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# Mean ergodic semigroups on $W^*$-algebras

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In 1966 I. KOVÁCS and J. SZŰCS [5] proved the following: If $G$ is a group of $^*$-automorphisms on a $W^*$-algebra $\mathscr{A}$ having a faithful family of normal $G$-invariant states, then the weak*-closed convex hull of $Gx$, $x \in \mathscr{A}$, contains a unique $G$-invariant element. As its forerunner, the von Neumann mean ergodic theorem, this result has found many applications in mathematical physics. For that reason it is interesting to ask whether the theorem may be generalized to semigroups of bounded operators on $\mathscr{A}$.

On the basis of an abstract mean ergodic theory (see [8], [14] or section 1 below) we will prove in section 2 that the desired result holds for semigroups of operators on $\mathscr{A}$ satisfying a certain contraction property. This answers a question raised in [9]. Our technique can be applied with particular success to semigroups of completely positive contractions on $\mathscr{A}$ (theorem 2.4). In section 3 we investigate the relation between mean ergodicity and compactness of such semigroups.

Some of the results have been announced in [6].

## 1. The abstract mean ergodic theory

In this section we recall the basic theory of mean ergodic operator semigroups. Let $E$ denote a Banach space, $E^*$ its (topological) dual and $\mathscr{L}(E)$ the space of all bounded linear operators on $E$. We call a semigroup $S \subset \mathscr{L}(E)$ *(norm)mean ergodic* if the strong (or weak) operator closure $\overline{\text{co}}\ S$ in $\mathscr{L}(E)$ of the convex hull of $S$ contains a projection $P$ satisfying

$$TP = PT = P \quad \text{for all} \quad T \in S.$$

Endow $E^*$ with the weak* topology $\sigma(E^*, E)$ and denote the weak* closed convex hull of the orbit of $\varphi \in E^*$ under the adjoint semigroup $S^*: = \{T^* \in \mathscr{L}(E^*): T \in S\}$

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by $\overline{\text{co}}\, S^*\varphi$. The following property of a mean ergodic semigroup is then immediate.

(1.1) Proposition. *Let* $S \subset \mathscr{L}(E)$ *be mean ergodic with projection* $P$. *Then* $P$ *(resp. its adjoint* $P^*$*) is a projection onto the* $S$*-fixed space* $F$ *in* $E$ *(resp.* $S^*$*-fixed space* $F^*$ *in* $E^*$*) and we have* $Px \in \overline{\text{co}}\, Sx$, $P^*\varphi \in \overline{\text{co}}\, S^*\varphi$ *for* $x \in E$, $\varphi \in E^*$. *Moreover, the dual of* $PE$ *is isomorphic to* $P^*E^*$.

For bounded semigroups we have the following useful characterization of mean ergodicity.

(1.2) Theorem [8]. *If the semigroup* $S \subset \mathscr{L}(E)$ *is bounded, the following conditions are equivalent:*

(a) $S$ *is mean ergodic.*

(b) $S^*$ *is weak* mean ergodic, i.e. the closed convex hull of* $S^*$ *with respect to the weak* operator topology* $\sigma(\mathscr{L}(E^*), E^* \otimes E)$ *contains a weak* continuous projection* $P^*$ *satisfying*

$$P^*T^* = T^*P^* = P^* \quad \text{for all} \quad T \in S.$$

(c) *The* $S$*-fixed space* $F$ *separates the* $S^*$*-fixed space* $F^*$ *and* $F^* \cap \overline{\text{co}}\, S^*\varphi \neq \emptyset$ *for all* $\varphi \in E^*$.

Remarks: 1. The proof of (c)$\Rightarrow$(a) follows as in [8], 1.7.

2. $\sigma(\mathscr{L}(E^*), E^* \otimes E)$ is the topology of pointwise convergence on $(E^*, \sigma(E^*, E))$.

3. Weak* mean ergodicity of $S^*$ should be distinguished from norm mean ergodicity of $S^*$ on the Banach space $E^*$.

The main objective of the theory is to show that certain semigroups on certain Banach spaces are mean ergodic. The oldest result in this direction is the von Neumann mean ergodic theorem which states that the semigroup $S := \{T^n : n \in \mathbb{N}\}$ generated by a unitary operator $T$ on a Hilbert space is mean ergodic. This has been generalized considerably with emphasis either on the geometry of the underlying Banach space or on particular properties of the semigroup. We quote two typical results (see [8], 1.4 and 1.9 or [14], III. 7.11 and 7.9).

