Acta Sci. Math., 41 (1979), 137—150

Sublattices of a distributive lattice

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At the Mini-Conference on Lattice Theory in Szeged, 1974, M. SEKANINA has formulated the following problem: Is it true that if a lattice B contains an arbitrarily large finite number of pairwise disjoint sublattices, isomorphic to a lattice A, then B also contains an infinite number of such sublattices? The aim of the present paper is to construct two countable distributive lattices A and B which are counter-examples, i.e. such that for any $m=1, 2, 3, \ldots$, B contains m disjoint copies of A, but it does not contain infinitely many such copies. An independent solution of Sekanina's problem was found by I. KOREC in a paper to appear (personal communication).

An analogous problem can be formulated for other structures than lattices and various concepts of subobject, e.g. summand. In the second part a general formulation of this problem is exhibited.

1. We recall that a graph (X, R) (i.e. $R\subset X\times X$) is bipartite if it is symmetric and there exists a subset M of X such that if $(x, y)\in R$ then $x\in M$ iff $y\notin M$.

Definition. A graph (X, R) is *strongly reduced* if for any distinct points $x, y\in X$ there exists at most one point z with $(z, x), (z, y)\in R$.

Convention. Denote by $\mathbf{N}$ the set of all natural numbers, by $\mathbf{Z}$ the set of all integers.

Construction 1.1. We shall construct countable, connected, strongly reduced, bipartite graphs (X_i, R_i) with $i\in\mathbf{N}$, $i>1$ such that

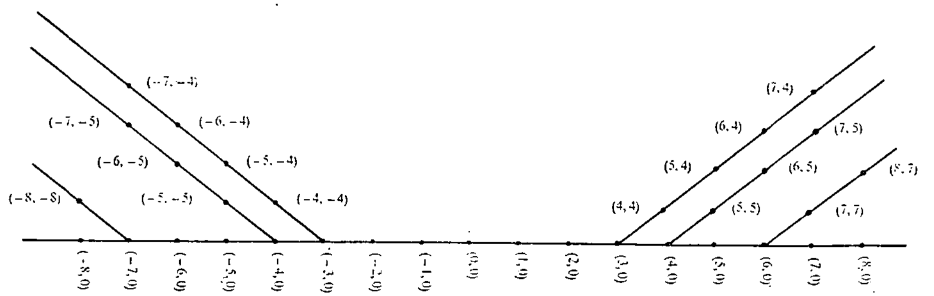
a) for every $x\in X_i$, card $\{z\colon (x, z)\in R_i\}\in\{2, 3\}$;

b) if $f\colon (X_i, R_i)\to(X_j, R_j)$ is a one-to-one compatible mapping then $i=j$ and f is the identity.

Received October 3, 1977.

Put

$$X_i = \{(x, y): (x, y \in \mathbf{Z}),\ (y \neq 0 \Rightarrow y \in \{i, -i\} \cup$$
$$\cup \{i+2k+1:\ k \in \mathbf{N}\} \cup \{-i-3k-1:\ k \in \mathbf{N}\})\ (\operatorname{sgn} x = \operatorname{sgn} y)\},$$
$$R_i = \{((x, 0), (x+1, 0)):\ x \in \mathbf{Z}\} \cup \{((x, 0), (x-1, 0)):\ x \in \mathbf{Z}\} \cup$$
$$\cup \{((y+s, y), (y+t, y)):\ (y \in \{i+2k+1:\ k \in \mathbf{N}\} \cup$$
$$\cup \{-i-3k-1:\ k \in \mathbf{N}\} \cup \{i, -i\}), (|s-t| = 1), (sy, ty \geqq 0)\} \cup$$
$$\cup \{((y-\frac{y}{|y|}, 0), (y, y)), ((y, y), (y-\frac{y}{|y|}, 0)):\ y \in \{i+2k+1:\ k \in \mathbf{N}\} \cup$$
$$\cup \{-i-3k-1:\ k \in \mathbf{N}\} \cup \{i, -i\}\}.$$



$(X_4,\ R_4)$

Fig. 1

It is clear that (X_i, R_i) is a countable, symmetric, strongly reduced graph. Set $M_i = \{(x, y) \in X_i : x \text{ is even}\}$, then $R_i \subset ((X_i - M_i) \times M_i) \cup (M_i \times (X_i - M_i))$ and therefore (X_i, R_i) is a bipartite graph. Further, for every $x \in X_i$,

$$\operatorname{card} \{z:\ (x, z) \in R_i\} \in \{2, 3\}.$$

We shall prove Property b). If $f:\ (X_i, R_i) \to (X_j, R_j)$ is a one-to-one compatible mapping then for $x \in \{i-1, 1-i\} \cup \{i+2k : k \in \mathbf{N}\} \cup \{-i-3k : k \in \mathbf{N}\}, f(x, 0) \in \{(j-1, 0), (1-j, 0)\} \cup \{(j+2k, 0) : k \in \mathbf{N}\} \cup \{(-j-3k, 0) : k \in \mathbf{N}\}$. Hence $f(\{(i-1, 0), (1-i, 0)\}) \in \in \{(j-1, 0), (1-j, 0)\}$ and therefore $i=j$. Further, $f(x, 0) \in \{(y, 0) : y \in \mathbf{Z}\}$. If $f(i-1, 0) = (1-i, 0)$ then $f(i+2k, 0) = (-i-2k, 0)$ but the latter is impossible, thus $f(i-1, 0) = (i-1, 0)$ and so is $f(x, 0) = (x, 0)$ for every $x \in \mathbf{Z}$. Hence f is the identity.

Let us introduce the notation $\mathfrak{X}_i = (X_i, R_i, M_i)$, $i \in \mathbf{N}$, $i > 1$.

Construction 1.2. Let $\mathfrak{X}=(X, R, M)$ where (X, R) is a bipartite graph and $M\subset X$ such that if $(x, y)\in R$ then $x\in M$ iff $y\notin M$. Set

$A_1^{\mathfrak{X}} = \{Z\subset X: (\exists (x, y)\in R)(x\in M \text{ and } Z = (M-\{x\})\cup\{y\})\}$;

$A_2^{\mathfrak{X}} = \{Z\subset X: (\exists x\in M)(\exists K\subset\{y: (y, x)\in R\})(K \text{ is finite and } Z = (M-\{x\})\cup K)\}$;

$A_3^{\mathfrak{X}} = \{Z\subset X: (\exists x\in X-M)(\exists K\subset \{y: (y, x)\in R\})(K \text{ is finite and } Z=(M-K)\cup\{x\})\}$;

$A_4^{\mathfrak{X}} = \{Z\subset X: (\exists K\subset M)(K \text{ is finite and } Z = M-K)\}$;

$A_5^{\mathfrak{X}} = \{Z\subset X: (\exists K\subset X-M)(K \text{ is finite and } Z = M\cup K)\}$.

Put $A^{\mathfrak{X}}=\bigcup_{i=1}^{5} A_i^{\mathfrak{X}}$, $B^{\mathfrak{X}}=A^{\mathfrak{X}}\cup\{\emptyset, X\}$. For $Z, V\in B^{\mathfrak{X}}$ define $Z\vee V=Z\cup V$, $Z\wedge V= =Z\cap V$, then it is easy to verify that $(A^{\mathfrak{X}}, \cup, \cap)$ and $(B^{\mathfrak{X}}, \cup, \cap)$ are lattices (and hence they are distributive). Moreover, $A^{\mathfrak{X}}$ and $B^{\mathfrak{X}}$ are countable iff X is countable.

