Acta Sci. Math., 41 (1979), 227—251

On structural properties of functions arising from strong approximation of Fourier series

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Introduction

Let $f(x)$ be an integrable and 2π-periodic function, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote by $s_n(x) = s_n(f; x)$ and $\omega(f; \delta)$ the n-th partial sum of (1) and the modulus of continuity of f, respectively; $\|\cdot\|$ always stays for the supremum norm.

FREUD [1] proved that

$$\left\| \sum_{k=1}^{\infty} |s_k - f|^p \right\| < \infty \quad \text{for some} \quad p > 1 \quad \text{implies} \quad f \in \mathrm{Lip}\, \frac{1}{p}.$$

An analogous problem with $p = 1$ was investigated by LEINDLER and NIKIŠIN [6], and this result was generalized by LEINDLER [4] as follows: If r is a nonnegative integer and

$$\left\| \sum_{k=1}^{\infty} k^r |s_k - f| \right\| < \infty,$$

then

$$|f^{(r)}(x+h) - f^{(r)}(x)| \leqq K \cdot h \cdot \log \frac{1}{h} \quad (x \in [0, 2\pi])$$

for all x, and this estimation is best possible.

From this result it follows that

$$\left\| \sum_{k=1}^{\infty} |s_k - f| \right\| < \infty \quad \text{does not imply} \quad f \in \mathrm{Lip}\, 1.$$

Received January 25, 1978.

LEINDLER raised the question whether the condition

$$\left\|\sum_{k=1}^{\infty}|s_k-f|^p\right\|<\infty \quad \textit{with some } p \quad (0<p<1) \quad \textit{implies} \quad f\in \mathrm{Lip}\, 1.$$

The answer was given in the affirmative by OSKOLKOV [7] and SZABADOS [8]. They also proved

Theorem A. *For an arbitrary modulus of continuity Ω,*

$$\left\|\sum_{k=1}^{\infty}\Omega(|s_k-f|)\right\|<\infty \tag{2}$$

and

$$\int_0^1 \frac{dx}{\Omega(x)}<\infty \tag{3}$$

imply $f\in \mathrm{Lip}\, 1$.

Under a certain restriction on Ω they also proved the necessity of condition (3). In [10] we proved the necessity of (3) without any further assumption, and more generally, we proved Theorem B (below).

In order to simplify our assertions, $\Omega(x)$ will always denote an increasing convex or concave function on $[0, \infty)$, with the properties

$$\Omega(x)>0\,(x>0) \quad \lim_{x\to 0+0}\Omega(x)=\Omega(0)=0, \tag{4}$$

and we suppose that the inverse of $\Omega(x)$ (denoted by $\bar{\Omega}(x)$) exists in the interval $[0; 1]$. With these notations we proved

Theorem B. *If f satisfies* (2), *then*

$$\omega(f;\delta)=O\left(\delta\int_\delta^1\frac{\bar{\Omega}(x)}{x^2}\,dx\right), \tag{5}$$

but no estimate better than this can be given. Moreover, if Ω is concave, then we can replace

$$\int_\delta^1 \frac{\bar{\Omega}(x)}{x^2}\,dx \quad \textit{by} \quad \int_{\Omega(\delta)}^1 \frac{dx}{\Omega(x)}.$$

The following theorem answers the analogous problem for the conjugate function.

Theorem 1. (i) *If Ω is concave, then* (2) *implies* $\tilde{f}\in \mathrm{Lip}\, 1$. (ii) *Let Ω be convex. From* (2) *the continuity of $\tilde{f}$ follows if and only if*

$$\int_0^1 \frac{\bar{\Omega}(x)}{x}\,dx<\infty. \tag{6}$$

If (6) *is fulfilled, then* (2) *implies that*

$$\omega(\tilde{f};\delta) = O\left(\int_0^\delta \frac{\bar{\Omega}(x)}{x}\,dx\right). \tag{7}$$

Furthermore, there exists a function f_0 *for which* (2) *is true, but*

$$\omega(\tilde{f}_0;\delta) \geqq c\int_0^\delta \frac{\bar{\Omega}(x)}{x}\,dx \quad (c>0). \tag{8}$$

We note that part (i) is a known result of LEINDLER [4].

Recently KROTOV and LEINDLER [2] investigated the problem to give a necessary and sufficient condition for a monotonic sequence $\{\lambda_k\}$ such that

$$\left\|\sum_{k=0}^\infty \lambda_k |s_k - f|^p\right\| < \infty \quad \text{with some } p \quad (0<p<\infty) \tag{9}$$

should imply $\omega(f;\delta) = O(\omega(\delta))$, where $\omega(\delta)$ is a fixed modulus of continuity. They proved

Theorem C. *Let* $\{\lambda_k\}$ *be a positive nondecreasing sequence,* $\omega(\delta)$ *be a modulus of continuity and* $0<p<\infty$. *Then* (9) *implies* $\omega(f;\delta)=O(\omega(\delta))$ *if and only if*

$$\sum_{k=1}^n (k\cdot\lambda_k)^{-\frac{1}{p}} = O\left(n\cdot\omega\left(\frac{1}{n}\right)\right). \tag{10}$$

As a common generalization of Theorem B and C we shall prove

Theorem 2. *Let* Ω *be a convex or concave function with properties* (4), *and let* $\{\lambda_k\}_0^\infty$, $\{\mu_k\}_0^\infty$ *be positive nondecreasing sequences. If*

$$\left\|\sum_{k=0}^\infty \lambda_k \Omega(\mu_k |s_k - f|)\right\| < \infty \tag{11}$$

then

$$\omega\left(f;\frac{1}{n}\right) = O\left(\frac{1}{n}\sum_{k=1}^n \frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\cdot\lambda_k}\right)\right). \tag{12}$$

Furthermore, there exists a function f_0 *satisfying* (11), *for which*

$$\omega\left(f_0;\frac{1}{n}\right) \geqq c\cdot\frac{1}{n}\sum_{k=1}^n \frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\cdot\lambda_k}\right) \quad (c>0). \tag{13}$$

Corollary 1. *Condition* (11) *implies* $f\in\mathrm{Lip}\,1$ *if and only if*

$$\sum_{k=1}^\infty \frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\cdot\lambda_k}\right) < \infty.$$

Corollary 2. *Let* $\gamma \geqq 0$. *Then*

$$\left\| \sum_{k=0}^{\infty} k^{\gamma} \Omega(|s_k - f|) \right\| < \infty$$

implies

$$\omega(f; \delta) = O\left(\delta \int_{\delta}^{1} \frac{\bar{\Omega}(x^{1+\gamma})}{x^2}\, dx\right).$$

It is easy to see that (12) reduces to (5) and (10) if $\lambda_k = \mu_k = 1$ and $\mu_k = 1$, $\Omega(x) = x^p$, respectively. Thus Theorem B and C, and hence all of the above results are consequences of Theorem 2.

We remark that for $\Omega(x) = x^p$ Leindler [5] proved some general statements of similar type.

It is a very interesting problem to find the analogue of Theorem 2 for the conjugate function.

We shall now generalize Theorem B in another direction. Let β be a nonnegative number and consider the condition

$$\left\| \sum_{k=0}^{\infty} k^{\beta} \Omega(|s_k - f|) \right\| < \infty \tag{14}$$

instead of (2). We ask for the differentiability properties of f and $\tilde{f}$. We prove

Theorem 3. *Let* Ω *be a concave function with properties* (4), *and let* $\beta \geqq 0$, $r = [\beta]$ *). (14) *implies that* $f, \tilde{f}$ *are* r *times differentiable, and if* r *is odd then*

$$f^{(r)} \in \mathrm{Lip}\, 1 \tag{15}$$

$$\omega(\tilde{f}^{(r)}; \delta) = O\left(\delta \int_{\delta}^{1} \frac{\bar{\Omega}(x^{1+\beta-r})}{x^2}\, dx\right), \tag{16}$$

while if r *is even then the role of* f *and* $\tilde{f}$ *in* (15) *and* (16) *must be inverted. Furthermore, there are functions* f_β *satisfying* (14) *with*

$$\omega(\tilde{f}_\beta^{(r)}; \delta) \quad \text{or} \quad \omega(f_\beta^{(r)}; \delta) \geqq c\delta \int_{\delta}^{1} \frac{\bar{\Omega}(x^{1+\beta-r})}{x^2}\, dx \quad (c > 0), \tag{17}$$

according as r *is odd or even.*

The example $\Omega(x) = e^{-\frac{1}{x}}$, $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ shows that for certain convex Ω condition (14) — with arbitrary large β — does not guarantee the differentiability

*) $[\beta]$ denotes the integral part of β.

of f. On this account for convex Ω we shall investigate the condition

(18) $$\left\|\sum_{k=0}^{\infty}\Omega(k^{\beta}|s_k-f|)\right\|<\infty$$

rather than (14).