(1.3) Examples. 1. Every contraction semigroup on a Hilbert space is mean ergodic (Alaoglu-Birkhoff).

2. A group $G$ in $\mathscr{L}(E)$ which is compact in the strong (or weak) operator topology is mean ergodic.

## 2. A non-commutative mean ergodic theorem

In what follows $\mathscr{A}$ shall always be a $W^*$-algebra with dual $\mathscr{A}^*$ and predual $\mathscr{A}_*$ (see [3] or [12]). On $\mathscr{A}$ we consider a bounded semigroup $S$ of weak* continuous linear operators whose preadjoints $S_* := \{T_* \in \mathscr{L}(\mathscr{A}_*) : T \in S\}$ then exist. For such a pair $(\mathscr{A}, S)$, called here a *dynamical system,* we state our main result.

(2.1) Theorem. *Let $(\mathscr{A}, S)$ be a dynamical system. If there exists a faithful family $\Phi$ of normal states on $\mathscr{A}$ satisfying*

$$(*) \qquad \varphi((Tx)^*(Tx)) \leqq \varphi(x^*x) \quad \textit{for all} \quad \varphi\in\Phi,\ T\in S,\ x\in\mathscr{A},$$

*then $S$ is weak* mean ergodic.*

The proof of the theorem will be based on example 1.3.1 and on the coincidence of certain topologies. For this purpose we denote by $\mathscr{T}_A$, A contained in $\mathscr{A}_*^+$, the set of all normal positive linear forms on $\mathscr{A}$, the topology on $\mathscr{A}$ generated by the seminorms

$$x \to \varphi(x^*x)^{1/2}, \quad \varphi\in A.$$

In particular, we write $\mathscr{T}_A = \mathscr{T}_\varphi$ if $A = \{\varphi\}$ and we have $\mathscr{T}_A = s(\mathscr{A}, \mathscr{A}_*)$, the strong topology, if $A = \mathscr{A}_*^+$ (see [12], 1.8.6). Take now $\varphi\in\mathscr{A}_*^+$ and denote its support by $p_\varphi$ and the orthogonal complement $1-p_\varphi$ by $p_\varphi^\perp$. Then we have $\mathscr{A} = K_\varphi \oplus L_\varphi$ setting

$$K_\varphi := \mathscr{A}p_\varphi \quad \text{and} \quad L_\varphi := \mathscr{A}p_\varphi^\perp = \{x\in\mathscr{A} : \varphi(x^*x) = o\}.$$

If $\mathscr{A}$ is a weakly closed *-subalgebra of $\mathscr{L}(H)$, $H$ Hilbert space, it follows from [3], chap. I, § 4, prop. 4 and [12], 1.15.2 that the topologies $\mathscr{T}_\varphi$, $s(\mathscr{A}, \mathscr{A}_*)$ and the strong operator topology coincide on norm bounded sets of $K_\varphi$.

Assume now that $A$ is a faithful family of normal states on $\mathscr{A}$. It follows from the considerations above that $\mathscr{T}_A$ coincides on norm bounded subsets of $\mathscr{A}$ with the topology of pointwise convergence on $\bigcup_{\varphi\in A} p_\varphi H$. By [3], chap. I, § 4, sect. 6 this set is total in $H$. Therefore we can apply [13], III.4.5. and have the following lemma.

