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A characterisation of binary geometries of types $K(3)$ and $K(4)$

L. KÁSZONYI

In connection with halfplanar geometries I introduced the property $K(r)$ of binary geometries (see [2]). The aim of this paper is to give a new characterisation of such geometries for $r=3$ and 4.

First I give some definitions.

Definition 1. Let us consider the binary geometry G, which is embedded in the binary projective geometry Γ (i.e. in the geometry over $GF(2)$, $r(G)=r(\Gamma)$, $G\subseteqq\Gamma$). A subspace (point, line, hyperplane) m of Γ is called an *outer subspace* (point, line, hyperplane) of G, if m is not spanned by G-points.

We note that by the homogeneity of Γ this definition depends only on G.

Definition 2. A binary geometry G of rank n has the *property* $K(r)$ if every subgeometry of G of rank $n-r$ is contained in an outer hyperplane ($n\geqq r$, $r\geqq 2$).

Definition 3. A set $H=\{h_1, h_2, \ldots, h_m\}$ ($m\geqq 3$) of hyperplanes of the binary projective geometry Γ is a *hypercircuit* if

$$r\left(\bigcap_{\substack{i=1\\ i\neq j}}^{m} h_i\right) = r\left(\bigcap_{i=1}^{m} h_i\right) = r(\Gamma)-m+1$$

holds for every $j\in\{1, 2, \ldots, m\}$; m is called the *length* of H.

We shall frequently use the following

Theorem 1. (Two Colour Theorem) *$b\cap G\neq\emptyset$ holds for all hyperplanes b of a binary projective geometry $\Gamma(\supset G)$ if and only if G contains an odd circuit.*

The geometrical dual of Theorem 1 may be formulated as follows:

Theorem 2. *A set $\mathscr{L}$ of hyperplanes of a binary projective geometry Γ covers all Γ-points if and only if $\mathscr{L}$ contains an odd hypercircuit.*

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Our theorem is the following:

Theorem 3. *A binary geometry G of rank n ($\geqq r+1$) is of type $K(r)$ ($r=3, 4$) if and only if for an arbitrary subspace a of G of rank $n-r-1$, the set $\mathcal{M}(a)$ of outer hyperplanes containing a contains an odd hypercircuit.*

First we prove

Lemma 1. *A binary geometry of rank 4 is of type $K(3)$ if and only if the set of its outer hyperplanes contains an odd hypercircuit.*

Proof. Sufficiency is clear by Theorem 2. We have to prove that if the set of outer planes covers the points of G then it covers the outer points of G as well (see Theorem 2).

Let us assume indirectly that the set $\mathcal{M}_3$ of outer planes of G covers the points in G but there is an outer point γ_1 which is not covered by $\mathcal{M}_3$. Consider the outer plane b_1 and let f be a line of b_1 which covers $b_1 \cap G$. The existence of such a line is trivial. Denote the planes incident in Γ to f and distinct from b_1, by d_1 and d_2. The set of planes $\{b_1, d_1, d_2\}$ covers all Γ-points. $\gamma_1 \notin b_1$, thus $\gamma_1 \in d_i \setminus f$ holds for an $i \in \{1, 2\}$ by our indirect hypothesis. Let for example $\gamma_1 \in d_1 \setminus f$ (see Fig. 1).