Before we state our result concerning (18), we need the following

Definition. If ω is a modulus of continuity for which $\sum_{k=0}^{\infty}\omega\left(\frac{1}{2^k}\right)<\infty$, or equivalently $\int_0^1\frac{\omega(x)}{x}dx<\infty$, let

$$\omega^*(\delta)=\sup_{\{\varepsilon_k\}}\sum_{k=0}^{\infty}\omega\left(\frac{\varepsilon_k\delta}{2^k}\right),$$

where the supremum is taken over the sequences $\{\varepsilon_k\}$ which satisfy the conditions:

$$\varepsilon_k\geqq 0\quad(k=0,1,\ldots),\quad \sum_{k=0}^{\infty}\varepsilon_k\leqq 1.$$

It is easy to verify that $\omega^*(\delta)$ is again a modulus of continuity, and that

$$\omega(\delta)\leqq\omega^*(\delta)\leqq\int_0^{\delta}\frac{\omega(x)}{x}dx.$$

With these notations we prove

Theorem 4.*) *Let Ω be convex with properties* (4), $\beta\geqq 0$, *and* $[\beta]=r$.

(i) *If* $\beta\neq[\beta]$ *then* (18) *implies*

(19) $$\omega(f^{(r)};\delta)=O(\bar{\Omega}(\delta)).$$

(ii) *Let* $\beta=[\beta]>0$. *From* (18) *it follows that*

(20) $$\omega(f^{(r-1)};\delta)=O\left(\delta\int_{\delta}^{1}\frac{\bar{\Omega}(x)}{x}dx\right)$$

and this estimation cannot be improved. Thus if (18) *implies the existence of* $f^{(r)}$ *then*

(21) $$\int_0^1\frac{\bar{\Omega}(x)}{x}dx<\infty.$$

In each of the above statements we can put $\tilde{f}$ *in place of* f.

*) We mention, that Krotov proved for a subclass of convex functions much more general results. His proofs are totally different from ours.

(iii) *Let us suppose that* (21) *is satisfied and* $r>0$. *Then* (18) *implies*

$$\omega(\tilde{f}^{(r)};\delta) = O\left(\int_0^\delta \frac{\bar{\Omega}(x)}{x}\,dx\right), \tag{22}$$

$$\omega(f^{(r)};\delta) = O\left(\bar{\Omega}^*(\delta)+\delta\int_\delta^1 \frac{\bar{\Omega}(x)}{x^2}\,dx\right), \tag{23}$$

if r is even, and the roles of $\tilde{f}$ and f must be interverted in the odd case. Furthermore there are functions f_r satisfying (18), *for which*

$$\omega(\tilde{f}_r^{(r)};\delta) \quad or \quad \omega(f_r^{(r)};\delta) \geqq c\int_0^\delta \frac{\bar{\Omega}(x)}{x}\,dx \quad (c>0) \tag{24}$$

according as r is even or odd.

Remark. *Estimation* (23) *is best possible also in the following sense: If*

$$\bar{\Omega}^*(\delta)+\delta\int_\delta^1 \frac{\bar{\Omega}(x)}{x^2}\,dx \neq O(\omega(\delta)), \tag{25}$$

where $\omega(\delta)$ is an arbitrary modulus of continuity, then there is an f satisfying (18), *but*

$$\omega(f^{(r)};\delta) \quad or \quad \omega(\tilde{f}^{(r)};\delta) \neq O(\omega(\delta)) \tag{26}$$

according as r is even or not.

We mention that from the proof of (i) the stronger estimation

$$\omega(f^{(r)};\delta) = O\left(\delta\int_\delta^1 \frac{\bar{\Omega}(x)}{x^{2-\beta+r}}\,dx\right)$$

also follows and with the aid of the function $f_0(x)=\sum_{n=1}^{\infty}\frac{1}{8n^{1+\beta}}\bar{\Omega}\left(\frac{1}{n}\right)\sin nx$ one can prove that this is the best possible if r is even, but we do not know what is the best estimation if r is odd.

I am grateful to Professor L. Leindler, who called my attention to these problems, and whose permanent interest and advises helped me very much in my work.

§ 1. Lemmas

Lemma 1 ([10], Lemma 2). *Let $\{\varrho_n\}$ be a decreasing sequence of positive numbers and let*

$$\varrho(x) = \sum_{n=1}^{\infty} \varrho_n \frac{1}{n}\sin nx.$$

Then

$$\varrho\left(\frac{\pi}{m}\right) \geqq \frac{1}{2}\frac{1}{m}\sum_{n=1}^{m}\varrho_n \quad (m=2,3,\ldots).$$

Lemma 2. *Let* $\omega(x)$ *be a modulus of continuity,* $\beta \geqq 0$, *and suppose that* $E_n(f) = O\left(\frac{1}{n^\beta}\omega\left(\frac{1}{n}\right)\right)$. *The following statements are true:*

(i) *if* $\beta > 0$ *then* $E_n(\tilde{f}) = O\left(\frac{1}{n^\beta}\omega\left(\frac{1}{n}\right)\right)$,

(ii) *if* $\beta = 0$, *and* $\int_0^1 \frac{\omega(x)}{x}dx < \infty$ *then* $E_n(\tilde{f}) = O\left(\int_0^{1/n} \frac{\omega(x)}{x}dx\right)$,

(iii) *if* $\beta > [\beta] = r$, *then* $E_n(f^{(r)}) = O\left(\frac{1}{n^{\beta-r}}\omega\left(\frac{1}{n}\right)\right)$,

(iv) *if* $\beta = [\beta] > 0$, *then* $E_n(f^{(\beta-1)}) = O\left(\frac{1}{n}\omega\left(\frac{1}{n}\right)\right)$,

(v) *if* $\beta = [\beta]$, *and* $\int_0^1 \frac{\omega(x)}{x}dx < \infty$ *then* $E_n(f^{(\beta)}) = O\left(\int_0^{1/n} \frac{\omega(x)}{x}dx\right)$.

These statements can be easily proved using the estimations below (see [9], pages 321 and 304):

$$E_n(\tilde{f}) \leqq c\left(E_n(f) + \sum_{\nu=n+1}^{\infty} \frac{1}{\nu} E_\nu(f)\right), \quad E_n(f^{(r)}) \leqq c_r\left(n^r E_n(f) + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)\right).$$

To prove (ii) and (v) use the inequality

$$\sum_{\nu=n}^{\infty} \frac{1}{\nu}\omega\left(\frac{1}{\nu}\right) \leqq \int_0^{1/n} \frac{\omega(x)}{x}dx.$$

We omit the details.

Lemma 3. *If* Ω *is concave, and* $\{\lambda_k\}_0^\infty$, $\{\mu_k\}_0^\infty$ *are nondecreasing positive sequences then*

$$\sum_{k=0}^{\infty} \lambda_k \Omega(\mu_k |s_k(x) - f(x)|) \leqq K \tag{1.1}$$

implies that

$$E_{4n} = O\left(\log n \left(n^2 \lambda_n^2 \mu_n \Omega\left(\frac{\log n}{n\lambda_n}\right)\right)^{-1}\right). \tag{1.2}$$

Proof. Using the known Lebesgue estimation

$$|s_n(x) - f(x)| \leqq 3E_n(f)\log n$$

and the inequality

$$\frac{\Omega(ay_1)}{y_1} = a\frac{\Omega(ay_1)}{ay_1} \geqq a\frac{\Omega(ay_2)}{ay_2} = \frac{\Omega(ay_2)}{y_2} \quad (a > 0;\ 0 < y_1 < y_2)$$

coming from the concavity of Ω, we get from (1.1)

$$K \geqq \left\| \sum_{k=n+1}^{2n} \lambda_k \Omega(\mu_k |s_k - f|) \right\| = \left\| \sum_{k=n+1}^{2n} \lambda_k |s_k - f| \frac{\Omega(\mu_k |s_k - f|)}{|s_k - f|} \right\| \geqq$$

$$\geqq \frac{\Omega(\mu_n E_n \log n)}{3E_n \log 2n} \lambda_n n \left\| \frac{\sum_{k=n+1}^{2n} |s_k - f|}{n} \right\| \geqq \frac{\Omega(\mu_n E_n \log n)}{6E_n \log n} \lambda_n n E_{2n}$$

i.e.