(2.2) Lemma. *Let $\mathscr{A}$ be a $W^*$-algebra in $\mathscr{L}(H)$, $H$ Hilbert space, with unit ball $\mathscr{A}_1$. If $\varphi\in\mathscr{A}_*^+$ (resp. if $A\subset\mathscr{A}_*^+$ faithful) then $\mathscr{T}_\varphi$ (resp. $\mathscr{T}_A$) coincides on $\mathscr{A}_1\cap K_\varphi$ (resp. $\mathscr{A}_1$) with the strong operator topology.*

Proof of the Theorem. *Step 1.* For $\varphi\in\Phi$ it follows from $(*)$ that $L_\varphi$ is $S$-invariant. Therefore $S$ induces an operator semigroup $S_\varphi$ on $K_\varphi$ by

$$T_\varphi(xp_\varphi) := T(xp_\varphi)p_\varphi = (Tx)p_\varphi \quad \text{for all} \quad T\in S.$$

The completion of $K_\varphi$ with respect to $\mathscr{T}_\varphi$ is a Hilbert space $H_\varphi$ on which $S_\varphi$ induces a contraction semigroup $\tilde{S}_\varphi$ (again by $(*)$). This contraction semigroup is mean ergodic by (1.3.1) with corresponding projection $P_\varphi \in \mathscr{L}(H_\varphi)$ and has the property that

$$x_\varphi^0 := P_\varphi x, \quad x \in H_\varphi,$$

is contained in the $\mathscr{T}_\varphi$-closed convex hull of $\tilde{S}_\varphi x$. Now take $x \in K_\varphi$. Then co $\tilde{S}_\varphi x$ is a norm bounded subset of $K_\varphi \subset \mathscr{A}$ whose strong operator closure is a complete subset of $K_\varphi$ ([3], chap. I, 3, sect. 1). From (2.2) it follows that $x_\varphi^0 \in K_\varphi$. Since the bounded convex subsets of $\mathscr{A}$ have the same closure for the strong operator topology, the weak operator topology and the weak* topology ([3], chap. I, § 3, th. 1), we have shown that for every $x \in K_\varphi$ there is a unique $S_\varphi$-fixed point $x_\varphi^0$ contained in the weak* closed convex hull $\overline{\text{co}}\, S_\varphi x$ of $S_\varphi x$.

*Step 2.* For $x \in \mathscr{A}$, $\varphi \in \Phi$ we define the non-empty, weak* compact set

$$Q_\varphi(x) := \{y \in \mathscr{A} \colon x_\varphi^0 = y p_\varphi \text{ and } \|y\| \leqq r\|x\|\}$$

where $r := \sup\{\|T\| \colon T \in S\}$. For $\varphi_1, \varphi_2 \in \Phi$ and $p_{\varphi_1} \leqq p_{\varphi_2}$ it follows from the construction above that $x_{\varphi_2}^0 p_{\varphi_1} = x_{\varphi_1}^0$ and therefore $Q_{\varphi_1}(x) \supset Q_{\varphi_2}(x)$. Since we may assume that $\Phi$ is convex, we obtain that $\{Q_\varphi(x) \colon \varphi \in \Phi\}$ is filtered downwards. Because of compactness there exists an element $x^0 \in \bigcap_{\varphi \in \Phi} Q_\varphi(x)$. From $(Tx^0 - x^0)p_\varphi = = T_\varphi(x^0 p_\varphi) - x^0 p_\varphi = 0$ for all $\varphi \in \Phi$ it follows that $x^0$ is an $S$-fixed point. Moreover, $x^0$ is contained in the $\mathscr{T}_\varphi$-closed, $\varphi \in \Phi$, hence in the $\mathscr{T}_\Phi$-closed convex hull of $Sx$. Again we conclude by (2.2) that this closure coincides with $\overline{\text{co}}\, Sx$ for the weak* topology on $\mathscr{A}$.

*Step 3.* Take $0 \neq x \in \mathscr{A}$ such that $Tx = x$ for all $T \in S$. Since $\Phi$ is faithful there exists $\varphi \in \Phi$ such that

$$0 \neq x p_\varphi =\colon x_\varphi = T_\varphi x_\varphi \in K_\varphi \quad \text{for all } T_\varphi \in S_\varphi.$$

Since $\tilde{S}_\varphi$ is mean ergodic, we can find a continuous linear form $\tilde{\psi}$ on the Hilbert space $H_\varphi$ such that $\tilde{\psi}(x_\varphi) \neq 0$ and $\tilde{\psi}(T_\varphi y_\varphi) = \tilde{\psi}(y_\varphi)$ for all $T \in S$, $y \in \mathscr{A}$ and $y_\varphi := := y p_\varphi \in K_\varphi$. The formula $\psi(y) := \tilde{\psi}(y_\varphi)$, $y \in \mathscr{A}$, defines an $s(\mathscr{A}, \mathscr{A}_*)$-continuous, hence a weak* continuous linear form on $\mathscr{A}$ (see [12], 1.8.10 and 1.8.12). Obviously $\psi$ is $S_*$-invariant and does not vanish on $x$. Therefore the fixed space of $S_*$ separates the fixed space of $S$ and $S$ is weak* mean ergodic by (1.2.c).

