$ is finite-dimensional, (since BK_2 is compact). Choose an A_0 in $\mathcal{A}$ which is not a multiple of the identity. Since A_0 has the finite-dimensional invariant subspace $\mathcal{A}x$, A_0 has a nontrivial eigenspace (if $\mathcal{A}x = \{0\}$, then A_0 has nullspace). Since an eigenspace of A_0 is invariant under all operators commuting with A_0, the commutant of $\mathcal{A}$ has a nontrivial invariant subspace.

Corollary 1. *If A is an operator for which there exist a bounded open set D containing $\sigma(A)$, an analytic function φ taking D into D and a nonzero compact operator K such that $AK = K\varphi(A)$, then A has a nontrivial hyperinvariant subspace (unless A is a multiple of the identity).*

Proof. Let $H^\infty(D)$ denote the Banach algebra of all bounded analytic functions on D, with supremum norm, and let

$$\mathcal{A} = \{f(A) \colon f \in H^\infty(D)\}.$$

Choose a fixed Cauchy domain S contained in D and containing $\sigma(A)$. Then for $f \in H^\infty(D)$

$$\|f(A)\| = \frac{1}{2\pi}\left\| \int_{\partial S} f(z)(z-A)^{-1}\,dz \right\| \leqq$$

$$\leqq \frac{1}{2\pi} \cdot (\text{length of } \partial S) \cdot \|f\|_\infty \cdot \sup_{z \in \partial S} \|(z-A)^{-1}\|.$$

Hence there is a constant M such that $\|f(A)\| \leqq M\|f\|_\infty$ for $f \in H^\infty(D)$, and it follows that $\mathcal{A}$ is the range of the operator $f \to f(A)$ (that $\mathcal{A}$ is an algebra follows from the fact that this map is an algebraic homomorphism).

Also, if $f \in H^\infty(D)$ then $f(A)K = Kf(\varphi(A))$. One way to verify this is to note that, regarded as operators on $\mathfrak{X} \oplus \mathfrak{X}$, $\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ commutes with $\begin{pmatrix} A & 0 \\ 0 & \varphi(A) \end{pmatrix}$, and hence with $f\left(\begin{pmatrix} A & 0 \\ 0 & \varphi(A) \end{pmatrix}\right) = \begin{pmatrix} f(A) & 0 \\ 0 & f(\varphi(A)) \end{pmatrix}$. Now $f(\varphi(A)) = (f \circ \varphi)(A)$ is again in $\mathscr{A}$, so Theorem 4 applies.

Corollary 2. *If A is power bounded (i.e., there exists a constant M such that $\|A^n\| \leqq M$ for all positive integers n), and if there exist an integer k and a nonzero compact operator K such that $AK = KA^k$, then A has a nontrivial hyperinvariant subspace (or is a multiple of the identity).*

Proof. Let $\mathscr{A} = \left\{ \sum_{n=0}^{\infty} a_n A^n \colon \sum_{n=0}^{\infty} |a_n| < \infty \right\}$. The fact that A is power bounded implies that the map of $\{a_n\}$ into $\sum_{n=0}^{\infty} a_n A^n$ is a continuous map of l^1 into the bounded operators, so $\mathscr{A}$ is an operator range. Note that $\mathscr{A}$ is an algebra, since l^1 is an algebra under convolution. Now $AK = KA^k$ yields $A^n K = KA^{nk}$ for all n, so $\left(\sum_{n=0}^{\infty} a_n A^n\right) K = K \left(\sum_{n=0}^{\infty} a_n A^{nk}\right)$ and Theorem 4 applies.

Note. Corollary 2 follows from Corollary 1 only under the additional assumption that $\sigma(A) \subset \{z \colon |z| < 1\}$, in which case the function $\varphi(z) = z^k$ will serve.

Examples. The hypotheses of Corollary 2 hold under various circumstances.

(i) Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for a Hilbert space H and let $\{k_n\}$ be a sequence converging to 0. If λ is a complex number of modulus 1 and A is defined by $Ae_n = \lambda^{2^n} e_n$, then $AK = KA^2$ where K is the compact weighted shift defined by $Ke_n = k_n e_{n+1}$. Then the unitary operator A satisfies the hypotheses of Corollary 2.

(ii) Let K_0 be a compact operator and B and C be power bounded operators such that $BK_0 = K_0 C^2$. If A is the operator $B \oplus C$ and K is the operator on $\mathfrak{X} \oplus \mathfrak{X}$ defined by $K(x_1 \oplus x_2) = K_0 x_2 \oplus 0$, then $AK = KA^2$, and A satisfies the hypotheses of Corollary 2.

A natural question is whether Theorems 3 and 4 hold if the intertwining takes place on the other side; i.e., if $K_1 \mathscr{A} \subset \mathscr{A} K_2$. Upon reading a preliminary version of this manuscript L. G. Brown discovered the following two theorems. We are grateful to him for permission to include them here. These results were also obtained independently by S. GRABINER [14].