(1.3) $$E_{4n} = O\big(E_{2n} \log n (n\lambda_n \Omega(\mu_{2n} E_{2n} \log n))^{-1}.$$

Now it follows from (1.1) that

$$\sum_{k=0}^{\infty} \lambda_k \mu_k |s_k(x) - f(x)| \leqq K',$$

and from this that $E_{2n} = O\big((n\lambda_n \mu_n)^{-1}\big)$. If we write this estimation in (1.3) we obtain (1.2)

Lemma 4. *If $\Omega(x)$ is convex, $\{\lambda_k\}$, $\{\mu_k\}$ are nondecreasing positive sequences, and*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{8n\mu_n} \bar{\Omega}\left(\frac{1}{n\lambda_n}\right) \sin nx$$

then

$$\left\| \sum_{k=0}^{\infty} \lambda_k \Omega(\mu_k |s_k - f|) \right\| < \infty.$$

Proof. We introduce the notation

$$A_n(x) = \frac{1}{8n\mu_n} \bar{\Omega}\left(\frac{1}{n\lambda_n}\right) \sin nx.$$

Since $f(x)$ is odd, it is enough to consider the case $x > 0$. Let $\frac{\pi}{N} < x \leqq \frac{\pi}{N-1}$, where N is an integer. With these notations we have

(1.4) $$\sum_{k=0}^{\infty} \lambda_k \Omega(\mu_k |s_k(x) - f(x)|) = \left(\sum_{k=0}^{N-1} + \sum_{k=N}^{\infty} \right) \lambda_k \Omega(\mu_k |s_k(x) - f(x)|) = B_1(x) + B_2(x).$$

Using the well-known estimation

$$\left| \sum_{l=p}^{\infty} a_l \sin lx \right| \leqq \frac{4}{x} a_p \quad (a_p \geqq a_{p+1} \geqq \ldots),$$

we get

(1.5)

$$B_2(x)=\sum_{k=N}^{\infty}\lambda_k\Omega\left(\mu_k\left|\sum_{n=k+1}^{\infty}A_n(x)\right|\right)\leqq\sum_{k=N}^{\infty}\lambda_k\Omega\left(\mu_k\frac{4}{x}\frac{1}{8(k+1)\mu_{k+1}}\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\leqq$$

$$\leqq\sum_{k=N}^{\infty}\lambda_k\Omega\left(\frac{1}{Nx}\frac{N}{k+1}\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\leqq\sum_{k=N}^{\infty}\lambda_k\Omega\left(\frac{N}{k+1}\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\leqq$$

$$\leqq\sum_{k=N}^{\infty}\lambda_k\frac{N}{k+1}\Omega\left(\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\leqq\sum_{k=N}^{\infty}\frac{N}{(k+1)^2}\leqq 1.$$

From the convexity of Ω it follows that

$$B_1(x)\leqq\sum_{k=0}^{N-1}\lambda_k\Omega\left(\mu_k\left|\sum_{n=k+1}^{N-1}A_n(x)\right|+\mu_k\left|\sum_{n=N}^{\infty}A_n(x)\right|\right)\leqq \tag{1.6}$$

$$\leqq\sum_{k=0}^{N-1}\frac{1}{2}\lambda_k\Omega\left(2\mu_k\left|\sum_{n=k+1}^{N-1}A_n(x)\right|\right)+\sum_{k=0}^{N-1}\frac{1}{2}\lambda_k\Omega\left(2\mu_k\left|\sum_{n=N}^{\infty}A_n(x)\right|\right)=B_{11}(x)+B_{12}(x).$$

Similarly to (1.5) we get

$$B_{12}(x)\leqq\sum_{k=0}^{N-1}\frac{1}{2}\lambda_k\Omega\left(2\mu_k\frac{4}{xN}\frac{1}{8\mu_N}\bar{\Omega}\left(\frac{1}{N\lambda_N}\right)\right)\leqq\frac{1}{2}\sum_{k=0}^{N-1}\lambda_N\Omega\left(\bar{\Omega}\left(\frac{1}{N\lambda_N}\right)\right)=\frac{1}{2}. \tag{1.7}$$

Finally, using the inequality $\sin x\leqq x$ $(x\geqq 0)$, we obtain

$$2B_{11}(x)\leqq\sum_{k=0}^{N-1}\lambda_k\Omega\left(2\mu_k\sum_{n=k+1}^{N-1}\frac{1}{8n\mu_n}\bar{\Omega}\left(\frac{1}{n\lambda_n}\right)nx\right)\leqq\sum_{k=0}^{N-2}\lambda_k\Omega\left(\frac{\sum_{n=k+1}^{N-1}\bar{\Omega}\left(\frac{1}{n\lambda_n}\right)}{N-1}\right)\leqq$$

$$\leqq\sum_{k=0}^{N-2}\lambda_k\frac{\sum_{n=k+1}^{N-1}\Omega\left(\bar{\Omega}\left(\frac{1}{n\lambda_n}\right)\right)}{N-1}\leqq\frac{1}{N-1}\sum_{k=0}^{N-2}\sum_{n=k+1}^{N-1}\frac{1}{n}=\frac{1}{N-1}\sum_{n=1}^{N-1}n\frac{1}{n}=1;$$

and this — together with (1.4)—(1.7) — verifies our Lemma.

Lemma 5. *If $\Omega(x)$ is concave, $\{\lambda_k\}$, $\{\mu_k\}$ are positive nondecreasing sequences and*

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{\mu_{n^2}}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)\sin nx,$$

then

$$\left\|\sum_{k=0}^{\infty}\lambda_k\Omega(\mu_k|s_k-f|)\right\|<\infty.$$

Proof. Let $A_n(x)=\frac{1}{\mu_{n^2}}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)\sin nx$ and $\frac{\pi}{N}<x\leqq\frac{\pi}{N-1}$. From the concavity of Ω we obtain

$$(1.8)\quad \sum_{k=0}^{\infty}\lambda_k\Omega(\mu_k|s_k(x)-f(x)|)=\left(\sum_{k=0}^{N-1}+\sum_{k=N}^{\infty}\right)\lambda_k\Omega(\mu_k|s_k(x)-f(x)|)=B_1(x)+B_2(x).$$

(1.9)

$$B_2(x)=\sum_{k=N}^{\infty}\lambda_k\Omega\left(\mu_k\left|\sum_{n=k+1}^{\infty}A_n(x)\right|\right)\leqq\sum_{k=N}^{\infty}\lambda_k\Omega\left(\mu_k\frac{4}{x}\frac{1}{\mu_{(k+1)^2}}\bar{\Omega}\left(\frac{1}{(k+1)^2\lambda_{(k+1)^2}}\right)\right)\leqq$$

$$\leqq\sum_{k=N}^{\infty}\lambda_k\Omega\left(\frac{4N}{\pi}\bar{\Omega}\left(\frac{1}{(k+1)^2\lambda_{(k+1)^2}}\right)\right)\leqq\sum_{k=N}^{\infty}\lambda_k\frac{4N}{\pi}\Omega\left(\bar{\Omega}\left(\frac{1}{(k+1)^2\lambda_{(k+1)^2}}\right)\right)\leqq$$

$$\leqq\sum_{k=N}^{\infty}\frac{4N}{\pi}\frac{1}{(k+1)^2}\leqq\frac{4}{\pi}.$$

$$(1.10)\quad B_1(x)\leqq\sum_{k=0}^{N-1}\lambda_k\Omega\left(\mu_k\sum_{n=k+1}^{N-1}A_n(x)\right)+\sum_{k=0}^{N-1}\lambda_k\Omega\left(\mu_k\left|\sum_{n=N}^{\infty}A_n(x)\right|\right)=B_{11}(x)+B_{12}(x).$$

Similarly to (1.9), we get

$$(1.11)\quad B_{12}(x)\leqq\sum_{k=0}^{N-1}\lambda_k\Omega\left(\mu_k\frac{4}{x}\frac{1}{\mu_{N^2}}\bar{\Omega}\left(\frac{1}{N^2\lambda_{N^2}}\right)\right)\leqq\frac{4N}{\pi}\lambda_N\sum_{k=0}^{N-1}\Omega\left(\bar{\Omega}\left(\frac{1}{N^2\lambda_{N^2}}\right)\right)\leqq\frac{4}{\pi}.$$