Remarks. 1. If $S$ satisfies the above assumptions, then the preadjoint semigroup $S_*$ is (norm)mean ergodic on $\mathscr{A}_*$ by (1.2).

2. The same result holds if $\Phi$ satisfies

$$(*) \qquad \varphi\big((Tx)(Tx)^*\big) \leqq \varphi(xx^*) \quad \text{for all} \quad \varphi \in \Phi,\ T \in S,\ x \in \mathscr{A}.$$

3. If $(\mathscr{A}_1, S_1)$ and $(\mathscr{A}_2, S_2)$ are dynamical systems having faithful families $\Phi_1$ and $\Phi_2$ of normal states which satisfy $(*)$ and if one defines the semigroup

$S_1 \otimes S_2$ on the $W^*$-tensor product $\mathscr{A}_1 \tilde{\otimes} \mathscr{A}_2$ in the usual way, then $\Phi_1 \otimes \Phi_2$ is a faithful family of normal states on $\mathscr{A}_1 \tilde{\otimes} \mathscr{A}_2$ satisfying $(*)$.

(2.3) Corollary. *Let $(\mathscr{A}, S)$ be a dynamical system with $\Phi$ as in* (2.1). *For $\varphi \in \Phi$, $x \in \mathscr{A}$ (and with the notation as in the proof of* (2.1)*) the following are equivalent:*

(a) *$x_\varphi$ is $\tilde{S}_\varphi$-invariant in $H_\varphi$.*

(b) *$y \mapsto \varphi(x^* y)$, $y \in \mathscr{A}$, defines an $S_*$-invariant, weak* continuous linear form on $\mathscr{A}$.*

Proof. (a)$\Rightarrow$(b): The contıaction semigroup $\tilde{S}_\varphi$ on $H_\varphi$ and its adjoint semigroup have the same fixed space containing $x_\varphi = x p_\varphi$. Therefore we obtain

$$\varphi(x^* y) = \varphi(p_\varphi x^* y p_\varphi) = \langle y_\varphi, x_\varphi \rangle_\varphi = \langle T_\varphi y_\varphi, x_\varphi \rangle_\varphi =$$
$$= \langle (Ty) p_\varphi, x_\varphi \rangle_\varphi = \varphi\big(p_\varphi x^* (Ty) p_\varphi\big) = \varphi(x^* Ty)$$

for all $T \in S$.

(b)$\Rightarrow$(a): Assume $T_\varphi(x p_\varphi) \neq x p_\varphi$ for some $T_\varphi \in \tilde{S}_\varphi$ and therefore $T_\varphi^*(x p_\varphi) \neq x p_\varphi$. Since $\mathscr{A} p_\varphi$ is dense in $H_\varphi$, there exists $y_\varphi = y p_\varphi$ such that

$$\varphi(x^* y) = \varphi(p_\varphi x^* y p_\varphi) = \langle y_\varphi, x_\varphi \rangle_\varphi \neq \langle y_\varphi, T_\varphi^* x_\varphi \rangle_\varphi =$$
$$= \langle T_\varphi y_\varphi, x_\varphi \rangle_\varphi = \varphi(x^* Ty).$$

The most important application of the above theorem will be to semıgroups of completely positive operators on $\mathscr{A}$ (see [16] for the definition). In particular we will obtain useful information on the corresponding mean ergodic projection and on the fixed space.