Let $\mathfrak{X}=(X, R, M)$, $\mathfrak{Y}=(Y, S, N)$ where (X, R), (Y, S) are bipartite graphs and for $(x, y)\in R$ (or $(x, y)\in S$), $x\in M$ iff $y\notin M$ (or $x\in N$ iff $y\notin N$, respectively). If $f\colon X\to Y$ such that $f(M)\subset N$ and $f\colon (X, R)\to(Y, S)$ is a one-to-one compatible mapping then $\varphi\colon B^{\mathfrak{X}}\to B^{\mathfrak{Y}}$ (or $\varphi/A^{\mathfrak{X}}\colon A^{\mathfrak{X}}\to A^{\mathfrak{Y}}$) is a one-to-one lattice homomorphism, where $\varphi(Z)=(f(Z)\cup N)-f(M-Z)$ if $Z\neq\emptyset, X$, $\varphi(\emptyset)=\emptyset$, $\varphi(X)=Y$. We shall write $\Psi\mathfrak{X}=(A^{\mathfrak{X}}, \cup, \cap)$, $\Psi f=\varphi/A^{\mathfrak{X}}$, $\Phi\mathfrak{X}=(B^{\mathfrak{X}}, \cup, \cap)$, $\Phi f=\varphi$.

Note 1.3. Denote by **Gr** the category whose objects are triples (X, R, M) where (X, R) is a bipartite graph and $M\subset X$ such that if $(x, y)\in R$ then $x\in M$ iff $y\notin M$ and whose morphisms $f\colon (X, R, M)\to(Y, N, S)$ are one-to-one mappings $f\colon (X, R)\to(Y, S)$ with $f(M)\subset N$. Denote by **DLat** the category of distributive lattices and one-to-one lattice homomorphisms. Then Φ, Ψ are faithful functors from **Gr** to **DLat**.

Definition. Let $\mathfrak{A}$ be a lattice. An element x of $\mathfrak{A}$ is called *meet-infinite* (or *join-infinite*) if there exists an infinite subset B of $\mathfrak{A}$ such that for any distinct points $a, b\in B$, $a\wedge b=x$ (or $a\vee b=x$, respectively).

Lemma 1.4. *Let $\mathfrak{X}=(X, R, M)$ be an object of* **Gr** *such that M and $X-M$ are infinite and for every $x\in X$ the set $\{y: (x, y)\in R\}$ is finite. Then for $Z\in A^{\mathfrak{X}}$ we have*

a) *Z is a meet-infinite element iff $Z\supset M$;*

b) *Z is a join-infinite element iff $Z\subset M$.*

Proof. If $V\supset M$ then it is clear that V is meet-infinite $(V=(V\cup\{x\})\cap(V\cup\{y\})$ for every $x\neq y$, $x, y\in X-V)$. Let V be meet-infinite. Let $\mathscr{B}\subset A^{\mathfrak{X}}$ be an infinite set with $W_1\cap W_2=V$ for every $W_1\neq W_2$, $W_1, W_2\in\mathscr{B}$. If $M-V\neq\emptyset$ then $M-W\neq \neq M-V$ holds only for finitely many $W\in\mathscr{B}$, and so $\mathscr{B}$ is finite because the set $\{y: (x, y)\in R\}$ is finite for every $x\in X$, a contradiction; thus $M-V=\emptyset$ and hence $V\supset M$. The proof of case b) is analogous.

Lemma 1.5. *Let $f: \mathfrak{A} \to \mathfrak{B}$ be a one-to-one lattice homomorphism. If $a \in \mathfrak{A}$ is a meet-infinite (join-infinite) element then $f(a)$ is meet-infinite (join-infinite), too.*

Proof. The proof is easy and is therefore omitted.

Lemma 1.6. *Let $\mathfrak{X}=(X, R, M)$ be an object of* **Gr** *such that for every $x \in X$ the set $\{y: (x, y) \in R\}$ is finite. Let $Z, V \in B^{\mathfrak{X}}$ be such that there exists an infinite set $\mathscr{B} \subset B^{\mathfrak{X}}$ with the following properties:* 1) *for every $W_1, W_2 \in \mathscr{B}$, $W_1 \cap W_2 = V$ (or $W_1 \cup W_2 = V$);* 2) *$Z \supset W$ (or $Z \subset W$) for every $W \in \mathscr{B}$. Then $Z = X$ (or $Z = \emptyset$, respectively).*

Proof. Clearly, X is finite iff $B^{\mathfrak{X}}$ is finite. If the set $\{y: (x, y) \in R\}$ is finite for every $x \in X$ then X is finite iff M and $X-M$ are finite. By Lemma 1.4 we get that $V \in A_5^{\mathfrak{X}}$ and therefore either $Z = X$ or $Z \in A_5^{\mathfrak{X}}$. If $Z \in A_5^{\mathfrak{X}}$, we have that $Z-V$ is finite and therefore $\mathscr{B}$ is not infinite, a contradiction.

Proposition 1.7. *Let $\mathfrak{X}=(X, R, M)$, $\mathfrak{Y}=(Y, S, N)$ be objects of* **Gr** *such that*

a) *(X, R), (Y, S) are strongly reduced;*

b) *for every $x \in X$ the set $\{y: (y, x) \in R\}$ is finite and has at least two points;*

c) *$M, X-M, N, Y-N$ are infinite.*

If $f: \Psi\mathfrak{X} \to \Psi\mathfrak{Y}$ (or $f: \Phi\mathfrak{X} \to \Phi\mathfrak{Y}$) is a one-to-one lattice homomorphism then there exists a morphism $g: (X, R, M) \to (Y, S, N)$ of **Gr** *with $\Psi g = f$ (or $\Phi g = f$, respectively).*

Proof. By Lemmas 1.4 and 1.5, $f(A_5^{\mathfrak{X}}) \subset A_5^{\mathfrak{Y}}$, $f(A_4^{\mathfrak{X}}) \subset A_4^{\mathfrak{Y}}$. Now we shall prove $f(A_1^{\mathfrak{X}}) \subset A_1^{\mathfrak{Y}}$. Since for every $Z \in A_1^{\mathfrak{X}}$, $Z-M$ and $M-Z$ are nonempty, we get that $f(Z) \in A_1^{\mathfrak{Y}} \cup A_2^{\mathfrak{Y}} \cup A_3^{\mathfrak{Y}}$, hence $f(A_1^{\mathfrak{X}}) \subset A_1^{\mathfrak{Y}} \cup A_2^{\mathfrak{Y}} \cup A_3^{\mathfrak{Y}}$. Assume that there exists $Z \in A_1^{\mathfrak{X}}$ with $f(Z) \in A_2^{\mathfrak{Y}}$. Then there exists $V_1 \in A_1^{\mathfrak{X}}$ with $V_1 \cup Z \in A_5^{\mathfrak{X}}$ and $V_1 \cap Z \notin A_4^{\mathfrak{X}}$. Then $f(V_1) \cup f(Z) \in A_5^{\mathfrak{Y}}$ and $f(V_1) \cap f(Z) \notin A_4^{\mathfrak{X}}$. Therefore $(f(V_1)-N) \cap$ $\cap (f(Z)-N) \neq \emptyset$ but $(N-f(V_1)) \cap (N-f(Z)) = \emptyset$. We shall prove $f(V_1)-N=$ $=f(Z)-N$, hence we get a contradiction because (Y, S) is strongly reduced. Choose $V_2 \in A_1^{\mathfrak{X}}$ with $V_2 \cup Z, V_2 \cup V_1 \in A_5^{\mathfrak{X}}$, $V_2 \cap Z$, $V_2 \cap V_1 \in A_4^{\mathfrak{X}}$. Then $V_2 \cup Z=$ $= V_2 \cup V_1 = Z \cup V_2 \cup V_1$ (we use that $V_1 \cap Z \notin A_4^{\mathfrak{X}}$ and therefore $V_1 - M = Z - M$). Then $f(V_2) \cup f(Z) = f(V_2) \cup f(V_1) = f(Z) \cup f(V_2) \cup f(V_1)$, hence $(f(V_2) \cup f(Z))-$ $-N = (f(V_2) \cup f(V_1)) - N$. Since $f(V_1) \cap f(V_2)$, $f(Z) \cap f(V_2) \in A_4^{\mathfrak{Y}}$, we have $(f(V_1)-N) \cap (f(V_2)-N) = \emptyset$ and $(f(Z)-N) \cap (f(V_2)-N) = \emptyset$. Thus $f(Z)-N=$ $=f(V_1)-N$. We obtain that $f(A_1^{\mathfrak{X}}) \subset A_1^{\mathfrak{Y}}$ because it can be proved analogously that $f(A_1^{\mathfrak{X}}) \cap A_3^{\mathfrak{Y}} = \emptyset$. Hence $f(A_2^{\mathfrak{X}}) \subset A_2^{\mathfrak{Y}}$, $f(A_3^{\mathfrak{X}}) \subset A_3^{\mathfrak{Y}}$. Define $g: X \to Y$ as follows:

for $x \in M$, $g(x) = y$ where $f(M - \{x\}) = N - \{y\}$,

for $x \notin M$, $g(x) = y$ where $f(M \cup \{x\}) = N \cup \{y\}$.

(Since $f(A_1^{\mathfrak{X}})\subset A_1^{\mathfrak{Y}}$, we get that for every $v\in M$, $f(M-\{v\})=N-\{w\}$ where $w\in N$ and for every $v\in X-M$, $f(M\cup\{v\})=N\cup\{w\}$ where $w\in Y-N$.) It is clear that $g(M)\subset N$ and g is one-to-one. If $(x, y)\in R$ with $x\in M$ then $Z=(M-\{x\})\cup\{y\}\in A_1^{\mathfrak{X}}$ and therefore $f(Z)\in A_1^{\mathfrak{Y}}$. Since $Z\supset M-\{x\}$, we get that $f(Z)\supset N-\{g(x)\}$ and since $Z\subset M\cup\{y\}$, we get that $f(Z)\subset N\cup\{g(y)\}$. Hence $f(Z)=(N-\{g(x)\})\cup \cup\{g(y)\}$ and so $(g(x), g(y))\in S$. It is clear that $\Psi g=f$. If $f\colon \Phi\mathfrak{X}\to\Phi\mathfrak{Y}$ then by Lemmas 1.4 and 1.5 $f(A_5^{\mathfrak{X}})\subset A_5^{\mathfrak{Y}}$, $f(A_4^{\mathfrak{X}})\subset A_4^{\mathfrak{Y}}$. Therefore by Lemma 1.6 $f(\emptyset)=\emptyset$, $f(X)=Y$ and the rest follows from the foregoing part of the proof.

Corollary 1.8. *Put* $\mathfrak{A}_i=\Psi\mathfrak{X}_i$, $\mathfrak{B}_i=\Phi\mathfrak{X}_i$ *(for* $\mathfrak{X}_i$, *see Construction 1.1). If* $f\colon \mathfrak{A}_i\to\mathfrak{A}_j$ *(or* $f\colon \mathfrak{B}_i\to\mathfrak{B}_j$*) is a one-to-one lattice homomorphism then* $i=j$ *and* f *is the identity.*

Construction 1.9. Let T be a set. Put

$$Y=\{Z\colon (Z\subset\exp T),\ (Z\neq\emptyset),\ (Z \text{ is finite}),\ (V\in Z\Rightarrow (V\neq\emptyset \text{ and } V \text{ or } T-V \text{ is finite})),$$
$$(\forall V_1, V_2\in Z)(V_1-V_2\neq\emptyset)\}.$$

Define a partial ordering $\leqq$ on Y as follows: $Z_1\leqq Z_2$ iff for every $V\in Z_1$ there exists $W\in Z_2$ with $V\supset W$. Clearly, $\leqq$ is a reflexive and transitive relation. Since for every $Z\in Y$, $V_1, V_2\in Z$ implies $V_1-V_2\neq\emptyset$, we get that $Z_1\leqq Z_2\leqq Z_1$ iff $Z_1=Z_2$; thus $\leqq$ is a partial ordering.

Now if we put

$$Z_1\vee Z_2=\{V\in Z_1\cup Z_2\colon (W\in Z_1\cup Z_2\Rightarrow W-V\neq\emptyset \text{ or } W=V)\};$$
$$Z_1\wedge Z_2=\{V\colon (\exists V_1\in Z_1)(\exists V_2\in Z_2)(V=V_1\cup V_2),$$
$$(\forall W_1\in Z_1)(\forall W_2\in Z_2)((W_1\cup W_2)\subset V\Rightarrow W_1\cup W_2=V)\},$$

we get that $(Y, \leqq)$ is a partial ordering induced by a lattice $(Y, \vee, \wedge)$ and it is easy to verify that $(Y, \wedge, \vee)$ is a distributive lattice. Put $\mathscr{D}(T)=(Y, \vee, \wedge)$. We shall identify $t\in T$ with $\{\{t\}\}\in Y$, i.e. $T\subset Y$. It is clear that the sublattice of $\mathscr{D}(T)$ generated by T is a free distributive lattice over T. Furthermore, no element Z of $\mathscr{D}(T)$ is join-infinite and $Z\in Y$ is meet-infinite iff there exists an infinite set $V\subset T$ with $V\in Z$.

Let U be a set and let $\{U_{i,j}\colon i, j\in\mathbf{N}\}$ be a cover of U. Define

$$\bar{Y}=\{Z\subset\exp U\colon (Z \text{ is finite}),\ (V\in Z\Rightarrow (V\neq\emptyset)),\ (V \text{ is finite or}$$
$$(\exists i, j\in\mathbf{N})(U_{i,j}-V \text{ is finite}))),\ (\forall V_1, V_2\in Z)(V_1-V_2\neq\emptyset)\}.$$

Define a partial ordering $\leqq$ on $\bar{Y}$ as follows: $Z_1\leqq Z_2$ iff for every $V\in Z_1$ there exists $W\in Z_2$ with $V\supset W$. Clearly, $\leqq$ is a partial ordering and if we put

$$Z_1\vee Z_2=\{V\in Z_1\cup Z_2\colon (\forall W\in Z_1\cup Z_2)(W\subset V\Rightarrow W=V)\};$$
$$Z_1\wedge Z_2=\{V\colon (\exists V_1\in Z_1)(\exists V_2\in Z_2)(V=V_1\cup V_2),\ (\forall W_1\in Z_1),$$
$$(\forall W_2\in Z_2)((W_1\cup W_2)\subset V\Rightarrow W_1\cup W_2=V)\}$$

then $(\bar{Y}, \vee, \wedge)$ is a distributive lattice induced by the ordering $\leqq$. Put

$$\bar{\bar{Y}}=\{Z\in\bar{Y}\colon V\in Z\Rightarrow (V \text{ is infinite, } (\exists i, j, m, n\in\mathbf{N})$$
$$((i,j)\neq(m,n),\ V-U_{i,j}\neq\emptyset,\ V-U_{m,n}\neq\emptyset))\},$$

then $\bar{\bar{Y}}$ is an ideal in $\bar{Y}$. Let $\sim$ be the congruence relation generated by $\bar{\bar{Y}}$. Then $Z_1\sim Z_2$ iff $V\in\bar{\bar{Y}}$ whenever $V\in(Z_1-Z_2)\cup(Z_2-Z_1)$. Hence if we put