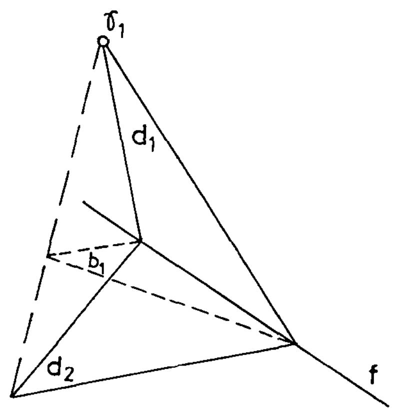

Fig. 1

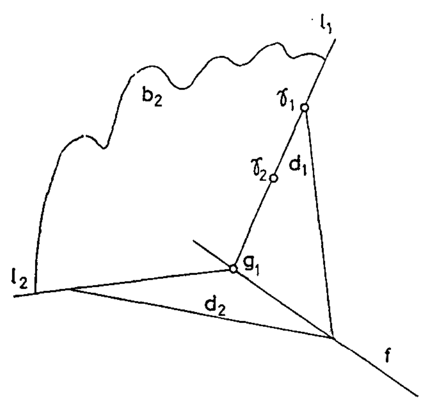

Fig. 2

We prove that for the set $U=d_1 \setminus f \setminus \{\gamma_1\}$, $U \subset G$ holds. Let us assume indirectly that U has a point γ_2 not in G. Consider the line $l_1=\sigma(\gamma_1, \gamma_2)$ *) and set $g_1=f \cap l_1$. Choose a point g of d_2 not on f and let $l_2=\sigma(g_1, g)$ (see Fig. 2). It is easy to see that the plane $b_2=\sigma(l_1 \cup l_2)$ is an outer plane containing γ_1, a contradiction.

*) $\sigma(\dots)$ denotes the subspace of Γ spanned by the set given in the parentheses.

We show that U is an oval of d_1. Let $u_1, u_2 \in U$ be arbitrary. $\sigma(u_1, u_2) \cap f \neq \emptyset$ and $f \cap U = \emptyset$, thus every line of U consits of two points. But $|U|=3$, therefore U spans d_1. It is easy to see that γ_1 is a nucleus of U (i.e. the common point of tangentials to U).

Let $u_1 \in U$ be arbitrary, consider the outer plane b_3 containing u_1. The line $f_1 = b_3 \cap d_1$ cannot be tangential to U, thus f_1 intersects U in two points, say at u_1 and u_2 (see Fig. 3). This means that f_1 covers all the points of b_3, and the points of $f_2 = d_2 \cap b_3$ and not on f are outer points. Therefore the plane $\sigma(f_2 \cup \{\gamma_1\})$ is an outer plane containing γ_1, a contradiction.

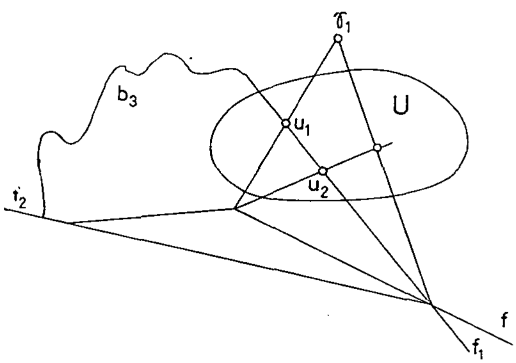

Fig. 3

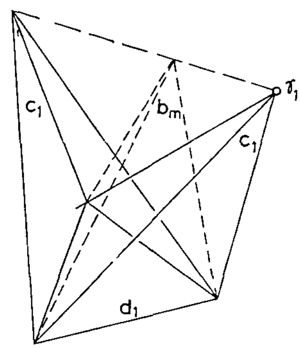

Fig. 4

Lemma 2. *A geometry G of rank 5 is of type $K(4)$ iff the set $\mathcal{M}_4$ of its outer hyperplanes contains an odd hypercircuit.*

Proof. We have to prove that if the elements of $\mathcal{M}_4$ cover G, then they cover the outer points as well. Let us assume indirectly that G is covered by the elements of $\mathcal{M}_4$ but G has an outer point γ_1 which is not covered by them. Let b_m be an element of $\mathcal{M}_4$ for which $|b_m \cap G|$ is maximal.

Let d_1 be a plane of b_m which covers the points of $b_m \cap G$. Let us denote by c_1 and c_2 the hyperplanes containing d_1 and distinct from b_m. The set $\{b_m, c_1, c_2\}$ covers all the Γ-points. Let $\gamma_1 \in c_1$ (see Fig. 4). Denote the set $(G \cap c_1) \setminus d_1$ by V. Let us project V from γ_1 to d_1. We prove that the projection V' meets all lines of d_1. Let us assume indirectly that d_1 has a line l for which $l \cap V' = \emptyset$ holds. Let g_2 be an arbitrary point of c_2 not on d_1. It is easy to see that $\sigma(\{g_2\} \cup l \cup \{\gamma_1\})$ is an outer hyperplane of G containg γ_1, a contradiction (see Fig. 5). Therefore V' contains a full line f_1. Let us consider the plane $d_2 = \sigma(f_1 \cup \{\gamma_1\})$. Making use of the fact that V' contains f_1, we can see that $U = V \cap d_2$ is an oval on d_2, the nucleous of which is γ_1.