(2.4) Theorem. *Let $(\mathscr{A}, S)$ be a dynamical system where $S$ consists of completely positive contractions. If there exists a faithful family $\Phi$ of $S_*$-subinvariant normal states on $\mathscr{A}$, then $S$ is weak* mean ergodic. Moreover, the corresponding projection $P$ is a conditional expectation onto the weak* closed fixed point subalgebra $P\mathscr{A}$ of $\mathscr{A}$.*

Proof. Completely positive contractions $T \in S$ satisfy a Schwarz inequality

$$(Tx)^*(Tx) \leqq T(x^* x) \quad \text{for all} \quad x \in \mathscr{A} \quad \text{(see [16])}.$$

For $\varphi \in \Phi$ we get

$$\varphi\big((Tx)^*(Tx)\big) \leqq \varphi\big(T(x^* x)\big) \leqq \varphi(x^* x) \quad \text{for all} \quad x \in \mathscr{A},$$

hence $(*)$ is satisfied and $S$ is weak* mean ergodic by (2.1). Obviously, the projection $P$ is positive and therefore $P(y^*) = (Py)^*$ for all $y \in \mathscr{A}$. Consequently, it suffices to show that $P(x_0 y)=x_0 Py$ for all $x_0 \in P\mathscr{A}$, $y \in \mathscr{A}$. Take $\varphi \in \Phi$, $x_0 \in P\mathscr{A}$ and denote the weak* continuous $S_*$-invariant linear form $x \mapsto \varphi(x_0^* x)$ by $L_{x_0}\varphi$ (use (2.3)). But $L_{x_0}\varphi$ is a linear combination of positive elements in $P_*\mathscr{A}_*$. Again by (2.3) we obtain $L_{z_0}\varphi \in P_*\mathscr{A}_*$ for $z_0 := y_0 x_0$ and $y_0 \in P\mathscr{A}$. Preserving the notation of (2.3) we remark that the mean ergodic projection $P_\varphi$ is self adjoint on $H_\varphi$. Therefore we obtain by an analogous computation that $\langle x_\varphi, P_\varphi z_{0\varphi}\rangle = \langle x_\varphi, z_{0\varphi}\rangle$ for all $x \in \mathscr{A}$. Since $\Phi$ is faithful we conclude $z_0 = y_0 x_0 \in P\mathscr{A}$. Consequently $x_0 Py \in P\mathscr{A}$, and for $\varphi \in P_*\mathscr{A}_*$ we have $\varphi(P(x_0 y)) = P_*\varphi(x_0 y) = \varphi(x_0 y) = L_{x_0^*}\varphi(y) = P_*(L_{x_0^*}\varphi)(y) = L_{x_0^*}\varphi(Py) = \varphi(x_0 Py)$. Since $P_*\mathscr{A}_*$ separates $P\mathscr{A}$, the assertion follows.

The known mean ergodic theorems for $W^*$-algebras follow from the fact that *-homomorphisms on arbitrary $W^*$-algebras and positive operators on commutative $W^*$-algebras are completely positive.

(2.5) Corollary. (Kovács—Szűcs [5], [1]) *Let $S$ be a semigroup of normal *-homomorphisms on a $W^*$-algebra $\mathscr{A}$. If there exists a faithful family $\Phi$ of $S_*$-invariant normal states on $\mathscr{A}$, then $S$ is weak* mean ergodic and the corresponding projection is a conditional expectation.*

(2.6) Corollary (Nagel [8]): *Let $S$ be a semigroup of weak* continuous positive contractions on a commutative $W^*$-algebra $\mathscr{A}$. If there exists a faithful normal $S_*$-subinvariant state on $\mathscr{A}$, then $S$ is weak* mean ergodic.*

(2.7) Corollary. *If $(\mathscr{A}, S)$ satisfies all assumptions of* (2.4), *the following are equivalent:*

(a) $T1=1$ *for all* $T \in S$.

(b) *The mean ergodic projection $P$ is strictly positive.*

(c) *$\Phi$ consists of $S$-invariant states.*

Proof. (a)⇒(c): $T1=1$ and $\varphi \in \Phi$ implies $Tp_\varphi \leqq p_\varphi$ and $T_\varphi p_\varphi = p_\varphi$. From (2.3) it follows that $x \mapsto \varphi(p_\varphi x) = \varphi(x)$ is $T_*$-invariant. (c)⇒(b) is trivial and (b)⇒(a) follows since $Py=0$ for $y := 1 - T1 \geqq 0$.