Theorem 5. *If $\mathscr{A}$ is an algebra of operators and $\mathscr{A}$ is an operator range, and if there exist a nonzero compact operator K_1 and an operator K_2 that is decomposable at 0 such that $K_1 \mathscr{A} \subset \mathscr{A} K_2$, then $\mathscr{A}$ has a nontrivial invariant subspace.*

Proof. If we suppose $\mathscr{A}$ has no invariant subspace, then, as in the proof of Theorem 3, Lomonosov's Lemma produces an A_0 in $\mathscr{A}$ with $1 \in \sigma(K_1 A_0)$. Hence $1 \in \sigma(A_0 K_1)$, and taking Banach space adjoints yields $1 \in \sigma(A_0^* K_1^*)$. Note that $\mathscr{A}^* = \{A^*: A \in \mathscr{A}\}$ is also an operator range, K_1^* is compact, K_2^* is decomposible at 0 and $\mathscr{A}^* K_1^* \subset K_2^* \mathscr{A}^*$. It follows as in the proof of Theorem 3, that there is a nonzero vector x^* in $\mathfrak{X}^*$ such that $\mathscr{A}^* x^*$ is not dense in $\mathfrak{X}^*$. In fact an examination of the proofs of Theorems 1 and 2 reveals that there is a nontrivial projection P on $\mathfrak{X}$ such that $\mathscr{A}^* x^*$ is included in the range of $1-P^*$. Since that range is weak* closed as well as nontrivial, there is a nonzero vector x in $\mathfrak{X}$ that annihilates $\mathscr{A}^* x^*$. Hence either $\mathscr{A}x = \{0\}$, in which case x spans a one dimensional invariant subspace of $\mathscr{A}$, or else the closure of $\mathscr{A}x$ is a proper invariant subspace of $\mathscr{A}$. The contradiction of the original supposition establishes the result.

Theorem 6. *If $\mathscr{A}$ is an algebra of operators wich is an operator range, if there exist compact operators K_1 and K_2 different from 0 such that $K_1 \mathscr{A} \subset \mathscr{A} K_2$, and if $\mathscr{A}$ contains an operator that is not a multiple of the identity, then the commutant of $\mathscr{A}$ has a nontrivial invariant subspace.*

Proof. Suppose the commutant of $\mathscr{A}$ has no invariant subspace. Then Lomonosov's Lemma implies the existence of a B commuting with $\mathscr{A}$ such that $1 \in \sigma(K_1 B)$, and hence $1 \in \sigma(B^* K_1^*)$. As in the proof of Theorem 4, there exists a nonzero vector x^* in $\mathfrak{X}^*$ such that $\mathscr{A}^* x^*$ is finite dimensional.

Choose an A_0 in $\mathscr{A}$ that is not a multiple of the identity. Either $\mathscr{A}^* x^* = \{0\}$, in wich case A_0^* has a nontrivial null space, or else $\mathscr{A}^* x^*$ is a finite dimensional invariant subspace of A_0^*. In either event A_0^* has an eigenvector. If λ is the corresponding eigenvalue, then it follovs that the closure of the range of $A_0 - \lambda$ is a nontrivial subspace of $\mathfrak{X}$ wich is invariant under the commutant of $\mathscr{A}$.

It might be worth noting that the compactness assumption on K_1 in Theorem 3 can be replaced by the hypothesis that K_1 has nonzero eigenvalues.

Theorem 7. *If $\mathscr{A}$ is an algebra of operators which is an operator range, if $\mathscr{A} K_1 \subset K_2 \mathscr{A}$ where K_2 is decomposable at 0 and K_1 has a nonzero eigenvalue, then $\mathscr{A}$ has a nontrivial invariant subspace.*

Proof. If $K_1 x_0 = \lambda x_0$ with $x_0 \neq 0$ and $\lambda \neq 0$, then, for any $A \in \mathscr{A}$,

$$A x_0 = \lambda^{-1} A K_1 x_0 = \lambda^{-1} K_1 A_1 x_0 \quad \text{for some} \quad A_1 \in \mathscr{A}.$$

Thus $\mathscr{A} x_0$ is contained in $K_1(\mathscr{A} x_0)$, so part (iii) of Theorem 3 implies $\mathscr{A} x_0$ is not dense.

Remarks. It is shown in [15] that there is an operator that does not satisfy the hypothesis of Lomonosov's invariant subspace theorem. In light of Theorem 4 above we can ask: if B is an operator on a Hilbert space must B commute with some uniformly closed algebra $\mathscr{A}$ (containing operators other than scalars) which intertwines two nonzero compact operators?

In [10] the following question was raised. If $\mathscr{A}$ is a uniformly closed algebra of operators such that $\mathscr{A}K \subset KB(\mathfrak{X})$ must $\mathscr{A}$ have a nontrivial invariant subspace? If $\mathscr{A}$ is not required to be closed but is merely required to be an operator range then the answer is no, as is seen by letting $\mathscr{A}=KB(\mathfrak{X})$ for an injective compact operator K with dense range.

Some other variants of Lomonosov's Theorem can be found in [3], [8] and [11]. We are grateful to L. Fialkow for providing us with a copy of [1], where it is shown that $AK=\lambda KA$ for K compact and λ a complex number implies A has a hyperinvariant subspace. In the case where $|\lambda| \leqq 1$ this follows from Corollary 1 above; when $|\lambda|>1$ it follows from the analogous corollary to Theorem 6.

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