$$\tilde{Y}=\{Z\in\bar{Y}\colon V\in Z\Rightarrow (V \text{ is finite or } (\exists i,j\in\mathbf{N})(U_{i,j}-V \text{ is finite, } V\subset U_{i,j}))\},$$

we get that $(\tilde{Y}, \leqq)$ induces operations sup and inf as follows: $\sup\{Z_1, Z_2\}=Z_1\vee Z_2$, $\inf\{Z_1, Z_2\}=Z_1\wedge Z_2$ if $Z_1\wedge Z_2\in\tilde{Y}$, $=\emptyset$ otherwise. Clearly, $(\tilde{Y}, \sup, \inf)$ is a lattice. Since $(\tilde{Y}, \sup, \inf)$ is isomorphic to $(\bar{Y}, \vee, \wedge)/\sim$, we get that it is distributive. We shall identify $u\in U$ with $\{\{u\}\}\in\tilde{Y}$, i.e. $U\subset\tilde{Y}$. Notice that the sublattice of $(\tilde{Y}, \sup, \inf)$ generated by U is a free distributive lattice over U. Introduce the notation $\mathscr{C}(U, U_{i,j}\colon i, j\in\mathbf{N})=(\tilde{Y}, \sup, \inf)$ (further on we shall write only $\vee$, $\wedge$ instead of sup, inf).

Lemma 1.10. *For every cover $\{U_{i,j}\colon i, j\in\mathbf{N}\}$ of U no element of $\mathscr{C}(U, U_{i,j}\colon i, j\in\mathbf{N}\}$ is join-infinite. An element Z of $\mathscr{C}(U, U_{i,j}\colon i, j\in\mathbf{N})$ is meet-infinite iff*

a) *either $Z\neq\emptyset$ and there exists $V\in Z$ such that V is infinite,*

b) *or $Z=\emptyset$ and there exist infinitely many $i, j\in\mathbf{N}$ such that $U_{i,j}$ is infinite.*

Proof. Let $Z\in\tilde{Y}$, we prove that it is not join-infinite. Let $\mathscr{T}$ be a subset of $\tilde{Y}$ such that $Z_1\vee Z_2=Z$ for any distinct $Z_1, Z_2\in\mathscr{T}$. Then $Z_1\cup Z_2\supset Z$ and for every $V\in(Z_1\cup Z_2)-Z$ there exists $W\in Z$ with $V\supset W$. Hence, if $V\in Z-Z_i$ for $Z_i\in\mathscr{T}$ then $V\in Z_j$ for every $Z_j\in\mathscr{T}-\{Z_i\}$ and if $Z_i\supset Z$ where $Z_i\in\mathscr{T}$ then $Z_i=Z$. Therefore we get that $\mathscr{T}$ is finite and Z is not join-infinite.

Let $Z\in\tilde{Y}$, $Z\neq\emptyset$ be such that every $V\in Z$ is finite. We shall prove that Z is not meet-infinite. Let $\mathscr{T}\subset\tilde{Y}$ be such that $Z_1\wedge Z_2=Z$ for any distinct $Z_1, Z_2\in\mathscr{T}$. Hence if $V\in Z$, $V_1\in Z_1$ with $V\supset V_1$ then for every $W_2\in Z_2$, $V\not\supset V_1\cup W_2$ and there exists $V_2\in Z_2$ with $V=V_1\cup V_2$. On the other hand, for every $V\in Z$ there exists $V_1\in Z_1$ with $V\supset V_1$. Now, for every $V\in Z$ and every $Z_i\in\mathscr{T}$ we choose $W_{V,i}\in Z_i$ with $W_{V,i}\subset V$. Then for $i\neq j$, $W_{V,i}\cup W_{V,j}=V$. Therefore for every $V\in Z$

the set $\{W_{V,i}: Z_i \in \mathcal{T}\}$ is finite and if $W_{V,i} \neq V$ then $W_{V,i} \neq W_{V,j}$ for every $Z_j \neq Z_i$, $Z_j \in \mathcal{T}$. Hence the set $\{Z_i \in \mathcal{T}: (\exists V \in Z)\ (W_{V,i} \neq V)\}$ is finite. Let $\mathcal{T}'$ be a subset of $\mathcal{T}$ with $Z_i \in \mathcal{T}'$ iff $Z_i \supset Z$. It suffices to prove that $\mathcal{T}'$ is finite. For any distinct $Z_1, Z_2 \in \mathcal{T}'$ and every $V_1 \in Z_1 - Z$, $V_2 \in Z_2 - Z$, there exists $V \in Z$ with $V_1 \cup V_2 \supset V$. For every $Z_i \in \mathcal{T}' - \{Z\}$, we choose $V_i \in Z_i - Z$ and put $W_i = V_i \cap \cap \bigcup_{V \in Z} V$. Now if $Z_i \neq Z_j$ then $W_i \cup W_j \subset V$ for some $V \in Z$. Since $\bigcup_{V \in Z} V$ is a finite set, we get that there exists only a finite set $\mathcal{T}'' \subset \mathcal{T}'$ such that if $Z_i \in \mathcal{T}''$ then $V - W_i \neq \emptyset$ for every $V \in Z$. Hence $\mathcal{T}'$ is finite because if $W_i \supset V$ for some $V \in Z$ then $W_i = V_i = V$, a contradiction (notice that $V \in Z_i$). Thus $\mathcal{T}$ is finite and Z is not meet infinite.

If there exists an infinite set $V \in Z$ then put $\mathcal{T} = \{\{W: W \in Z - \{V\}\} \cup \cup \{V - \{x\}\}: x \in V\}$. Clearly, if $Z_1, Z_2 \in \mathcal{T}$, $Z_1 \neq Z_2$ then $Z_1 \wedge Z_2 = Z$ and $\mathcal{T}$ is infinite since V is infinite.

If $Z = \emptyset$ and $\mathbf{M} = \{(i, j): U_{i,j}$ is infinite$\}$ is infinite, then put $\mathcal{T} = \{\{U_{i,j}\}: (i, j) \in \mathbf{M}\}$. Then $\mathcal{T}$ is infinite and for distinct $Z_1, Z_2 \in \mathcal{T}$, $Z_1 \wedge Z_2 = \emptyset = Z$.

Let $\mathcal{T}$ be an infinite subset of $\mathscr{C}(U, U_{i,j}: i, j \in \mathbf{N})$ such that for distinct $Z_1, Z_2 \in \mathcal{T}$, $Z_1 \wedge Z_2 \neq \emptyset$. Then for every $Z_i \in \mathcal{T} - \{\emptyset\}$ there exists an infinite set $V_i \in Z_i$ and if $Z_i \neq Z_j$ then $V_i \cup V_j$ is not a subset of any $U_{m,n}$, $m, n \in \mathbf{N}$, but every V_i is a subset of some U_{m_i,n_i}. Hence if $Z_i \neq Z_j$ then $(m_i, n_i) \neq (m_j, n_j)$ and U_{m_i,n_i} is infinite.

Construction 1.11. Choose countably infinite sets T and U and a covering $\{U_{i,j}: i, j \in \mathbf{N}\}$ of U such that $U_{i,j}$ is infinite, and if $i+j = m+n$ then $U_{i,j} \cap U_{m,n} = \emptyset$, otherwise the intersection is a singleton. Choose a mapping $\varepsilon: U \to T$ such that $\varepsilon | U_{i,j}: U_{i,j} \to T$ is a bijection for every $(i, j) \in \mathbf{N} \times \mathbf{N}$ and choose a bijection $\mu: \mathbf{N} \to T$. Set $\mathbf{K} = \left\{(p, q): (p \in \mathbf{N}), \left(q \in \varepsilon^{-1}\left(\mu\left(\left[\frac{p}{2}\right]\right)\right)\right)\right\}$.