## 3. Compactness and mean ergodic properties

Compactness in some form underlies many ergodic theorems. In the case of automorphism groups on $W^*$-algebras this has been studied by several authors (e.g. [15], [4], [7], [10]). We will generalize these results to bounded semigroups.

As in the previous section, let $\mathscr{A}$ denote a $W^*$-algebra with predual $\mathscr{A}_*$. On $\mathscr{L}(\mathscr{A})$ (resp. $\mathscr{L}(\mathscr{A}_*)$) we consider the weak* operator topology $\sigma(\mathscr{L}(\mathscr{A}), \mathscr{A} \otimes \mathscr{A}_*)$

(resp. the weak operator topology $\sigma(\mathscr{L}(\mathscr{A}_*), \mathscr{A}_*\otimes\mathscr{A})$) and recall that the unit ball in $\mathscr{L}(\mathscr{A})$ is weak* operator compact.

(3.1) Proposition. *For a dynamical system* $(\mathscr{A}, S)$ *the following are equivalent:*
(a) $S_*$ *(and* co $S_*$*) are relatively compact.*
(b) *The closure of* $S$ *(and of* co $S$*) contains only weak* continuous operators.*
(c) $S_*(W)$ *is relatively weakly compact for any weakly compact set* $W\subset\mathscr{A}_*$.
(d) $S$ *is equicontinuous for the Mackey topology* $\tau(\mathscr{A}, \mathscr{A}_*)$.

Proof. The implications (a)$\Leftrightarrow$(b)$\Leftarrow$(c)$\Leftrightarrow$(d) hold in any (dual) Banach space and follow from topological vector space theory (use [13], IV.3.2, IV.11.4).

(a)$\Rightarrow$(c): By Eberlein's theorem ([13], IV.11.2) it suffices to show that $\{T_*\psi_i\colon T\in S, i\in\mathbf{N}\}$ is relatively weakly compact for any weakly convergent sequence $\{\psi_i\}$ in $\mathscr{A}_*$. To that purpose we choose a sequence $\{p_n\}$ of mutually orthogonal projections in $\mathscr{A}$ and show that $\lim_{n\to\infty}\psi_i(Tp_n)=0$ uniformly for $i\in\mathbf{N}$, $T\in S$ (use the compactness criterion from [2]). Take $\varepsilon>0$ and denote the limit of $\{\psi_i\}$ by $\psi_0$ and the $r$-ball in $\mathscr{A}$ for $r:=\sup\{\|T\|\colon T\in S\}$ by $\mathscr{A}_r$. Then define

$$Q_i := \{x\in\mathscr{A}_r\colon |(\psi_j-\psi_0)(x)| \leqq \varepsilon/4 \text{ for all } j\geqq i\}.$$

Each $Q_i$ is weak* closed and $\mathscr{A}_r=\bigcup_{i=1}^{\infty} Q_i$. Since $\mathscr{A}_r$ is weak* compact, there exists $i_0\in\mathbf{N}$ such that $Q_{i_0}$ contains a weak* open set in $\mathscr{A}_r$, i.e. there exists $x_0\in\mathscr{A}_r$ and $\varphi_1, \dots, \varphi_m\in\mathscr{A}_*$ such that $\bigcap_{i=1}^{m}\{x\in\mathscr{A}_r\colon |\varphi_i(x)-\varphi_i(x_0)|<1\}$ is contained in $Q_{i_0}$. By (a) the set $\{T_*\varphi_i\colon T\in S, 1\leqq i\leqq m\}$ is relatively weakly compact, hence there exists $n_1\in\mathbf{N}$ such that $|\varphi_i(Tp_n)|<1$ for all $T\in S$, $1\leqq i\leqq m$ and $n\geqq n_1$ (use [2] again). Then $(Tp_n+x_0)\in Q_{i_0}$ and, since $x_0\in Q_{i_0}$,

$$|(\psi_j-\psi_0)(Tp_n)| \leqq \varepsilon/2 \quad \text{for} \quad T\in S,\ j\geqq i_0 \quad \text{and} \quad n\geqq n_1. \tag{1}$$