In order to estimate $B_{11}(x)$ let $2^{m-1}\leqq N-1<2^m$ and $m_k=[\underset{2}{\log}(k+1)]$. Using these notations we have

$$(1.12)\qquad B_{11}(x)\leqq\sum_{k=0}^{N-2}\lambda_k\Omega\left(\mu_k\sum_{n=k+1}^{N-1}\frac{1}{\mu_{n^2}}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)nx\right)\leqq$$

$$\leqq\sum_{k=0}^{2^m-1}\lambda_k\Omega\left(\sum_{n=2^{m_k}}^{2^m-1}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)n\frac{\pi}{2^{m-1}}\right)\leqq 2\pi\sum_{k=0}^{2^m-1}\lambda_k\Omega\left(\sum_{l=m_k}^{m-1}\sum_{n=2^l}^{2^{l+1}-1}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)\frac{n}{2^m}\right)\leqq$$

$$\leqq 2\pi\sum_{k=0}^{2^m-1}\lambda_k\sum_{l=m_k}^{m-1}\Omega\left(\sum_{n=2^l}^{2^{l+1}-1}\bar{\Omega}\left(\frac{1}{n^2\lambda_{n^2}}\right)\frac{n}{2^m}\right)\leqq$$

$$\leqq 2\pi\sum_{l=0}^{m-1}\sum_{m_k\leqq l}\lambda_k\Omega\left(\bar{\Omega}\left(\frac{1}{2^{2l}\lambda_{2^{2l}}}\right)2^{2l+1-m}\right)\leqq 2\pi\sum_{l=0}^{m-1}2^{l+1}\lambda_{2^{l+1}}\Omega\left(\bar{\Omega}\left(\frac{1}{2^{2l}\lambda_{2^{2l}}}\right)2^{2l+1-m}\right)=$$

$$=2\pi\sum_{l=0}^{\frac{m-1}{2}}+2\pi\sum_{l=\frac{m+1}{2}}^{m-1}\leqq 2\pi\sum_{l=0}^{\frac{m-1}{2}}2^{l+1}\lambda_{2^{l+1}}\Omega\left(\bar{\Omega}\left(\frac{1}{2^{2l}\lambda_{2^{2l}}}\right)\right)+$$

$$+2\pi\sum_{l=\frac{m+1}{2}}^{m-1}2^{l+1}\lambda_{2^{l+1}}2^{2l+1-m}\Omega\left(\bar{\Omega}\left(\frac{1}{2^{2l}\lambda_{2^{2l}}}\right)\right)\leqq 12\pi+4\pi\frac{\lambda_2}{\lambda_1}.$$

(1.8)—(1.12) verify the assertion.

Lemma 6. *Let* $r\geqq 1$ *and* Ω *concave. If*

$$\left\|\sum_{k=0}^{\infty} k^r \Omega(|s_k(f)-f|)\right\| < \infty \tag{1.13}$$

then

$$\left\|\sum_{k=0}^{\infty} k^{r-1} \Omega(|s_k(\tilde{f}')-\tilde{f}'|)\right\| < \infty. \tag{1.14}$$

Proof. Let $f\sim\sum_{k=0}^{\infty} A_k(x)$. Taking into account the concavity of Ω and $r\geqq 1$, (1.13) gives that

$$\sum_{k=0}^{\infty} k|A_k(x)| \leqq \sum_{k=0}^{\infty} k(|s_{k-1}(x)-f(x)|+|s_k(x)-f(x)|) = O\left(\sum_{k=0}^{\infty} k^r\, \Omega(|s_k(x)-f(x)|)\right),$$

i.e. $\sum_{k=0}^{\infty} kA_k(x)$ is absolutely convergent. From this it follows that $\tilde{f}'(x)=\sum_{k=0}^{\infty} kA_k(x)$, and hence

$$\sum_{k=0}^{\infty} k^{r-1}\Omega(|s_k(\tilde{f}';\, x)-\tilde{f}'(x)|) = \sum_{k=0}^{\infty} k^{r-1}\Omega\left(\left|\sum_{n=k+1}^{\infty} nA_n(x)\right|\right) =$$

$$= \sum_{k=0}^{\infty} k^{r-1}\Omega\left(\left|k(s_k(x)-f(x))+\sum_{n=k}^{\infty}(s_n(x)-f(x))\right|\right) \leqq \sum_{k=0}^{\infty} k^{r-1}\Omega(k|s_k(x)-f(x)|)+$$

$$+\sum_{k=0}^{\infty} k^{r-1}\sum_{n=k}^{\infty}\Omega(|s_n(x)-f(x)|) \leqq \sum_{k=0}^{\infty} k^r\,\Omega(|s_k(x)-f(x)|)+\sum_{n=0}^{\infty}\Omega(|s_n(x)-f(x)|)\sum_{k=0}^{n} k^{r-1},$$

from which, using (1.13), we obtain (1.14).

Lemma 7. *Let* $R_n(r, f)=R_n(r, f; x)=\sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^r\right)A_k(x)$, *where* $f(x)\sim\sim\sum_{k=0}^{\infty} A_k(x)$. *If* $|f|\leqq\delta$ *and* $r\geqq 1$, *then*

$$|R_n(r, f)| \leqq C_r\delta$$

where C_r *depends only on* r.

Proof. Denote $D_k(t)$ and $K_k(t)$ the k-th Dirichlet and Fejér kernel, respectively. Using the nonnegativity of $K_k(t)$ we get by an Abel rearrangement

$$|R_n(r,f;x)| = \frac{1}{(n+1)^r}\left|\sum_{k=0}^{n} s_k(x)\big((k+1)^r-k^r\big)\right| =$$

$$= \frac{1}{(n+1)^r}\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\left\{\sum_{k=0}^{n} D_k(u)\big((k+1)^r-k^r\big)\right\}du\right| =$$

$$= \frac{1}{(n+1)^r}\frac{1}{\pi}\left|\int_{-\pi}^{\pi} f(x+u)\left\{\sum_{k=0}^{n-1}(k+1)K_k(u)\big(2(k+1)^r-k^r-(k+2)^r\big)+\right.\right.$$

$$\left.\left.+(n+1)K_n(u)\big((n+1)^r-n^r\big)\right\}du\right| \leqq$$

$$\leqq \frac{\delta}{(n+1)^r}\frac{1}{\pi}\left\{\sum_{k=0}^{n-1}(k+1)\big(k^r+(k+2)^r-2(k+1)^r\big)+\big((n+1)^r-n^r\big)(n+1)\right\} =$$

$$= O\left(\frac{\delta}{(n+1)^r}\left\{\sum_{k=1}^{n-1}(k+1)k^{r-2}+(n+1)^r\right\}\right) = O(\delta),$$

and this proves our lemma.

Lemma 8. *For*

$$\tau_n(r,f) = \tau_n(r,f;x) = \frac{2^r R_{2n-1}(r,f;x)-R_{n-1}(r,f;x)}{2^r-1} \quad (r \geqq 1)$$

we have

$$|\tau_n(r,f)-f| \leqq c_r' E_n(f).$$

Proof.

$$|\tau_n(r,f)-f| = \left|\frac{\sum_{k=n}^{2n-1}(s_k-f)\big((k+1)^r-k^r\big)}{n^r(2^r-1)}\right| \leqq \frac{\sum_{k=n}^{2n-1}|s_k-f|\big((k+1)^r-k^r\big)}{n^r(2^r-1)} \leqq$$

$$\leqq \frac{r}{n}\sum_{k=n}^{2n-1}|s_k-f| = O\big(E_n(f)\big).$$

In the last step we used one of the results of Leindler [3].

Lemma 9. *Let Ω be a convex function, for which*

$$\int_0^1 \frac{\bar{\Omega}(x)}{x}\,dx < \infty,$$

and let $a_n \geqq 0$ *such that*

$$\sum_{k=1}^{\infty}\Omega(ka_k) \leqq K \quad \textit{for some} \quad K \geqq 1.$$

Then

$$\sum_{k=n}^{\infty} a_k \leqq K\bar{\Omega}^*\left(\frac{1}{n}\right)$$

($\bar{\Omega}^*(\delta)$ was defined in the Definition).

Proof. It is enough to prove Lemma 9 for $K=1$, namely if $K>1$ we can apply the case $K=1$ to the sequence $\frac{a_n}{K}$, using the inequality $\Omega\left(\frac{x}{K}\right) \leqq \frac{\Omega(x)}{K}$.

For $K=1$ the proof is very simple:

$$\Omega\left(\sum_{k=2^s n}^{2^{s+1}n-1} a_k\right) \leqq \Omega\left(\frac{\sum_{k=2^s n}^{2^{s+1}n-1} k a_k}{2^s n}\right) \leqq \frac{\sum_{k=2^s n}^{2^{s+1}n-1} \Omega(k a_k)}{2^s n} := \frac{\varepsilon_s}{2^s n}$$

i.e.

$$\sum_{k=2^s n}^{2^{s+1}n-1} a_k \leqq \bar{\Omega}\left(\frac{\varepsilon_s}{2^s n}\right);$$

and if we sum these inequalities for $s=0, 1, \ldots$ we get the required inequality.

