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Type sets and nilpotent multiplications

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Introduction. The nilstufe, $\nu(G)$, of a torsion-free abelian group was defined by SZELE [7] to be the largest positive integer n such that there is an associative ring $(G, \circ)$ on G having a non-zero product of n elements. If no such integer exists then $\nu(G)$ is set equal to ∞. FEIGELSTOCK [2] defines the strong nilstufe, $N(G)$, in a similar manner but allows non-associative ring structures on G. In § 2 we define the *solvable degree* $\varrho(G)$ in an analogous way.

Several authors [1, 4, 5, 6, 7, 9, 10] have studied related problems of nilpotency in torsion-free rings. They have mainly restricted their attention to associative ring structures and have often demanded that the group G be completely decomposable. In [8] WEBB showed that if G is torsion-free with finite rank r then either $\nu(G)=\infty$ or $\nu(G)\leqq r$, and either $N(G)=\infty$ or $N(G)\leqq 2^{(r-1)}$.

In this note we obtain improved bounds on both $\nu(G)$ and $N(G)$ under certain conditions on the type set, $\mathbf{T}(G)$, of G. Here the *type set* of G means the partially ordered set of types $\mathbf{t}(g)$ of non-zero elements g in G. Our new bounds are expressed in terms of the *length* $l(G)$ of G by which we mean the length of the longest chain in $\mathbf{T}(G)$. If no longest chain exists we put $l(G)=\infty$, and observe the usual conventions about the ordering on $\mathbf{Z}\cup\{\infty\}$. We observe that if G has finite rank r then $l\leqq r$ (cf. FUCHS [3], page 112, Ex. 10).

We require the following notions. If α, β are types we say that α *absorbs* β if $\alpha\beta=\alpha$. If in particular α is self-absorbing then we say that α is *idempotent;* (many authors have used the term non-nil for this last notion, we prefer the word idempotent, for the existence of idempotent types in the type set of the additive group of a ring is closely related to the existence of idempotent elements in the ring itself).

Throughout the remainder of this note G denotes a torsion-free group of rank r and length l (both of which may be ∞). With these conventions we have:

Proposition 1.1. *If* $\mathbf{T}(G)$ *contains no absorbing elements, then*

$$\text{i) } N(G)\leqq 2^{l-1},\quad \text{ii) } \nu(G)\leqq l,\quad \text{iii) } \varrho(G)\leqq l.$$

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Proposition 1.2. *If* $\mathbf{T}(G)$ *contains no idempotent elements, then*

$$\nu(G) \leqq \min\{2^l-1, r\}, \quad \varrho(G) \leqq l.$$

Proposition 1.3. *If* $\mathbf{T}(G)$ *contains no absorbing elements and* G *has length* 2, *then every ring on* G *is associative, and nilpotent of degree at most two.*

Basic ideas.

Lemma 2.1 *Suppose that* $(G, *)$ *is a non-associative ring on* G, *and that* $g_i \in G$ $(i=1, 2)$ *are such that* $g_1 * g_2 \neq 0$.

(i) *If neither* $\mathbf{t}(g_i)$ *absorbs the other then*

$$\mathbf{t}(g_1 * g_2) > \mathbf{t}(g_i) \quad (i = 1, 2).$$

(ii) *If neither* $\mathbf{t}(g_i)$ *is idempotent then either*

$$\mathbf{t}(g_1 * g_2) > \mathbf{t}(g_1) \quad \textit{or} \quad \mathbf{t}(g_1 * g_2) > \mathbf{t}(g_2).$$

Proof. Clearly $\mathbf{t}(g_1 * g_2) \geqq \mathbf{t}(g_1)\mathbf{t}(g_2) \geqq \mathbf{t}(g_i)$ $(i=1, 2)$. If $\mathbf{t}(g_1 * g_2) = \mathbf{t}(g_1)$, then

$$\text{(A)} \qquad \mathbf{t}(g_1) = \mathbf{t}(g_1)\mathbf{t}(g_2) \geqq \mathbf{t}(g_2)$$

and $\mathbf{t}(g_1)$ absorbs $\mathbf{t}(g_2)$. This proves (i).