Let $u \in U$ be arbitrary, denote the outer hyperplane containg the point u by b_u. We prove the validity of the following three statements:

(i) $d_2 \cap b_u$ is a line and a secant to U (i.e. $|U \cap (d_2 \cap b_u)| = 2$),

(ii) $b_u \cap c_1 \cap G \subseteqq d_2$,

(iii) $|b_u \cap G| \geqq 3$.

Part (i). $\gamma_1 \notin b_u$, thus $d_2 \not\subset b_u$, therefore $d_2 \cap b_u$ is a line. The lines of d_2 containing u are either secants or tangentials to U. The lines which are tangential to U contain γ_1, thus $d_2 \cap b_u$ is a secant.

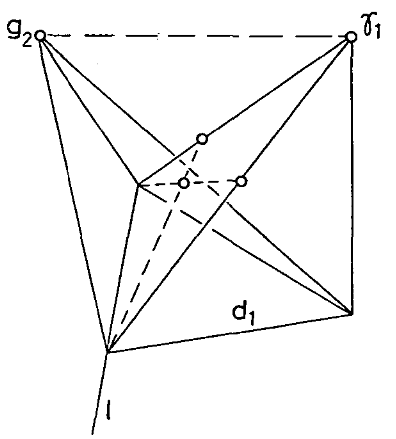

Fig. 5

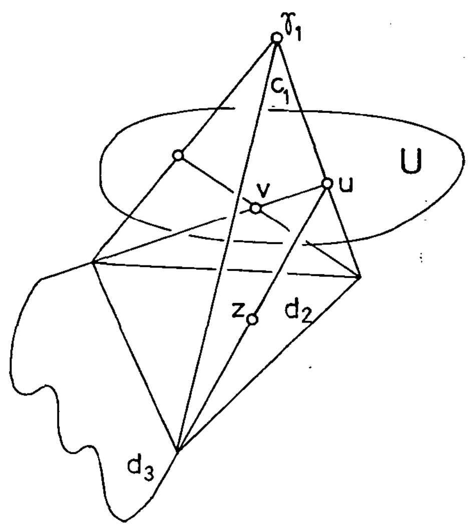

Fig. 6

Part (ii). Let $U \cap b_u = \{u, v\}$; assume indirectly that $b_u \cap G \cap c_1$ contains a point z not in U (see Fig. 6). The plane $d_3 = \sigma(u, v, z)$ covers $G \cap b_u$, because $b_u \in \mathcal{M}_4$. Therefore $d_3 \cap d_1$ covers $c_2 \cap b_u$. Using this fact we can see that $\sigma(c_2 \cap b_u \cup \{\gamma_1\})$ is an outer hyperplane, a contradiction (see Fig. 7).

Part (iii). $|b_u \cap U| = 2$, thus we have to prove $c_2 \cap b_u \cap G \neq \emptyset$. Assuming $c_2 \cap b_u \cap G = \emptyset$ we can see that $\sigma(c_2 \cap b_u \cup \{\gamma_1\})$ is an outer hyperplane.

Let $U = \{u_1, u_2, u_3\}$, and assume that $U \cap b_{u_2} = \{u_1, u_2\}$ and $U \cap b_{u_3} = \{u_1, u_3\}$. Set $\sigma(u_1, u_2) = s_1$, $\sigma(u_1, u_3) = s_2$, $c_1 \cap b_{u_2} = d_4$, $c_1 \cap b_{u_3} = d_5$, $d_1 \cap d_4 = s_4$, $d_1 \cap d_5 = s_5$, $g_{45} = s_4 \cap s_5$ (see Fig. 8). We prove that the plane $\sigma(s_4 \cup \{\gamma_1\})$ contains a point g_6 not on $\sigma(u_3, \gamma_1)$. Let us assume indirectly that

$$\{\sigma(s_4 \cup \{\gamma_1\}) \cap G\} \setminus \{u_3, \gamma_1\} = \emptyset.$$