Lemma 10. *If ω is concave and $E_n(f)=O\left(\omega^*\left(\frac{1}{n}\right)\right)$, then*

$$\omega(f;\delta) = O\left(\delta \int_{\delta}^{1} \frac{\omega(x)}{x^2}\,dx + \omega^*(\delta)\right). \tag{1.15}$$

Proof. It is enough to prove (1.15) for $\delta=\frac{1}{2^m}$. We shall use the following inequality (see [9], page 333).

$$\omega\left(f;\frac{1}{n}\right) \leqq K\left(\frac{\sum_{k=0}^{n} E_k(f)}{n+1}\right). \tag{1.16}$$

From the definition of ω^* it follows that there are sequences $\{\varepsilon_s^{(r)}\}_{s=0}^{\infty}$ $(r=0, 1, \ldots, m-1)$, for which

$$\omega\left(f;\frac{1}{2^m}\right) = O\left(2^{-m}\sum_{k=1}^{2^m}\omega^*\left(\frac{1}{k}\right)\right) = O\left(2^{-m}\sum_{r=0}^{m-1} 2^r\omega^*\left(\frac{1}{2^r}\right)\right) =$$

$$= O\left(2^{-m}\sum_{r=0}^{m-1} 2^r \sum_{s=0}^{\infty}\omega\left(\frac{\varepsilon_s^{(r)}}{2^{r+s}}\right)\right) = O\left(2^{-m}\sum_{r=0}^{m-1} 2^r\left\{\sum_{s=0}^{m-r-1}\omega\left(\frac{\varepsilon_s^{(r)}}{2^{r+s}}\right)+\omega^*\left(\frac{1}{2^m}\right)\right\}\right) =$$

$$= O\left(2^{-m}\sum_{r=0}^{m-1} 2^r \sum_{s=0}^{m-r-1}\omega\left(\frac{1}{2^{r+s}}\right)+\omega^*\left(\frac{1}{2^m}\right)\right) = O\left(2^{-m}\sum_{t=0}^{m-1} 2^{t+1}\omega\left(\frac{1}{2^t}\right)+\omega^*\left(\frac{1}{2^m}\right)\right) =$$

$$= O\left(2^{-m}\int_{2^{-m}}^{1}\frac{\omega(x)}{x^2}\,dx+\omega^*\left(\frac{1}{2^m}\right)\right)$$

and this proves (1.15).

§ 2. Proof of the theorems

Proof of Theorem 1. It is enough to prove the theorem for convex Ω, namely if Ω is concave, then (2) implies

$$\left\| \sum_{k=0}^{\infty} |s_k - f| \right\| < \infty,$$

and if we apply the second part of Theorem 1 to the convex function $\Omega(x)=x$ we get that

$$\omega(\tilde{f}; \delta) = O\left(\int_0^{\delta} \frac{x}{x}\, dx \right) = O(\delta)$$

i.e. $\tilde{f} \in \mathrm{Lip}\, 1$.

Let thus Ω be convex. First we prove (7). Let us denote by $\sigma_n(f)=\sigma_n(f; x)$ the n-th $(C, 1)$-mean of the Fourier series of f, and let

$$\tau_n(f) = \tau_n(f; x) = 2\sigma_{2n-1}(f; x) - \sigma_{n-1}(f; x) = \frac{\sum_{k=n}^{2n-1} s_k(x)}{n}.$$

From (2), using the convexity of Ω we get

$$|\sigma_n(f)-f| = \bar{\Omega}(\Omega(|\sigma_n(f)-f|)) \leqq \bar{\Omega}\left(\Omega\left(\frac{\sum_{k=0}^{n} |s_k - f|}{n+1} \right) \right) \leqq \tag{2.1}$$

$$\leqq \bar{\Omega}\left(\frac{\sum_{k=0}^{n} \Omega(|s_k - f|)}{n+1} \right) = O\left(\bar{\Omega}\left(\frac{1}{n} \right) \right).$$

With the notation

$$f - \sigma_n(f) = g_n(f)$$

we have

$$\sigma_n(f) - f = (\sigma_n(\sigma_n(f)) - \sigma_n(f)) + (\sigma_n(g_n(f)) - g_n(f)). \tag{2.2}$$

We can write (2.1) in the form $g_n(f) = O\left(\bar{\Omega}\left(\frac{1}{n} \right) \right)$, from which $\sigma_n(g_n(f)) = O\left(\bar{\Omega}\left(\frac{1}{n} \right) \right)$, and so (2.2) implies

$$\sigma_n(\sigma_n(f)) - \sigma_n(f) = O\left(\bar{\Omega}\left(\frac{1}{n} \right) \right). \tag{2.3}$$

If we keep in view the expression of $\sigma_n(f)$, it is easy to see that

$$\sigma_n(\sigma_n(f)) - \sigma_n(f) = -\frac{(\tilde{\sigma}_n(f))'}{n+1},$$

so (2.3) implies $\tilde{\sigma}_n'(f)=O\left[n\bar{\Omega}\left(\frac{1}{n}\right)\right]$, and together with this

$$(2.4)\qquad (\tilde{\tau}_n(f))' = (\tau_n(\tilde{f}))' = O\left[n\bar{\Omega}\left(\frac{1}{n}\right)\right].$$

Now (2.1) gives $E_n(f)=O\left[\bar{\Omega}\left(\frac{1}{n}\right)\right]$, from which by Lemma 2 (ii) it follows $E_n(\tilde{f})=O\left(\int_0^{1/n}\frac{\bar{\Omega}(x)}{x}\,dx\right)$. It is known (see e.g. Lemma 8) that $|\tau_n(g)-g|\leqq \leqq KE_n(g)$; and hence, also using the previous estimation, we get

$$(2.5)\qquad |\tau_n(\tilde{f})-\tilde{f}| = O\left(\int_0^{1/n}\frac{\bar{\Omega}(x)}{x}\,dx\right).$$

Now we are ready to prove (7). If $|h|\leqq\frac{1}{n}$, then (2.4) and (2.5) give

$$|\tilde{f}(x)-\tilde{f}(x+h)|\leqq|\tilde{f}(x)-\tau_n(\tilde{f};x)|+|\tau_n(\tilde{f};x)-\tau_n(\tilde{f};x+h)|+|\tau_n(\tilde{f};x+h)-\tilde{f}(x+h)|=$$
$$=O\left(\int_0^{1/n}\frac{\bar{\Omega}(x)}{x}\,dx+|h\tau_n'(\tilde{f};\,x+\vartheta h)|\right)=$$
$$=O\left(\int_0^{1/n}\frac{\bar{\Omega}(x)}{x}\,dx+|h|n\bar{\Omega}\left(\frac{1}{n}\right)\right)=O\left(\int_0^{1/n}\frac{\bar{\Omega}(x)}{x}\,dx\right),$$

and this is equivalent to (7).

By Lemma 4, (2) is satisfied by the function

$$f_0(x)=\sum_{n=1}^{\infty}\frac{1}{\gamma_n}\bar{\Omega}\left(\frac{1}{n}\right)\sin nx.$$

Then,

$$\tilde{f}_0(x)=-\sum_{=1}^{\infty}\frac{1}{8n}\bar{\Omega}\left(\frac{1}{n}\right)\cos nx,$$

and here the right hand side is the Fourier series of a continuous function only if

$$\sum_{n=1}^{\infty}\frac{1}{n}\bar{\Omega}\left(\frac{1}{n}\right)<\infty$$

(for the $(C, 1)$ means of this series must then be bounded), and this is the same as (6). The statement, that in case (6) $\tilde{f}$ is continuous is a direct consequence of (7), proved above.

Let $h=\frac{\pi}{2^{k+1}}$; then

$$\tilde{f}_0(h)-\tilde{f}_0(0)=\sum_{n=2^k+1}^{\infty}\frac{1}{8n}\bar{\Omega}\left(\frac{1}{n}\right)-\sum_{n=2^k+1}^{\infty}\frac{1}{8n}\bar{\Omega}\left(\frac{1}{n}\right)\cos nh+\sum_{n=1}^{2^k}\frac{1}{8n}\bar{\Omega}\left(\frac{1}{n}\right)2\sin^2 n\frac{h}{2}.$$

It is easy to see that

$$\sum_{n=2^k+1}^{\infty} \frac{1}{8n} \bar{\Omega}\left(\frac{1}{n}\right) \cos nh \leqq 0,$$

and so

$$\tilde{f}_0(h) - \tilde{f}_0(0) \geqq \sum_{n=2^k+1}^{\infty} \frac{1}{8n} \bar{\Omega}\left(\frac{1}{n}\right) \geqq c \int_0^{1/2^k} \frac{\bar{\Omega}(x)}{x}\, dx,$$

and hence (8) follows by a standard argument.