Let $\mathfrak{M}$ be the sublattice of $\mathscr{D}(T) \times \prod_{i \in \mathbf{N}} \mathfrak{B}_i$ (for $\mathfrak{B}_i$ see Corollary 1.8) generated by the set

$$S = \left\{(t, \{a_i\}_{i \in \mathbf{N}}): (t \in T), \left((i \text{ is odd}), \left(\mu\left(\left[\frac{i}{2}\right]\right) \neq t\right) \Rightarrow a_i = \emptyset\right),\right.$$

$$\left.\left((i \text{ is even}), \left(\mu\left(\left[\frac{i}{2}\right]\right) \neq t\right) \Rightarrow a_i = X_{\left[\frac{i}{2}\right]}\right), \quad \left(\mu\left(\left[\frac{i}{2}\right]\right) = t \Rightarrow a_i \in \mathfrak{B}_j\right)\right\} \cup$$

$$\cup \{(\{V\}, \{a_i\}_{i \in \mathbf{N}}): (T - V \text{ is finite}), (\forall i \in \mathbf{N})(a_i = \emptyset)\}.$$

It is clear that S is a countable set and therefore $\mathfrak{M}$ is a countable distributive lattice. For $t \in T$ set $\alpha(t) = (t, \{a_i\}_{i \in \mathbf{N}})$ where $\alpha(t) \in S$ and $a_i = M_i$ if $\mu\left(\left[\frac{i}{2}\right]\right) = t$.

Let $\mathfrak{N}$ be the sublattice of $\mathscr{C}(U, U_{i,j}: i, j\in\mathbf{N})\times\prod_{i\in\mathbf{N}}\mathfrak{B}_i^{s_i}$ with $s_i = \operatorname{card}\varepsilon^{-1}\left(\mu\left(\left[\frac{i}{2}\right]\right)\right)$ generated by the set

$$Q = \{(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})\colon (u\in U), ((p \text{ is odd}), (q\neq u)\Rightarrow$$
$$\Rightarrow a_{p,q}=\emptyset), ((p \text{ is even}), (q\neq u)\Rightarrow a_{p,q}=X_{\left[\frac{p}{2}\right]}),$$
$$(q=u\Rightarrow a_{p,q}\in\mathfrak{B}_{\left[\frac{p}{2}\right]})\}\cup\{(\{V\}, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})\colon$$
$$(\exists i, j\in\mathbf{N})((U_{i,j}-V \text{ is finite}), (V\subset U_{i,j})),$$
$$(\forall(p,q)\in\mathbf{K}(a_{p,q}=\emptyset))\}.$$

Since Q is a countable set, $\mathfrak{N}$ is a distributive lattice. For $u\in U$, put $\beta(u) = (u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})\in Q$ where $a_{p,q}=M_{\left[\frac{p}{2}\right]}$ if $q=u$.

Lemma 1.12. *Let $(t, \{a_i\}_{i\in\mathbf{N}})$ be a point where $t\in\mathscr{D}(T)$ and $a_i\in\mathfrak{B}_{[i/2]}$ for every $i\in\mathbf{N}$. Then $(t, \{a_i\}_{i\in\mathbf{N}})$ is a point of $\mathfrak{M}$ iff the following conditions hold:*

a) *if i is odd and $a_i\neq\emptyset$ then t is greater than or equal to $\mu\left(\left[\frac{i}{2}\right]\right)$;*

b) *if i is even and $a_i\neq X_{[i/2]}$ then either t is less than or equal to $\mu\left(\left[\frac{i}{2}\right]\right)$ or for every $i\in\mathbf{N}$, $a_i=\emptyset$ and every set $V\in t$ is infinite.*

Let $(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ be a point where $u\in\mathscr{C}(U, U_{i,j}: i, j\in\mathbf{N})$ and $a_{p,q}\in\mathfrak{B}_{[p/2]}$ for every $(p,q)\in\mathbf{K}$. Then $(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ is a point of $\mathfrak{N}$ iff the following conditions hold:

a) *if p is odd and $a_{p,q}\neq\emptyset$ then u is greater than or equal to q;*

b) *if p is even and $a_{p,q}\neq X_{[p/2]}$ then either u is less than or equal to q or for every $(p,q)\in\mathbf{K}$, $a_{p,q}=\emptyset$ and either $u=\emptyset$ or every set $V\in u$ is infinite.*

Proof. Easy.

Notice, if $(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ is a point of $\mathfrak{N}$ then there exist only finitely many $(p,q)\in\mathbf{K}$ with $a_{p,q}=\emptyset, X_{[p/2]}$. Hence we get

Corollary 1.13. *An element $(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ of $\mathfrak{N}$ is meet-infinite iff either there exists an infinite set $V\subset U$ with $V\in u$ or $u=\emptyset$, or there exists $(p,q)\in\mathbf{K}$ with $X_{[p/2]}\neq a_{p,q}\supset M_{[p/2]}$. An element $(u, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ of $\mathfrak{N}$ is join-infinite iff there exists $(p,q)\in\mathbf{K}$ with $M_{[p/2]}\supset a_{p,q}\neq\emptyset$.*

Proof. The statement follows from Lemmas 1.4, 1.10, 1.12 and the fact that if $Z_1\cap Z_2=\emptyset$ then either $Z_1=\emptyset$ or $Z_2=\emptyset$ and if $Z_1\cup Z_2=X_i$ then either $Z_1=X_i$ or $Z_2=X_i$ in each $\mathfrak{B}_i$.

Proposition 1.14. *Let $\varphi\colon\mathfrak{M}\to\mathfrak{N}$ be a one-to-one lattice homomorphism. Then for every $t\in T$, $\varphi(\alpha(t))=\beta(u)$ where $\varepsilon(u)=t$.*

Proof. Set $\varphi(\alpha(t))=(u^t, \{a^t_{p,q}\}_{(p,q)\in K})$. Since $\alpha(t)$ is join-infinite, we get according to Corollary 1.13 and Lemma 1.12 that there exists $\bar{u}^t\in U$ such that either $u^t\leqq\bar{u}^t$ or $u^t\geqq\bar{u}^t$.

a) First we prove that $u^t=\bar{u}^t$. Assume the contrary, i.e. for some $t_0\in T$, $u^{t_0}\neq\bar{u}^{t_0}$. We know that there exists a finite set $W\subset U$ with $W\in u^{t_0}$. Now if $u^{t_0}<\bar{u}^{t_0}$ (in the case $u^{t_0}>\bar{u}^{t_0}$, the proof is analogous) then put $L_t=\{u\in U: u>u^t\}$ for $t\in T$. Clearly, L_t is a finite set for every $t\in T$. Now there exists a finite subset $T'\subset T$ with $\bigcap_{t\in T} L_t=\bigcap_{t\in T'} L_t$. Then $\bigvee\{\alpha(t): t\in T'\}$ is join-infinite and therefore $\bigcap_{t\in T'} L_t\neq\emptyset$ (see Corollary 1.13 and Lemma 1.12). For $t\in T-T'$ put

$$E_t=\left\{(t, \{a_i\}_{i\in N})\in\mathfrak{M}: \left(\left(\mu\left(\left[\frac{i}{2}\right]\right)=t\right), (i \text{ is even})\Rightarrow a_t=M_{[i/2]}\right),\right.$$

$$\left.(i \text{ is odd}\,(\exists x\in M_{[i/2]})(a_i=M_{[i/2]}-\{x\}))\right\}.$$

