Covering branchings

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In a previous paper [4] we proved, among others, a min-max theorem concerning cuts of a directed graph. Now this theorem will be applied in order to get some new min-max theorems about branchings and arborescences. For example, a good characterization is given for the problem of the existence of k branchings covering all of the edges of a directed graph. This theorem can be considered as a directed counterpart of a theorem of Nash-Williams about covering forests.

Another corollary is a directed analogue of Tutte's theorem about edge disjoint spanning trees. A directed graph has k edge disjoint spanning arborescences (possibly rooted at different vertices) if and only if, for every family of t disjoint subsets of vertices, the sum of their indegrees is at least $k(t-1)$. This theorem differs from Edmonds' one concerning the existence of k edge disjoint spanning arborescences rooted at a fixed vertex. However we shall use Edmonds' result in the proof.

Let $G=(V, E)$ be a finite directed graph with vertex set V and edge set E. Multiple edges are allowed, loops are excluded. Let r be a distinguished vertex of G. We use the notation $U=V\setminus\{r\}$.

An *arborescence* a is a directed tree such that every edge is directed toward a different vertex. It is well known that an arborescence has a unique vertex (of indegree 0) from which every other vertex can be reached by a directed path. This vertex is called the *root* of a. A spanning arborescence of G rooted at r is called an *r-arborescence*.

A *branching* b is a directed forest, the components of which are arborescences.

We say that a directed edge e *enters* a set X of vertices if the head of e is in X but its tail is not. We say that a subset E' of edges *enters* X if at least one element of E' enters X.

The *indegree* $\varrho_G(X)$ of a subset X of V is the number of edges entering X. The following inequality is straightforward: $\varrho_G(X)+\varrho_G(Y)\geqq\varrho_G(X\cup Y)+\varrho_G(X\cap Y)$.

Received December 30, 1977.

For an arbitrary set X, $X' \sqsubset X$ means that X' is a family of not necessarily distinct elements of X.

A family $\mathcal{F}$ of subsets of U is called *laminar* if at least one of $X \setminus Y$, $Y \setminus X$, $X \cap Y$ is empty for any two members of $\mathcal{F}$.

Let $\mathbf{f}$ be a non-negative integer valued function defined on the subsets of U. $\mathbf{f}$ is called *weakly supermodular* if $X, Y \subseteq U$, $\mathbf{f}(X), \mathbf{f}(Y) > 0$ and $X \cap Y \neq \emptyset$ imply $\mathbf{f}(X) + \mathbf{f}(Y) \leqq \mathbf{f}(X \cup Y) + \mathbf{f}(X \cap Y)$. If $X, Y \subseteq U$ and $X \cap Y \neq \emptyset$ already imply it then $\mathbf{f}$ is called *supermodular*.

A family E' of not necessarily distinct edges of G (i.e. $E' \sqsubset E$) is called $\mathbf{f}$-*entering* if in the graph $G' = (V, E')$ the indegree of every subset X is at least $\mathbf{f}(X)$.

Let $\mathbf{c}$ be a non-negative integer valued function on E. A family $\mathcal{F}$ of not necessarily distinct subsets of U is called $\mathbf{c}$-*edge-independent* if each edge e of G enters at most $\mathbf{c}(e)$ members of $\mathcal{F}$.

The following theorem was proved in a slightly other form in [4].

Theorem 1. *If* $\mathbf{f}$ *is weakly supermodular and* $\varrho(Y) = 0$ *implies* $\mathbf{f}(Y) = 0$ *then*

$$\max_{\mathcal{F}} \sum_{X \in \mathcal{F}} \mathbf{f}(X) = \min_{E' \sqsubset E} \sum_{e \in E'} \mathbf{c}(e)$$

where $\mathcal{F}$ *is* $\mathbf{c}$-*edge-independent* ($\mathcal{F} \sqsubset 2^U$) *and* $E' \sqsubset E$ *is* $\mathbf{f}$-*entering. The maximum can be realized by a laminar* $\mathcal{F}$.

Let k be a natural number and $F \subseteq E$.

Problem 1. What is the maximum number M of edges of F which can be covered by k r-arborescences of G?

The case $F = E$ was discussed in [4]. We formulate this problem in another form.

Problem 1a. What is the minimum number m of not necessarily distinct edges of G which, together with F, contain k edge disjoint r-arborescences?

The two problems are equivalent because $M \geqq k(|V|-1) - m$ and $m \leqq k(|V|-1) - M$, hence

$$m + M = k(|V|-1). \tag{1}$$

By a theorem of J. Edmonds [3, 5] a digraph has k edge disjoint r-arborescences if and only if the indegree of every subset of $V \setminus \{r\}$ is at least k. Therefore $m = \min_{E' \sqsubset E} |E'|$ where E' is $\mathbf{f}$-entering and the function $\mathbf{f}$ is defined as follows:

$$\mathbf{f}(X) = \max\left(0, k - \varrho_H(X)\right) \quad \text{for} \quad X \subseteq U$$

where $\varrho_H(X)$ is the indegree of X in the subgraph $H = (V, F)$. Obviously $\mathbf{f}$ is weakly supermodular. (Observe that F is used only to define $\mathbf{f}$). Applying Theorem 1 to G and to this function $\mathbf{f}$, with the choice $\mathbf{c}(e) = 1$ ($e \in E$), we get $m = \max_{\mathcal{F}} \sum_{X \in \mathcal{F}} \mathbf{f}(X)$ where $\mathcal{F}$ is 1-edge-independent. This, together with (1), proves

Theorem 2. *If $H=(V, F)$ is a subgraph of $G=(V, E)$ then the maximum number of edges of H which can be covered by k r-arborescences of G is equal to*

$$\min\left[k(|V|-1-t)+\sum_{i=1}^{t}\varrho_H(V_i)\right]$$

where the minimum is taken over all 1-edge-independent laminar families $\mathcal{F}=\{V_1, V_2, \ldots, V_t\}$ $(V_i \subseteqq U)$.