We have completed our proof.

Proof of Theorem 2. We have to consider two cases separately

Case I: Ω is convex. Let

$$\sum_{k=0}^{\infty} \lambda_k \Omega(\mu_k |s_k(x) - f(x)|) \leqq K.$$

We have

$$\Omega(\mu_n E_{2n}) \leqq \Omega\left(\mu_n \left\| \frac{\sum_{k=n+1}^{2n} |s_k - f|}{n} \right\|\right) \leqq \Omega\left(\left\| \frac{\sum_{k=n+1}^{2n} \mu_k |s_k - f|}{n} \right\|\right) =$$

$$= \left\| \Omega\left(\frac{\sum_{k=n+1}^{2n} \mu_k |s_k - f|}{n}\right) \right\| \leqq \left\| \frac{\sum_{k=n+1}^{2n} \Omega(\mu_k |s_k - f|)}{n} \right\| \leqq$$

$$\leqq \left\| \frac{\sum_{k=n+1}^{2n} \lambda_k \Omega(\mu_k |s_k - f|)}{n\lambda_n} \right\| \leqq \frac{K}{n\lambda_n},$$

i.e.

$$E_{2n}(f) = O\left(\frac{1}{\mu_n} \bar{\Omega}\left(\frac{1}{n\lambda_n}\right)\right),$$

and hence, using the inequality (1.16),

$$\omega\left(f; \frac{1}{n}\right) = O\left(\frac{\sum_{k=0}^{n} E_k}{n+1}\right) = O\left(\frac{\sum_{k=0}^{n} E_{2k}}{n+1}\right) = O\left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\mu_k} \bar{\Omega}\left(\frac{1}{k\lambda_k}\right)\right)$$

and this is (12).

Let

$$f_0(x) = \sum_{n=1}^{\infty} \frac{1}{8n\mu_n} \bar{\Omega}\left(\frac{1}{n\lambda_n}\right) \sin nx.$$

By Lemma 4, f_0 satisfies (11). Now applying Lemma 1 to f_0 we get

$$f_0\left(\frac{\pi}{n}\right)-f_0(0) \geqq \frac{1}{2}\frac{1}{8}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right),$$

and this proves (13).

Case II: Ω is concave. By Lemma 3 we have

$$E_{4n}(f) = O\left[\log n\left(n^2\lambda_n^2\mu_n\Omega\left(\frac{\log n}{n\lambda_n}\right)\right)^{-1}\right]. \tag{2.6}$$

Let m_k resp. n_k the least and the greatest n (if any), for which

$$\frac{1}{(k+1)\lambda_{k+1}} < \Omega\left(\frac{\log n}{n\lambda_n}\right) \leqq \frac{1}{k\lambda_k}. \tag{2.7}$$

Ω is concave, so there is a $c>0$ for which

$$\Omega\left(\frac{\log k}{k\lambda_k}\right) \geqq c\frac{\log k}{k\lambda_k} > \frac{1}{k\lambda_k}$$

if k is large enough. From this and (2.7) it follows at once for $k \geqq k_0$ that

$$m_k \geqq k+1, \quad \lambda_{m_k} \geqq \lambda_{k+1}, \quad \mu_{m_k} \geqq \mu_k \tag{2.8}$$

$$\frac{\log m_k}{m_k\lambda_{m_k}} \leqq \bar{\Omega}\left(\frac{1}{k\lambda_k}\right), \quad \frac{\log n_k}{n_k\lambda_{n_k}} \geqq \bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right). \tag{2.9}$$

We shall show that for $k \geqq k_0$

(2.10)

$$\sum_{n=m_k}^{n_k}\log n\left(n^2\lambda_n^2\mu_n\Omega\left(\frac{\log n}{n\lambda_n}\right)\right)^{-1} = O\left[(k+1)\left(\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)-\frac{1}{\mu_{k+1}}\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right].$$

First we consider the case $n_k=m_k=n$. Using the inequalities

$$\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right) \leqq \bar{\Omega}\left(\frac{1}{(k+1)\lambda_k}\right) \leqq \frac{k}{k+1}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right), \quad \frac{\Omega(x)}{x} \geqq C$$

coming from the concavity of Ω, we obtain for $k \geqq k_0$

$$\log n\left(n^2\lambda_n^2\mu_n\Omega\left(\frac{\log n}{n\lambda_n}\right)\right)^{-1} = O\left(\frac{1}{n\lambda_n\mu_n}\right) = O\left(\frac{\log n}{n\lambda_n\mu_n}\right) = O\left(\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)\right) =$$

$$= O\left[\frac{(k+1)}{\mu_k}\left(\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)-\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right] =$$

$$= O\left[(k+1)\left(\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)-\frac{1}{\mu_{k+1}}\bar{\Omega}\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right].$$

If, however, $n_k>m_k$ and $k\geqq k_0$, then

$$\sum_{n=m_k}^{n_k}\log n\left\{n^2\lambda_n^2\mu_n\Omega\left(\frac{\log n}{n\lambda_n}\right)\right\}^{-1}=O\left(\frac{(k+1)\lambda_{k+1}}{\lambda_{m_k}^2\mu_{m_k}}\sum_{n=m_k}^{n_k}\frac{\log n}{n^2}\right)=$$

$$=O\left(\frac{(k+1)}{\lambda_{m_k}\mu_{m_k}}\int_{m_k}^{n_k}\frac{\log x-1}{x^2}\,dx\right)=O\left(\frac{(k+1)}{\lambda_{m_k}\mu_{m_k}}\left(\frac{\log m_k}{m_k}-\frac{\log n_k}{n_k}\right)\right)=$$

$$=O\left(\frac{(k+1)}{\mu_k}\left(\frac{\log m_k}{m_k\lambda_{m_k}}-\frac{\log n_k}{n_k\lambda_{n_k}}\right)\right)=O\left(\frac{(k+1)}{\mu_k}\left(\bar\Omega\left(\frac{1}{k\lambda_k}\right)-\bar\Omega\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right)=$$

$$=O\left((k+1)\left(\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right)-\frac{1}{\mu_{k+1}}\bar\Omega\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right).$$

Thus we have proved (2.10) for $k\geqq k_0$.

Let now $m_i\leqq m\leqq n_i$. Using (2.6) and (2.10) we get

$$\omega\left(f;\frac{1}{m}\right)=O\left(\frac{1}{m}\sum_{k=0}^{m}E_k(f)\right)=O\left(\frac{1}{m}\sum_{k=0}^{m}E_{4k}(f)\right)=$$

$$=O\left(\frac{1}{m}\sum_{k=1}^{i}\sum_{n=m_k}^{n_k}\log n\left(n^2\lambda_n^2\mu_n\Omega\left(\frac{\log n}{n\lambda_n}\right)\right)^{-1}\right)=O\left(\frac{1}{m}\left(\sum_{k=1}^{k_0-1}+\sum_{k=k_0}^{i}\right)\right)=$$

$$=O\left(\frac{1}{m}+\frac{1}{m}\sum_{k=k_0}^{i}(k+1)\left(\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right)-\frac{1}{\mu_{k+1}}\bar\Omega\left(\frac{1}{(k+1)\lambda_{k+1}}\right)\right)\right)=$$

$$=O\left(\frac{1}{m}\sum_{k=1}^{i}\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right)\right)=O\left(\frac{1}{m}\sum_{k=1}^{m}\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right)\right),$$

which proves (12). Let

$$f_0(x)=\sum_{n=1}^{\infty}\frac{1}{\mu_{n^2}}\bar\Omega\left(\frac{1}{n^2\lambda_{n^2}}\right)\sin nx.$$

By Lemma 5, f_0 satisfies (11). Applying again Lemma 1 (it is easy to see that it is applicable), we obtain

$$f_0\left(\frac{\pi}{n}\right)-f_0(0)\geqq\frac{1}{2}\frac{1}{n}\sum_{k=1}^{n}k\frac{1}{\mu_{k^2}}\bar\Omega\left(\frac{1}{k^2\lambda_{k^2}}\right)\geqq\frac{1}{6}\frac{1}{n}\sum_{k=1}^{n}(2k+1)\frac{1}{\mu_{k^2}}\bar\Omega\left(\frac{1}{k^2\lambda_{k^2}}\right)\geqq$$

$$\geqq\frac{1}{6}\frac{1}{n}\sum_{k=1}^{n^2-1}\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right)\geqq\frac{1}{6}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\mu_k}\bar\Omega\left(\frac{1}{k\lambda_k}\right),$$

and this is (13). — The proof of Theorem 2 is thus completed.