Using (ii) we can see that $\sigma(\{c_2 \cap b_{u_2}\} \cup \{\gamma_1\})$ is an outer hyperplane. It is easy to see as well that g_6 is the point of the line $\sigma(\gamma_1, g_{45})$ distinct from γ_1 and g_{45}. Consider b_{g_6}. The line $b_{g_6} \cap d_2$ cannot be a secant to U by (ii), and cannot be tangential to U by $\gamma_1 \notin b_{g_6}$, therefore $b_{g_6} \cap d_2 = f_1$.

We prove that $|f_1 \cap G| \leqq 1$. Let us assume indirectly that $|f_1 \cap G| \geqq 2$. Let $g_7, g_8 \in f_1 \cap G$. Then the plane $\sigma(g_6, g_7, g_8)$ covers $b_{g_6} \cap G$. Using this fact we can see that the hyperplane $\sigma(\{c_2 \cap b_{g_6}\} \cup \{\gamma_1\})$ is an outer hyperplane, a contradiction. Therefore $|\{f_1 \cup s_4 \cup s_5\} \cap G| \leqq 1$, thus $|b_m \cap G| = |d_1 \cap G| \leqq 2$ holds, contradicting (iii) and the maximality property of b_m.

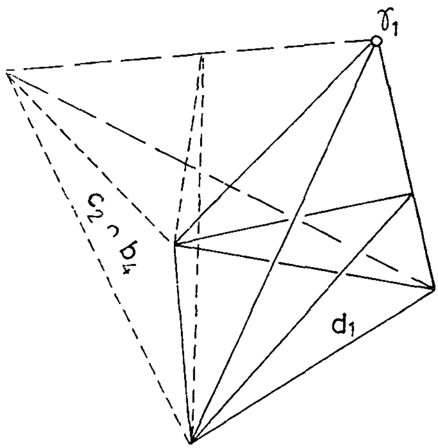

Fig. 7

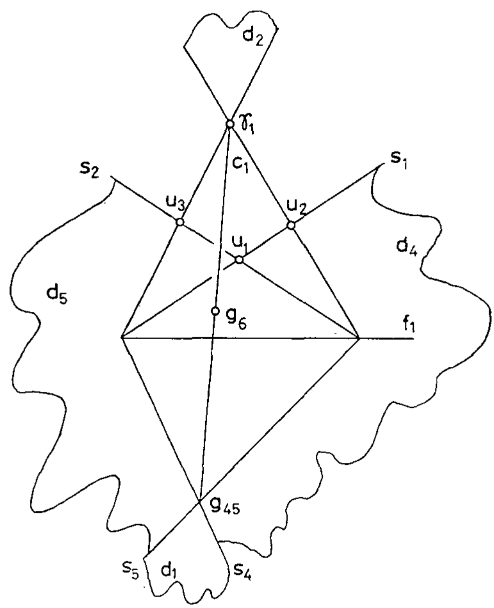

Fig. 8

Our Theorem 3 is now an easy consequence of Lemmas 1 and 2 and of the "Scum Theorem" (see [1]).

The following assertion may be proved by induction on $n = r(G)$ for $r = 2, 3, 4$: if the outer hyperplanes of the binary geometry G cover all of its subspaces with rank $n-r$ then they cover all those of Γ as well.

To prove it for general r it would suffice to settle the case $n = r+1$:

Conjecture. If the outer hyperplanes of a binary geometry G cover all G-points, then they cover all Γ-points as well.

The conjecture is proved for $r(G)=4$ and $r(G)=5$ in Lemmas 1 and 2. The case $r(G)=3$ is trivial. For $r(G)>5$ the proof (if it exists) seems to be hard.

References

[1] H. Crapo—G. C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, M.I.T. Press (Cambridge, Mass., 1970).

[2] L. Kászonyi, On the structure of binary geometries having the $K(3)$-property, *Ann. Univ. Sci. Budapest, Sect. Math.*, to appear.

DEPARTMENT OF PROBABILITY THEORY
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÚT 6—8.
1088 BUDAPEST, HUNGARY