If $\mathbf{t}(g_1)$ is not idempotent then (A) implies that $\mathbf{t}(g_1) > \mathbf{t}(g_2)$ and (ii) follows.

For each positive integer k, let $V_k = \{x \in G |$ there is a chain $\mathbf{t}(x) > \tau_2 > \ldots > \tau_k$ of types in $\mathbf{T}(G)\}$ and let G_k be the subgroup of G generated by V_k. We clearly have a descending chain $G = G_1 \supset G_2 \supset G_3 \supset \ldots$ of subgroups of G.

Corollary 2.2. *Under the same hypothesis on* G *as in Lemma* 2.1 *we have:*

(i) *If* $\mathbf{T}(G)$ *has no absorbing elements then*

$$G * G_i \subset G_{i+1} \quad \textit{and} \quad G_i * G \subset G_{i+1} \quad \textit{for all positive integers } i.$$

(ii) *If* $\mathbf{T}(G)$ *has no idempotents then*

$$G_i * G_i \subset G_{i+1} \quad \textit{for all positive integers } i.$$

Remark. In both cases $(G_i, *)$ is a subring of $(G, *)$ and in case (i) $(G_i, *)$ is an ideal in $(G, *)$.

Let R be a non-associative ring. For each positive integer k we may define four 'powers' of R as follows.

(i) $R^{(k)}$ is the subring of R generated by all products of k elements in R, however the products are associated.

(ii) $R^{[1]} = R, \quad R^{[k]} = R^{[k-1]}R^{[k-1]}$ for all $k > 1$.

(iii) $\vec{R}^1 = R, \quad \vec{R}^k = \vec{R}^{k-1}R$ for all $k > 1$.

(iv) $\overleftarrow{R}^1 = R, \quad \overleftarrow{R}^k = R\overleftarrow{R}^{k-1}$ for all $k > 1$.

We observe that each of these 'k-th powers' is contained in $R^{(k)}$. A simple induction shows that

(2.3) $$R^{[k]} \subset R^{(2^{k-1})} \quad \text{for all integers } k>1.$$

Recall that R is *nilpotent* if there is an index k such that $R^{(k+1)}=0$ and R is *solvable* if $R^{[k+1]}=0$. If G is a group we say that the *solvable degree,* $\varrho(G)$, of G is k if $[G, *]^{[k+1]}=0$ for all multiplications $*$ on G and there is a multiplication $\circ$ with $[G, \circ]^{[k]} \neq 0$.

The following inclusions are an easy consequence of Corollary 2.2.

Proposition 2.4.

(i) *If* $\mathbf{T}(G)$ *has no absorbing types then*

$$(\overleftarrow{G, *})^n \subseteqq G_n \quad \textit{and} \quad (\overrightarrow{G, *})^n \subseteqq G_n \quad \textit{for all positive integers } n.$$

(ii) *If* $\mathbf{T}(G)$ *has no idempotents then*

$$(G, *)^{[n]} \subseteqq G_n \quad \textit{for all positive integers } n.$$

In order to obtain information about $(G, *)^{(k)}$ we need a further notion. Denote by $F(R)$ the subring of the (associative) ring $E(R)$ of endomorphisms of the additive group of R, generated by the left and right multiplications L_a, R_a, $a \in R$ where

$$xL_a = ax; \quad xR_a = xa \quad \text{for all} \quad x \in R.$$

Lemma 2.5. *Let R be a torsion-free ring. Let n and k be positive integers satisfying $k > 2^{n-1}$. Then*

$$R^{(k)} \subseteqq R[F(R)]^n.$$

Proof. We proceed by induction on n the result being clear when $n=1$. Suppose that the result holds for $n=m \geqq 1$ and that $k>2^m$. Let x be the product of k elements in R. Then $x=uv$ where at least one of u or v, u say, is the product of at least 2^{m-1} elements of R. Thus by hypothesis u belongs to $R[F(R)]^m$ and $uv \in R[(F(R)]^{m+1}$.