Hence, if e_1, e_2 are distinct points of E_t then $e_1\vee e_2=\alpha(t)$ and $e_1\vee c\neq e_2\vee c$, $e_1\wedge c=e_2\wedge c$ where $c=\bigvee\{\alpha(t): t\in T'\}$. For $w\in\mathfrak{M}$, let $\varphi(w)=\{(v^w, b^w_{p,q})\}_{(p,q)\in K}$. Since no element of $\mathscr{C}(U, U_{i,j}: i, j\in N)$ is join-infinite, the set $\bar{E}_t=\{e\in E_t: v^e=u^t\}$ is infinite for every $t\in T-T'$ (because for infinitely many $e_1, e_2\in E_t$, $v^{e_1}=v^{e_2}$ and then necessarily $v^{e_1}=u^t$). Hence for $e\in\bar{E}_t$, $v^{e\vee c}=v^{\alpha(t)\vee c}$. Now, for distinct t_1, $t_2\in T-T'$ and for $e_1\in\bar{E}_{t_1}$, $e_2\in\bar{E}_{t_2}$, we have $e_1\wedge e_2=\alpha(t_1)\wedge\alpha(t_2)$. Thus, for every distinct points $t_1, t_2\in T-T'$,

(a) $e_1\vee c\neq e_2\vee c$ for any $e_1, e_2\in\bar{E}_{t_1}$ and $v^{e_1\vee c}=v^{e_2\vee c}$,

(b) $e\wedge\bar{e}=\alpha(t_1)\wedge\alpha(t_2)$ for every $e\in\bar{E}_{t_1}$, $\bar{e}\in\bar{E}_{t_2}$.

Since for every $t\in T-T'$ there exists only a finite subset $\mathbf{K}_t\subset\mathbf{K}$ such that $(p, q)\in\mathbf{K}_t$ whenever $b_{p,q}\neq\emptyset$, $X_{[p/2]}$ and $(v^{e\vee c}, \{b_{p,q}\}_{(p,q)\in K})\in\mathfrak{N}$ where $e\in E_t$, therefore there exists $(p_t, q_t)\in\mathbf{K}$ and an infinite set $\tilde{E}_t\subset\bar{E}_t$ such that $b^{e_1\vee c}_{p_t,q_t}\neq b^{e_2\vee c}_{p_t,q_t}$ whenever e_1, e_2 are distinct points of $\tilde{E}_t$. Since $\alpha(t)\wedge\alpha(t')\leqq e\vee c$ for every $t, t'\in T-T'$ and $e\in E_t$, we get that $b^{\alpha(t)\wedge\alpha(t')}_{p_t,q_t}=\emptyset$ (see Lemma 1.6) and since in $\mathfrak{B}_i$ $Z_1\cap Z_2=\emptyset$ implies either $Z_1=\emptyset$ or $Z_2=\emptyset$, we have that for every $t'\in T-T'$, $t\neq t'$ and every $e\in\bar{E}_{t'}$, $b^e_{p_t,q_t}=\emptyset$. Since $q_{\bar{t}}\in\bigcap_{t\in T} L_t$ for every $\bar{t}\in T-T'$ and since $\bigcap_{t\in T} L_t$ is finite, we get a contradiction. Hence $u^{t_0}=\bar{u}^{t_0}$.

b) Now, we prove that $a_{\bar{p},u}=M_{[\bar{p}/2]}$. Assume the contrary, i.e. $a_{\bar{p},u}=Z\neq M_{[\bar{p}/2]}$. If $\bar{p}$ is odd then for $t', t''\in T$, $t'\neq t''\neq t$, $t'\neq t$, the element $e=(\alpha(t)\vee\alpha(t'))\wedge\wedge(\alpha(t)\vee\alpha(t''))$ is both meet- and join-infinite. On the other hand, if $\varphi(e)==(v^e, \{b^e_{p,q}\}_{(p,q)\in K})$ then $b^e_{p,q}=\emptyset$ or $X_{[p/2]}$ if $(p, q)\neq(\bar{p}, u)$ and $b^e_{\bar{p},u}=Z$, which contradicts Lemmas 1.4 and 1.5. If $\bar{p}$ is even, the proof is analogous. Thus $a_{\bar{p},u}=M_{[\bar{p}/2]}$.

c) Now we prove that $\varepsilon(u^t)=t$. Let i_0 be an odd natural number with $\mu\left(\left[\frac{i_0}{2}\right]\right)=t$, let p_0 be an odd natural number with $\mu\left(\left[\frac{p_0}{2}\right]\right)=\varepsilon(u^t)$. It suffices to prove that $i_0=p_0$. Define $\psi\colon \mathfrak{B}_{i_0}\to\mathfrak{B}_{p_0}$ as follows: $\psi(Z)=b^{e_Z}_{p,u^t}$ where $e_Z=(t, \{a_i\}_{i\in \mathrm{N}})$ and if $\mu\left(\left[\frac{i}{2}\right]\right)=t$ and i is odd then $a_i=Z$, while if i is even then $a_i=M_{[i/2]}$ (recall that $\varphi(e_Z)=(v^{e_Z}, \{b^{e_Z}_{p,q}\}_{(p,q)\in \mathrm{K}})$). It is clear that ψ is a lattice homomorphism (it is a composition of the embedding of $\mathfrak{B}_{i_0}$ into $\mathfrak{M}$, of φ and of the projection from $\mathfrak{N}$ to $\mathfrak{B}_{p_0}$). We shall prove that ψ is one-to-one. By Lemma 1.6 it suffices to prove that $\psi|\mathfrak{A}_{i_0}$ is one-to-one. First we shall prove that for every $Z\in\mathfrak{A}_{i_0}$, $v^{e_Z}=u^t$. Hence, it follows immediately that ψ is one-to-one and by Corollary 1.8, $i_0=p_0$. Put

$$E_1=\left\{(t, \{a_i\}_{i\in\mathrm{N}})\colon \left(\left(\mu\left(\left[\frac{i}{2}\right]\right)=t\right), (i \text{ is even})\Rightarrow a_i=M_{[i/2]}\right),\right.$$
$$\left.\left(i \text{ is odd}\Rightarrow (\exists x\in M_{[i/2]})(a_i=M_{[i/2]}-\{x\})\right)\right\},$$

$$E_2=\left\{(t, \{a_i\}_{i\in\mathrm{N}})\colon \left(\left(\mu\left(\left[\frac{i}{2}\right]\right)=t\right), (i \text{ is even})\Rightarrow a_i=M_{[i/2]}\right),\right.$$
$$\left.\left(i \text{ is odd}\Rightarrow (\exists x\in X_{[i/2]}-M_{[i/2]})(a_i=M_{[i/2]}\cup\{x\})\right)\right\}.$$

Clearly, if we verify that for $e\in E_1\cup E_2$, $v^e=u^t$, then for every $Z\in\mathfrak{A}_{i_0}$, $v^{e_Z}=u^t$. Since for any distinct $e_1, e_2\in E_1$ $(e_1, e_2\in E_2)$, $e_1\vee e_2=\alpha(t)$ $(e_1\wedge e_2=\alpha(t)$, resp.) we get that there exists at most one $e_1\in E_1$ (or $e_2\in E_2$) with $v^{e_1}\neq u^t$ (or $v^{e_2}\neq u^t$) because for $u\in U$, if $u_1\vee u_2=u$ (or $u_1\wedge u_2=u$) then either $u_1=u$ or $u_2=u$. Then necessarily $v^{e_1}\leqq u^t\leqq v^{e_2}$. Choose a homomorphism $\sigma\colon \mathcal{D}(T)\to\mathfrak{M}$ such that $\sigma(t)=\alpha(t)$ for every $t\in T$ (clearly, such a homomorphism exists). Now we can choose $t'\in\mathcal{D}(T)$ such that $t'>t$ and $v^{\sigma(t')}$ and v^{e_2} are incomparable. Then $\sigma(t')\wedge e_2=\alpha(t)$ (observe that if $\sigma(t')=(t, \{a_i\}_{i\in\mathrm{N}})$ then for an odd i with $\mu\left(\left[\frac{i}{2}\right]\right)=t$, $a_i=M_{[i/2]}$), but $\varphi(\sigma(t'))\wedge\varphi(e_2)\neq\varphi(\alpha(t))$, a contradiction. Thus for every $e\in E_2$, $v^e=u^t$. Analogously, we prove that $v^{e_1}=u^t$. The proof is concluded.