Problem 2. Let $H=(U, F)$ be a directed graph (there is no distinguished vertex). What is the maximum number M of edges which can be covered by k branchings?

Complete H by a new vertex r and by $|U|$ new edges which are joined from r to all other vertices of U, i.e. $V=U\cup\{r\}$ and $E=F\cup\{(\overrightarrow{r, x}): x\in U\}$. It is easy to check that the maximum number of edges of H which can be covered by k r-arborescences of $G=(V, E)$ is M. Apply Theorem 2 and observe that in this case a laminar family of subsets of U consists of pairwise disjoint subsets. Thus we have

Theorem 3. *The maximum number of edges of $H=(U, F)$ which can be covered by k branchings is equal to*

$$\min\left[k(|U|-t)+\sum_{i=1}^{t}\varrho_H(V_i)\right]$$

where the minimum is taken over all families of disjoint subsets V_i $(i=1, 2, \ldots, t)$ of U.

A simple application of this theorem provides an analogue of Tutte's disjoint spanning trees theorem [8].

Theorem 4. *$H=(U, F)$ has k edge disjoint spanning arborescences (possibly rooted at different vertices) if and only if*

$$\sum_{i=1}^{t}\varrho_H(V_i) \geqq k(t-1) \tag{2}$$

for every family of disjoint subsets V_i $(i=1, 2, \ldots, t)$ of U.

Proof. H has k edge disjoint spanning arborescences if and only if at least $k(|U|-1)$ edges of H can be covered by k branchings, i.e., by Theorem 3, $k(|U|-t)+\sum_{i=1}^{t}\varrho_H(V_i)\geqq k(|U|-1)$, which is equivalent to (2). □

Another consequence of Theorem 3 is

Theorem 5. *The edges of H can be covered by k branchings if and only if*

(3) $$k(|U|-t) \geqq e_t$$

for every family of disjoint subsets $V_1, V_2, \ldots, V_t$ of U, where e_t denotes the number of edges not entering any V_i.

Proof. By Theorem 3 we have to assure that $k(|U|-t)+\sum_{i=1}^{t} \varrho_H(V_i) \geqq |F|$. But this is equivalent to (3), because $e_t+\sum_{i=1}^{t} \varrho_H(V_i)=|F|$. □

Theorem 5a. *The edges of H can be covered by k branchings if and only if*
(4a) *the indegree of every vertex is at most k, and*
(4b) *the edges of H (in the undirected sense) can be covered by k forests.*

Proof. The necessity of the conditions is obvious. For the sufficiency we verify that (4a) and (4b) imply (3). Let $V_1, V_2, \ldots, V_t$ be disjoint subsets of U. Let $V_0=U \setminus \bigcup_{i=1}^{t} V_i$ (V_0 may be empty) and let $e(X)$ denote the number of edges with tails and heads both in X. Then

$$e_t = \sum_{x \in V_0} \varrho_H(x)+\sum_{i=1}^{t} e(V_i) \leqq k|V_0|+\sum_{i=1}^{t} k(|V_i|-1) = k(|U|-t). \quad \square$$

Remark. The last theorem can be considered as a new "linking" theorem. Let $\mathcal{M}_1$ denote the circuit matroid (on F) of H considering H as an undirected graph. Let $\mathcal{M}_2$ denote the matroid on F in which a subset is defined to be independent if it contains no two edges directed toward the same vertex. Now Theorem 5a states that if F can be covered by k independent sets of $\mathcal{M}_1$ and can be covered by k independent sets of $\mathcal{M}_2$ then F can be covered by k sets which are independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$.

Another special case of this statement, when $\mathcal{M}_1$ and $\mathcal{M}_2$ are transversal matroids, was proved by Brualdi [2]. However, this statement is not true in general: Let $\mathcal{M}_1$ be the circuit matroid of K_4 (the complete graph on 4 vertices) and $\mathcal{M}_2$ be defined such that a subset in independent if it contains no disjoint edges of K_4.

Now we prove a Vizing type theorem which is due to Mosesyan [6] for $\gamma=1$.

Theorem 6. *If in $H=(U, F)$ the indegree of every vertex is at most K and H does not contain $\gamma+1$ edges with the same heads and tails then F can be covered by $k=K+\gamma$ branchings.*

Proof. (4a) holds obviously. To prove (4b) we have to verify that $e(X) \leqq \leqq k(|X|-1)$ for $X \subseteq U$. This condition is equivalent to (4b) by a well-known theorem of NASH-WILLIAMS [7]. If $|X|\gamma \leqq k$ then $e(X) \leqq |X|(|X|-1)\gamma \leqq \leqq k(|X|-1)$. If in turn $|X|\gamma \geqq k$ then $e(X) \leqq |X| \cdot K = |X|(k-\gamma) \leqq k(|X|-1)$. □

Finally, a theorem is stated which is also a consequence of Theorem 1. The proof is left to the reader.

Theorem 7. *The edges of $H=(U, F)$ can be covered by k spanning arborescences if and only if $k(|U|-1-t+d) \geqq e_t$ for every 1-edge-independent laminar family $\mathcal{F}=\{V_1, \ldots, V_t\}$, where e_t is the number of edges not entering any V_i and d denotes the maximum number of V_i's containing any vertex.*

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