Corollary 2.6. *Let G be a torsion-free group whose type set contains no absorbing elements. Let $(G, *)$ be a (non-associative) ring on G. Let n and k be positive integers satisfying $k > 2^{n-1}$. Then*

$$(G, *)^{(k)} \subset G_{n+1}.$$

Proof. In fact we show that $G[F(G, *)]^{(n)} \subset G_n$, for all positive integers n. We may assume without loss of generality that $G[F(G, *)]^{(n)}$ is non null. In particular there exists non-zero monomials in $G[F(G, *)]^{(n)}$. Recalling that $F(G, *)$ is associative we see that such a monomial may be written in the form

$$\xi = (\dots((gX_1)X_2)\dots X_n) \neq 0$$

where $g \in G$ and, for each i, X_i denotes $*$ multiplication on the left or right by an element of G. It follows from Lemma 2.1 (i) that

$$\mathbf{t}(g) < \mathbf{t}(gX_1) < \mathbf{t}((gX_1)X_2) < \dots < \mathbf{t}(\xi)$$

is a strictly ascending chain in $\mathbf{T}(G)$ of length $n+1$ and so $\xi \subset V_{n+1}$. However, the monomials generate $G[F(G, *)]^{(n)}$ and the corollary follows.

Proof of Propositions 1.1, 1.2 and 1.3. Suppose that G has finite length l. Then, by definition, $G_{l+1}=0$. If $\mathbf{T}(G)$ has no absorbing types Proposition 2.4 gives

(a) $$(\overrightarrow{G, *})^{l+1} = (\overleftarrow{G, *})^{l+1} = 0$$

whilst Corollary 2.6 yields

(b) $$(G, *)^{(2^{l-1}+1)} = 0.$$

Since $*$ is an arbitrary multiplication on G we conclude from (a) that $\nu(G) \leqq l$ and from (b) that $N(G) \leqq 2^{l-1}$. Furthermore putting $k=l+1$ in equation 2.3 gives

$$(G, *)^{[l+1]} \subset (G, *)^{(2^l)} \subset (G, *)^{(2^{l-1}+1)} = 0$$

and we deduce that $\varrho(G) \leqq l$. This proves Proposition 1.1 (if l is infinite the result is trivial!). Substitution of $l=2$ in (b) gives $(G, *)^{(3)}=0$, and we deduce that in this case $(G, *)$ is always associative thus proving Proposition 1.3.

If all we know about $\mathbf{T}(G)$ is that it contains no idempotents then Proposition 2.4 (ii) gives $(G, *)^{[l+1]}=0$ and we have $\varrho(G) \leqq l$. If $*$ is an associative multiplication then

$$0 = (G, *)^{[l+1]} = (G, *)^{2^l}$$

whence $\nu(G) \leqq 2^l - 1$. Webb [8] has shown that $\nu(G) \leqq r$ and we have proved Proposition 1.2.

Finally we construct a group G which has

(i) finite type set, (ii) no idempotent types, (iii) $N(G)=\infty$.

Let $R_1 \subset R_2$ be subgroups of the rationals containing 1. Suppose that neither R_1 nor R_2 is a subring of Q, but that $R_1 R_2 = R_2$, the multiplication being the usual one on Q. Put $G = Rx \oplus Ry$, then G satisfies conditions (i) and (ii) above. We put two multiplications $*$ and $\circ$ on G as follows

$*$	x	y
x	0	y
y	y	0

$\circ$	x	y
x	y	0
y	0	0

If n is a positive integer then

$$x*(\ldots*(x*(x*y))\ldots) = y \neq 0,$$

x appearing n times. It follows that $N(G)=\infty$, and reference to Proposition 1.2 shows that $(G, *)$ is nonassociative.

It is easily checked that

$$0 \neq (G, *)^{[2]} \subset R_2 y, \quad (G, *)^{[3]} = 0$$

so that $\varrho(G)=2=l(G)$. Lastly we see that $(G, \circ)$ is associative, indeed $(G, \circ)^{(3)}=0$, so $\nu(G)=2$.

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