We apply now the hypothesis (a) and the compactness criterion to the set $\{T_*(\psi_j-\psi_0)\colon T\in S,\ 1\leqq j\leqq i_0-1\}$ and find $n_2\in\mathbf{N}$ such that

$$|(\psi_j-\psi_0)(Tp_n)| \leqq \varepsilon/2 \quad \text{for} \quad T\in S,\quad 1\leqq j\leqq i_0-1,\quad n\geqq n_2. \tag{2}$$

Finally, there exists $n_3\in\mathbf{N}$ such that

$$|\psi_0(Tp_n)| \leqq \varepsilon/2 \quad \text{for} \quad T\in S \quad \text{and} \quad n\geqq n_3. \tag{3}$$

Define $n_0:=\max\{n_1, n_2, n_3\}$ and conclude from (1), (2) and (3) that

$$|\psi_j(Tp_n)| \leqq \varepsilon \quad \text{for} \quad T\in S,\quad j\in\mathbf{N} \quad \text{and} \quad n\geqq n_0.$$

Remarks 1. SAITÔ [10] proved the equivalence of (a) and (c) for mean ergodic groups of *-automorphisms.

2. Instead of the weak* operator topology we could use equally well the topology of pointwise convergence on $\mathscr{A}$ where $\mathscr{A}\subset\mathscr{L}(H)$ is endowed with the weak operator topology.

Our next result shows once more the usefulness of condition $(*)$ of section 2. Not only does it imply mean ergodicity but also compactness.

(3.2) Proposition. *Let $(\mathscr{A}, S)$ be a dynamical system. If there exists a faithful family $\Phi$ of normal states on $\mathscr{A}$ satisfying*

$$(*) \qquad \varphi\big((Tx)^*(Tx)\big) \leqq \varphi(x^*x) \quad \text{for all} \quad T\in S, \quad x\in\mathscr{A}, \quad \varphi\in\Phi,$$

*then $S$ satisfies* (a)–(d) *of* (3.1).

Proof. Take $\psi\in\mathscr{A}_*$ and assume that $S_*\psi$ is not relatively weakly compact. By [2] there exists $\varepsilon>0$, a sequence $\{T_n\}\subset S$ and a sequence $\{p_n\}$ of mutually orthogonal projections in $\mathscr{A}$ such that $|\psi(T_n p_n)|\geqq\varepsilon$ for all $n\in\mathbf{N}$. But for $\varphi\in\Phi$ we have $\lim\limits_{n\to\infty}\varphi(p_n)=0$ and $\lim\limits_{n\to\infty}\varphi\big((T_n p_n)^*(T_n p_n)\big)=0$ (apply $(*)$), i.e. $T_n p_n$ converges to 0 for the topology $\mathscr{T}_\Phi$. Since this topology is stronger than the weak* topology on norm bounded sets of $\mathscr{A}$ (use (2.2)), we obtain the contradiction that $\psi(T_n p_n)$ converges to 0.

Remark. Examples show that for arbitrary dynamical systems weak compactness does not imply weak* mean ergodicity [10] nor does weak* mean ergodicity imply compactness.

(3.3) Corollary. *Let $(\mathscr{A}, S)$ be a dynamical system where $S$ is a group of $*$-automorphisms on $\mathscr{A}$. The following properties are equivalent:*

(a) *There exists a faithful family of $S_*$-invariant normal states.*

(b) *$S$ is weak* mean ergodic.*

(c) *$S_*$ is relatively weak operator compact in $\mathscr{L}(\mathscr{A}_*)$.*

Proof. The implications (a)⇒(b) and (a)⇒(c) follow from (2.1) and (3.2), while a proof of (c)⇒(a) can be found in [15]. Assume now that $S$ is weak* mean ergodic with projection $P$. To prove (a) it suffices to show that $P$ is strictly positive: Assume $P(x^*x)=0$ for some non-zero $x\in\mathscr{A}$. Since $P$ is normal we can find a maximal projection $0\not\equiv p\in\mathscr{A}$ such that $P(p)=0$. This and the assumption $T1=1$ imply $Tp\leqq p$ for all $T\in S$. But $S$ is a group and therefore $Tp=p$ for all $T\in S$, which is a contradiction.

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