Theorem 1.15. *For every natural number i, there exist pairwise disjoint sublattices $\mathfrak{N}_0, \mathfrak{N}_1, \dots, \mathfrak{N}_{i-1}$ of the lattice $\mathfrak{N}$ which are isomorphic to $\mathfrak{M}$, but there are not infinitely many pairwise disjoint sublattices $\mathfrak{N}_0, \mathfrak{N}_1, \dots$ of $\mathfrak{N}$ which are isomorphic to $\mathfrak{M}$.*

Proof. Let $\{\varphi_k\}_{k\in\mathrm{N}}$ be a sequence of one-to-one lattice homomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$. Then for arbitrary $k\in\mathrm{N}$ and $t\in T$, $\varphi_k(\alpha(t))=\beta(u)$ where $\varepsilon(u)=t$. Further, for every finite set $T'\subset T$ there exists a point $\gamma(T')\in\mathfrak{M}$ such that $\gamma(T')\leqq\alpha(t)$ iff $t\in T-T'$. On the other hand, if $U'\subset U$ is an infinite set and $U'-U_{i,j}\neq\emptyset$ for any $i, j\in\mathbf{N}$, then $e\leqq\beta(u)$ for every $u\in U'$ iff $e=(\emptyset, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ where $a_{p,q}=\emptyset$ for every $(p,q)\in\mathbf{K}$. Therefore there exist $i_k, j_k\in\mathbf{N}$ with $\varphi_k(\alpha(t))=(u^t, \{a_{p,q}\}_{(p,q)\in\mathbf{K}})$ where $\{u^t\}=\varepsilon^{-1}(t)\cap U_{i_k,j_k}$ for every $t\in T$. Therefore there exist $k_1\neq k_2$ with $i_{k_1}+j_{k_1}\neq i_{k_2}+j_{k_2}$. Put $\{u\}=U_{i_{k_1},j_{k_1}}\cap U_{i_{k_2},j_{k_2}}$ $\varepsilon(u)=t$. Then $\varphi_{k_1}(\alpha(t))=\varphi_{k_2}(\alpha(t))$ and $\{\varphi_k(\mathfrak{M})\}_{k\in\mathbf{N}}$ are not pairwise disjoint.

Let k be a natural number. For every $j\leqq k$ define $\psi_j\colon \mathscr{D}(T)\to\mathscr{C}(U, U_{i,j}\colon i,j\in\mathbf{N})$ as follows: $\psi_j(Z)=\{\varepsilon^{-1}(V)\cap U_{(k-j),j}\colon V\in Z\}$. Clearly, the ψ_j's are one-to-one homomorphisms and $\{\psi_j(\mathscr{D}(T))\}_{j\leqq k}$ are pairwise disjoint. Define $\varphi_j\colon\mathfrak{M}\to\mathfrak{N}$, $\varphi_j(t, \{a_i\}_{i\in\mathbf{N}})=(\psi_j(t), \{b_{p,q}\}_{(p,q)\in\mathbf{K}})$ where $b_{i,\psi_j(t)}=a_i$. Then $\{\varphi_j\colon j\leqq k\}$ is a family of pairwise disjoint one-to-one lattice homomorphisms. The proof is concluded.

2. Let us formulate the above problem in a general category with a class $\mathfrak{M}$ of its morphisms.

Definition. Let $\mathscr{K}$ be a category with a cosingleton $\emptyset$. Let $f, g\colon A\to B$ be morphisms of $\mathscr{K}$. We shall say that f, g are *disjoint* if

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\;g\;} & B \end{array}$$

is a pull back.

Definition. Let $\mathscr{K}$ be a category, let $\mathfrak{M}$ be a class of its morphisms. A pair (A, B) of objects is said to have the property $(S_{\mathfrak{M}})$ if for every $n=1,2,\dots$ there exist n pairwise disjoint $\mathfrak{M}$-morphisms from A to B, but there do not exist infinitely many such morphisms. We say that $\mathscr{K}$ fulfils Sekanina's axiom with respect to $\mathfrak{M}$ if no pair of objects has the property $(S_{\mathfrak{M}})$.

Now we can formulate the foregoing result as follows: The pair $(\mathfrak{M}, \mathfrak{N})$ of countable distributive lattices has the property $(S_{\mathscr{M}})$ with $\mathscr{M}$ the class of all monomorphisms.

Now we establish some other results:

Theorem 2.1. *The category of sets, the category of vector spaces and the category of unary algebras with one operation fulfil Sekanina's axiom with respect to $\mathfrak{M}$ for every class $\mathfrak{M}$ containing all monomorphisms.*

Proof. Easy.

Theorem 2.2. *The category of complete, completely distributive Boolean algebras fulfils Sekanina's axiom with respect to the class of all monomorphisms.*

Proof. The statement follows immediately from the well-known fact that every complete, completely distributive Boolean algebra is the algebra of all subsets of some set.

Theorem 2.3. *The category of graphs or unary algebras with α operations (α is a cardinal, $\alpha>0$) fulfils Sekanina's axiom with respect to the class of all summands.*

Proof. The statement follows from the fact that $f: A\to B$ is sumand iff A is isomorphic to the sum of some components of B.

Now we recall that a monomorphism f in a category $\mathcal{K}$ is an *extremal monomorphism* if any epimorphism e is an isomorphism whenever $f=g\circ e$ for some morphism g of $\mathcal{K}$. In the category of graphs or topological spaces extremal monomorphisms are embeddings to full subgraphs or subspaces.

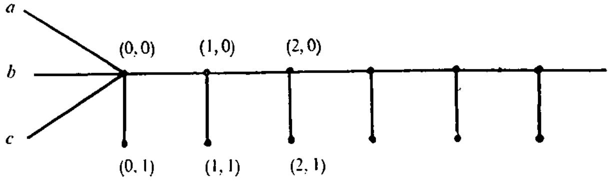
I. Korec showed that there exists a pair (A, B) of countable graphs or countable unary algebras with two operations which have the property $(S_{\mathfrak{M}})$ where $\mathfrak{M}$ is the class of all extremal monomorphisms.