Proof of Theorem 3. We shall consider only the case when r is odd, the other case could be treated similarly.

If we apply Lemma 6 r-times, we gett hat (14) implies

$$\left\| \sum_{k=0}^{\infty} k^{\beta-r} \Omega\left(|s_k(\tilde{f}^{(r)}) - \tilde{f}^{(r)}|\right) \right\| < \infty,$$

and hence, using the assertion (i) of Theorem 1, we get $\tilde{\tilde{f}}^{(r)} = f^{(r)} \in \mathrm{Lip}\, 1$, while using Corollary 2 of Theorem 2 we obtain

$$\omega(\tilde{f}^{(r)}; \delta) = O\left(\delta \int_{\delta}^{1} \frac{\bar{\Omega}(x^{1+\beta-r})}{x^2}\, dx\right)$$

as it was proposed in (16).

Let

$$f_\beta(x) = \sum_{n=1}^{\infty} \bar{\Omega}\left(\frac{1}{n^{2+\beta}}\right) \sin nx.$$

If we run through the proof of Lemma 5 we can see that its proof equally works for f_β, so f_β satisfies (14). Keeping in mind that $\bar{\Omega}$ is convex, we have

$$n^{\beta+2} \bar{\Omega}\left(\frac{1}{n^{\beta+2}}\right) \geqq (n+1)^{\beta+2} \bar{\Omega}\left(\frac{1}{(n+1)^{\beta+2}}\right),$$

and this implies that

$$n^{r+1} \bar{\Omega}\left(\frac{1}{n^{\beta+2}}\right) \geqq (n+1)^{r+1} \bar{\Omega}\left(\frac{1}{(n+1)^{\beta+2}}\right),$$

so we can apply Lemma 1 to $\tilde{f}_\beta^{(r)}$, and this gives

$$\left|\tilde{f}_\beta^{(r)}\left(\frac{\pi}{n}\right) - \tilde{f}_\beta(0)\right| \geqq \frac{1}{2}\frac{1}{n} \sum_{k=1}^{n} k^{r+1} \bar{\Omega}\left(\frac{1}{k^{\beta+2}}\right) \geqq c \frac{1}{n} \int_{1/n}^{1} \frac{\bar{\Omega}(x^{\beta+2})}{x^{r+3}}\, dx = \tag{2.11}$$

$$= c' \frac{1}{n} \int_{n^{-\frac{2+\beta}{1+\beta-r}}}^{1} \frac{\bar{\Omega}(u^{1+\beta-r})}{u^{\gamma}}\, du,$$

where

$$\gamma = \frac{1+\beta-r}{2+\beta}\left(r+3+\frac{1+r}{1+\beta-r}\right) \geqq 2.$$

Also $\dfrac{2+\beta}{1+\beta-r} \geqq 1$, so we get from (2.11) that

$$\left|\tilde{f}_\beta^{(r)}\left(\frac{\pi}{n}\right) - \tilde{f}_\beta^{(r)}(0)\right| \geqq c \frac{1}{n} \int_{1/n}^{1} \frac{\bar{\Omega}(x^{1+\beta-r})}{x^2}\, dx,$$

which was to be proved.

Thus we have completed our proof.

Proof of Theorem 4. Let $f(x)\sim\sum_{k=0}^{\infty}A_k(x)$ and

$$R_n(\beta, f;\, x)=\sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{\beta}\right)A_k(x).$$

Using an Abel rearrangement we get from (18)

$$\Omega\left(\frac{(n+1)^{\beta}}{2^{\beta}(\beta+1)}|R_n(\beta+1, f)-f|\right)=\Omega\left(\frac{(n+1)^{\beta}}{2^{\beta}(\beta+1)}\left|\frac{\sum_{k=0}^{n}s_k((k+1)^{\beta+1}-k^{\beta+1})}{(n+1)^{\beta+1}}-f\right|\right)=$$

$$=\Omega\left(\frac{(n+1)^{\beta}}{2^{\beta}(\beta+1)}\left|\frac{\sum_{k=0}^{n}(s_k-f)((k+1)^{\beta+1}-k^{\beta+1})}{(n+1)^{\beta+1}}\right|\right)\leqq\Omega\left(\frac{|s_0-f|+\sum_{k=1}^{n}k^{\beta}|s_k-f|}{n+1}\right)\leqq$$

$$\leqq\frac{\Omega(|s_0-f|)+\sum_{k=1}^{n}\Omega(k^{\beta}|s_k-f|)}{n+1}\leqq\frac{K}{n+1},$$

which implies

$$|R_n(\beta+1, f)-f|=O\left(\frac{1}{n^{\beta}}\bar{\Omega}\left(\frac{1}{n}\right)\right). \tag{2.12}$$

Now $R_n(\beta+1, f)$ is a trigonometric polinomial of order at most n, so (2.12) implies

$$E_n(f)=O\left(\frac{1}{n^{\beta}}\bar{\Omega}\left(\frac{1}{n}\right)\right). \tag{2.13}$$

We shall treat after that the cases (i)—(iii) separately.

Case (i). By Lemma 2 (iii) from (2.13) it follows that $E_n(f^{(r)})=O\left(\frac{1}{n^{\beta-r}}\bar{\Omega}\left(\frac{1}{n}\right)\right)$, and this, connecting with inequality (1.16) gives

$$\omega\left(f^{(r)};\,\frac{1}{n}\right)=O\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^{\beta-r}}\bar{\Omega}\left(\frac{1}{k}\right)\right)\quad\left(=O\left(\frac{1}{n}\int_{1/n}^{1}\frac{\bar{\Omega}(x)}{x^{2-\beta+r}}\,dx\right)\right).$$

From the concavity of $\bar{\Omega}$ it follows that $\bar{\Omega}\left(\frac{1}{k}\right)\leqq\frac{n}{k}\bar{\Omega}\left(\frac{1}{n}\right)$ $(n\geqq k)$ and so

$$\omega\left(f^{(r)};\,\frac{1}{n}\right)=O\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^{\beta-r}}\frac{n}{k}\bar{\Omega}\left(\frac{1}{n}\right)\right)=O\left(\bar{\Omega}\left(\frac{1}{n}\right)\right),$$

and this proves (19).

Case (ii) According to Lemma 2 (iv), (2.13) implies

$$E_n(f^{(r-1)})=O\left(\frac{1}{n}\bar{\Omega}\left(\frac{1}{n}\right)\right),$$

and so

$$\omega\left(f^{(r-1)};\frac{1}{n}\right)=O\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\bar{\Omega}\left(\frac{1}{k}\right)\right)=O\left(\frac{1}{n}\int_{1/n}^{1}\frac{\bar{\Omega}(x)}{x}dx\right),$$

from which (20) already follows.

Let r e.g. even, and

$$f_0(x)=\sum_{n=1}^{\infty}\frac{1}{n^{r+1}}\bar{\Omega}\left(\frac{1}{n^2}\right)\cos nx$$

(if r is odd then we must take $\sin x$ in place of $\cos x$). f_0 satisfies (18):

$$\sum_{k=0}^{\infty}\Omega\left(k^r\left|\sum_{n=k+1}^{\infty}\frac{1}{n^{r+1}}\bar{\Omega}\left(\frac{1}{n^2}\right)\cos nx\right|\right)\leqq\sum_{k=0}^{\infty}\Omega\left(k^r\frac{1}{(k+1)^r}\bar{\Omega}\left(\frac{1}{(k+1)^2}\right)\right)\leqq\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}.$$

$$f_0^{(r-1)}(x)=(-1)^{r/2}\sum_{n=1}^{\infty}\frac{1}{n^2}\bar{\Omega}\left(\frac{1}{n^2}\right)\sin nx,$$

and so using Lemma 1 we get

$$\left|f_0^{(r-1)}\left(\frac{\pi}{n}\right)-f_0^{(r-1)}(0)\right|\geqq\frac{1}{2}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\bar{\Omega}\left(\frac{1}{k^2}\right)\geqq\frac{1}{6}\frac{1}{n}\sum_{k=1}^{n}(2k+1)\frac{1}{k^2}\bar{\Omega}\left(\frac{1}{k^2}\right)\geqq$$

$$\geqq\frac{1}{6}\frac{1}{n}\sum_{k=1}^{n^2-1}\frac{1}{k}\bar{\Omega}\left(\frac{1}{k}\right)\geqq c\frac{1}{n}\int_{1/n}^{1}\frac{\bar{\Omega}(x)}{x}dx,$$

and this proves that (20) is best possible in general.