Theorem 2.4. *There exists a pair (A, B) of connected, countable, bipartite graphs with the property $(S_{\mathfrak{M}})$ where $\mathfrak{M}$ is an arbitrary class of monomorphisms containing all extremal monomorphisms.*

Theorem 2.5. *There exists a pair (A, B) of continua with the property $(S_{\mathfrak{M}})$ where A is a subcontinuum of the plane, B is a subcontinuum of the cube and $\mathfrak{M}$ is an arbitrary class of monomorphisms containing all extremal monomorphisms.*

Proof of Theorems 2.4 and 2.5. Put $X=\{a, b, c\}\cup(\mathbf{N}\times\{0, 1\})$,

$$R = \{((0,0),a),((0,0),b),((0,0),c),(a,(0,0)),(b,(0,0)),(c,(0,0))\}\cup$$
$$\cup\{((i,0),(i,1)),((i,1),(i,0))\colon i\in\mathbf{N}\}\cup$$
$$\cup\{((i,0),(i+1,0)),((i+1,0),(i,0))\colon i\in\mathbf{N}\}.$$



(X, R)

Fig. 2

Clearly, (X, R) is a connected, countable bipartite graph. Choose a bijection φ from $\mathbf{L}=\{(x, y, z, v):(x, y, z, v\in\mathbf{N}), (x+y\neq z+v)\}$ to $\mathbf{N}$. Put $(Y, S)=(X, R)\times\times(\mathbf{N}\times\mathbf{N}, \Delta)/\sim$ where $(\mathbf{N}\times\mathbf{N}, \Delta)$ is the smallest reflexive relation on $\mathbf{N}\times\mathbf{N}$ and $\sim$ is the smallest equivalence relation on $X\times\mathbf{N}\times\mathbf{N}$ with

$$(k, 1, x, y) \sim (k, 1, z, v) \quad \text{whenever} \quad \varphi(x, y, z, v) = k.$$

Clearly, (Y, S) is a connected, countable graph. To verify that it is bipartite, it suffices to put $M=\{(k, i, x, y)\in Y\colon k+i \text{ is even}\}/\sim$. Let k be a natural number, $i\leqq k$. Define $f_i^k\colon (X, R)\to(Y, S)$ as follows: $f_i^k(x)$ is the $\sim$-class containing $(x, k, k-i)$. Clearly, $f_i^k, i=0, 1, \ldots, k$, are pairwise disjoint extremal monomorphisms. Let $\{f_i\}$ be a sequence of one-to-one morphisms from (X, R) to (Y, S). Since $\operatorname{card}\{y\colon(y, (0, 0))\in R\}=4$, we get that for every i there exists $(p_i, q_i)\in\mathbf{N}\times\mathbf{N}$ such that $f_i(0, 0)$ is the $\sim$-class containing $(0, 0, p_i, q_i)$. Hence we easily get that $f_i(j, 0)$ is the $\sim$-class containing $(j, 0, p_i, q_i)$ and $f_i(j, 1)$ is the $\sim$-class containing $(j, 1, p_i, q_i)$. Further, there exist i_0, i_1 with $p_{i_0}+q_{i_0}\neq p_{i_1}+q_{i_1}$. Let $k=\varphi(p_{i_0}, q_{i_0}, p_{i_1}, q_{i_1})$. Then $f_{i_0}(k, 1)=f_{i_1}(k, 1)$ and therefore f_{i_0} and f_{i_1} are not disjoint. If we set $A=(X, R)$, $B=(Y, S)$, then the proof of Theorem 2.4 is concluded.

Let K be a circle with the usual topology. Choose two distinct points $a, b\in K$. Let $S=\{\{x, y\}\colon (x, y)\in R\}$ be equipped with the discrete topology where $R\subset X\times X$ is the relation defined above. Let P_1 be the one-point compactification of $K\times S/\sim$ with $\sim$ standing for the smallest equivalence relation such that:

$$(a, \{x, y\}) \sim (a, \{x, z\}) \quad \text{for every} \quad \{x, y\}, \{x, z\}\in S \quad \text{with} \quad x\in\{(i, j)\colon i+j \text{ is even}\};$$

$$(b, \{x, y\}) \sim (b, \{x, z\}) \quad \text{for every} \quad \{x, y\}, \{x, z\}\in S \quad \text{with} \quad x\in\{(i, j)\colon i+j \text{ is odd}\}.$$

Clearly, P_1 is a subcontinuum of the plane. We shall assume that $\mathbf{N}$ has the discrete topology. Let P_2 be the one-point compactification of $P_1\times\mathbf{N}\times\mathbf{N}/\approx$ where $\approx$ is the smallest equivalence relation such that if $\varphi(x, y, z, v)=k$ then

$$\big((a, \{(k, 0), (k, 1)\}], x, y\big) \approx \big([a, \{(k, 0), (k, 1)\}], z, v\big) \quad \text{if } k \text{ is odd},$$

$$\big([b, \{(k, 0), (k, 1)\}], x, y\big) \approx \big([b, \{(k, 0), (k, 1)\}], z, v\big) \quad \text{if } k \text{ is even},$$

where $[x]$ denotes the $\sim$-class containing x. Clearly, P_2 is a subcontinuum of the cube. The proof that (P_1, P_2) has the property $(S_{\mathfrak{M}})$ is analogous to that of the similar statement for (X, R) and (Y, S). It suffices to realize that if $f\colon K\to K$ is one-to-one then f is a homeomorphism.

Theorem 2.6. *There exists a pair (A, B) of 0-dimensional compact Hausdorff spaces on sets of power $\aleph_1$, which has the property $(S_{\mathfrak{M}})$ where $\mathfrak{M}$ is the class of all summands.*

Proof. Define topological spaces S_n by induction as follows: S_1 is the one-point compactification of a countable discrete set; S_n is the one-point compactification of $S_{n-1}\times N$ where N has the discrete topology. Put R_n to be the one-point compactification of $\aleph_1$ copies of S_n. Let T_1 be the one-point compactification of the disjoint union of $R_1, R_2, \dots$. Clearly, T_1 is a 0-dimensional compact Hausdorff space on a set of power $\aleph_1$.

Let U be a countable set and let $\{U_{i,j}: i, j\in N\}$ be a cover of U such that every $U_{i,j}$ is infinite and $U_{i,j}\cap U_{m,n}=\emptyset$ if $i+j=m+n$, $U_{i,j}\cap U_{m,n}$ is infinite if $i+j\neq m+n$. Choose a mapping $\psi: U\to\{1, 2, 3, \dots\}$ such that $\psi|U_{i,j}$ is a bijection from $U_{i,j}$ onto $\{1, 2, 3, \dots\}$ for every $i, j\in N$.

Let T_2 be the one-point compactification of $T_1\times N\times N/\approx$ ($N\times N$ has the discrete topology) where $\approx$ is the smallest equivalence relation such that $(x, i, j)\approx(x, m, n)$ if $x\in R_p$ and $p\in\psi(U_{i,j}\cap U_{m,n})$. Clearly, T_2 is a 0-dimensional compact Hausdorff space on a set of power $\aleph_1$.

Let k be a natural number, $i=0, 1, \dots, k$. Define $f_i^k: T_1\to T_2$, by $f_i^k(x)$ being the $\approx$-class containing $(x, k, k-i)$. It is easy to verify that f_i^k is a summand and that f_i^k and f_j^k are disjoint whenever $i\neq j$.

Let $\{f_i: T_1\to T_2\}$ be a sequence of summands. Then for every i, there exist $j_i, k_i\in N$ and $i_1, i_2, \dots, i_n\in N$ such that $f_i(T_1-\bigcup_{m=1}^{n} R_{i_m})\subset T_1\times\{(j_i, k_i)\}/\approx$, therefore if $j_{i_0}+k_{i_0}\neq j_{i_1}+k_{i_1}$, we get that $\mathrm{Im} f_{i_0}\cap\mathrm{Im} f_{i_1}\neq\emptyset$ and thus f_{i_0} and f_{i_1} are not disjoint. On the other hand, there exist i_0, i_1 such that either $j_{i_0}+k_{i_0}\neq j_{i_1}+k_{i_1}$ or $(j_{i_0}, k_{i_0})=(j_{i_1}, k_{i_1})$. Hence if we set $A=T_1, B=T_2$, the proof of the theorem is complete.

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