Lemma 2 (i) and the above proofs show that all of the above statements are true for the conjugate function, too.

Case (iii). We shall consider the case when r is even. Let

$$f=R_n(r+1,f)+g_n(f).$$

With this notation

(2.14)
$$R_n(r+1,f)-f=\big(R_n(r+1,R_n(r+1,f))-R_n(r+1,f)\big)+\big(R_n(r+1,\,g_n(f))-g_n(f)\big).$$

By (2.12) $g_n(f)=O\left(\frac{1}{n^r}\bar{\Omega}\left(\frac{1}{n}\right)\right)$, and this implies by Lemma 7 that $R_n(r+1,g_n(f))=O\left(\frac{1}{n^r}\bar{\Omega}\left(\frac{1}{n}\right)\right)$, and so from (2.14) it follows that

(2.15)
$$R_n(r+1,\,R_n(r+1,f))-R_n(r+1,f)=O\left(\frac{1}{n^r}\bar{\Omega}\left(\frac{1}{n}\right)\right).$$

Let

$$R_n(r+1,f)\sim\sum_{k=0}^{n}A_k(x).$$

Then

$$R_n(r+1, R_n(r+1, f)) - R_n(r+1, f) = \sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{r+1}\right)A_k(x) - \sum_{k=0}^{n} A_k(x) =$$

$$= -\frac{1}{(n+1)^{r+1}}\sum_{k=0}^{n} k^{r+1}A_k(x) = \frac{(-1)^{r/2+1}}{(n+1)^{r+1}}(\tilde{R}_n(r+1, f))^{(r+1)}.$$

This equality together with (2.15) gives

$$\tilde{R}_n^{(r+1)}(r+1, f) = O\left(n\bar{\Omega}\left(\frac{1}{n}\right)\right),$$

from which

$$\tilde{\tau}_n^{(r+1)}(r+1, f) = \tau_n^{(r+1)}(r+1, \tilde{f}) = \tau_n'(r+1, \tilde{f}^{(r)}) = O\left(n\bar{\Omega}\left(\frac{1}{n}\right)\right) \tag{2.16}$$

follows at once ($\tau_n(r, f)$ was defined in Lemma 8).

(2.13) implies by Lemma 2 (i) and (v) and by Lemma 8 that

$$|\tau_n(r+1, \tilde{f}^{(r)}) - \tilde{f}^{(r)}| = O\left(\int_0^{1/n} \frac{\bar{\Omega}(x)}{x}\,dx\right). \tag{2.17}$$

Now we get (22) from (2.16) and (2.17) as we got (7) in Theorem 1 from (2.4) and (2.5).

Before proving (23) we show that $f^{(r)}$ is the sum of its Fourier series. Because of the continuity of $f^{(r)}$ it is enough to prove that its Fourier series everywhere convergent. With the usual notations

$$f^{(r)}(x) \sim (-1)^{r/2}\sum_{k=0}^{\infty} k^r A_k(x),$$

$$\sum_{k=m}^{n} k^r A_k(x) = \sum_{k=m}^{n-1}\left(k^r-(k+1)^r\right)s_k(x) - m^r s_{m-1}(x) + n^r s_n(x) = \tag{2.18}$$

$$= O\left(\sum_{k=m}^{n-1} k^{r-1}|s_k(x)-f(x)| + m^r|s_{m-1}(x)-f(x)| + n^r|s_n(x)-f(x)|\right).$$

Lemma 9 shows by (18) that

$$\sum_{k=m}^{n-1} k^{r-1}|s_k(x)-f(x)| \to 0 \quad \text{as} \quad n, m \to \infty,$$

moreover $\Omega(n^r|s_n(x)-f(x)) \to 0$ $(n\to\infty)$, and this implies $n^r|s_n(x)-f(x)| \to 0$ as $n\to\infty$. Thus (2.18) gives the convergence of $\sum_{k=0}^{\infty} k^r A_k(x)$, and so

$$f^{(r)}(x) = (-1)^{r/2}\sum_{k=0}^{\infty} k^r A_k(x).$$

On this account the following transformations are legitimate, and (2.12), as well as Lemma 9 give

$$(-1)^{r/2}(\sigma_n(f^{(r)})-f^{(r)}) = \sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)k^r A_k - \sum_{k=0}^{\infty} k^r A_k = (n+1)^r(R_n(r+1,f)-f)-$$

$$-\sum_{k=n+1}^{\infty}(k^r-(n+1)^r)A_k = (n+1)^r(R_n(r+1,f)-f)+\sum_{k=n+2}^{\infty}(s_{k-1}-f)(k^r-(k-1)^r) =$$

$$= O\left[(n+1)^r\frac{1}{n^r}\bar{\Omega}\left(\frac{1}{n}\right)+\sum_{k=n+2}^{\infty}|s_{k-1}-f|(k-1)^{r-1}\right] = O\left[\bar{\Omega}\left(\frac{1}{n}\right)+\bar{\Omega}^*\left(\frac{1}{n}\right)\right] =$$

$= O\left[\bar{\Omega}^*\left(\frac{1}{n}\right)\right]$, from which $E_n(f^{(r)})=O\left[\bar{\Omega}^*\left(\frac{1}{n}\right)\right]$ follows at once. Now we can apply Lemma 10, and we get (23).

To prove (24) let

$$f_r(x) = \sum_{n=1}^{\infty}\frac{1}{8n^{r+1}}\bar{\Omega}\left(\frac{1}{n}\right)\sin nx.$$

By Lemma 4, f_r satisfies (18). Now

$$\tilde{f}_r^{(r)}(x) = (-1)^{r/2+1}\sum_{n=1}^{\infty}\frac{1}{8n}\bar{\Omega}\left(\frac{1}{n}\right)\cos nx$$

and in the proof of Theorem 1 we have already seen that for this function (24) is true.

The proof of Theorem 4 is thus completed.

Proof of Remark. Let r be, for example, an odd number.

We separate the proof into two cases.

1. $\delta\int_{\delta}^{1}\frac{\bar{\Omega}(x)}{x^2}dx \neq O(\omega(\delta))$. In this case by the aid of the above defined function

$$f_r(x) = \sum_{n=1}^{\infty}\frac{1}{8n^{r+1}}\bar{\Omega}\left(\frac{1}{n}\right)\sin nx$$

the proof can be easily carried out.

2. If $\delta\int_{\delta}^{1}\frac{\bar{\Omega}(x)}{x^2}dx = O(\omega(\delta))$, then there is a sequence of natural numbers $\{n_m\}$, for whicl.

$$\omega\left(\frac{\pi}{2n_m}\right) < \frac{1}{4^m}\bar{\Omega}^*\left(\frac{\pi}{2n_m}\right). \tag{2.19}$$

Let n be a fixed natural number, and $\varepsilon_0 \geqq \varepsilon_1 \geqq \dots$; $\sum_{k=0}^{\infty} \varepsilon_k \leqq 1$. Let $c_m = \bar{\Omega}\left(\frac{\varepsilon_k}{2^k n}\right)$ if $2^k n \leqq m < 2^{k+1} n$, and

$$f(x) = f_{\{\varepsilon_k\},n}(x) = (-1)^{\frac{r-1}{2}} \sum_{m=n}^{\infty} \frac{1}{2m^{r+1}} c_m \cos mx.$$

With the aid of (21) we get that $f^{(r)}$ exists, and less than a bound independent from $\{\varepsilon_k\}$ and n. We show that f satisfies (18).

$$(2.20)\quad \sum_{k=0}^{\infty} \Omega(k^r |s_k(x)-f(x)|) \leqq \sum_{k=1}^{n-1} \Omega\left(k^r \sum_{m=n}^{\infty} \frac{1}{2m^{r+1}} c_m\right) + \sum_{k=n}^{\infty} \Omega\left(k^r \sum_{m=k+1}^{\infty} \frac{1}{2m^{r+1}} c_m\right) \leqq$$

$$\leqq \sum_{k=1}^{n-1} \Omega\left(k^r \frac{1}{2n^r} c_n\right) + \sum_{k=n}^{\infty} \Omega\left(k^r \frac{1}{2k^r} c_k\right) \leqq \frac{1}{2} \sum_{k=1}^{n-1} \Omega\left[\bar{\Omega}\left(\frac{\varepsilon_0}{n}\right)\right] +$$

$$+ \frac{1}{2} \sum_{k=0}^{\infty} 2^k n \Omega\left[\bar{\Omega}\left(\frac{\varepsilon_k}{2^k n}\right)\right] \leqq \frac{\varepsilon_0}{2} + \frac{\sum_{k=0}^{\infty} \varepsilon_k}{2} \leqq 1.$$

Now

$$f^{(r)}(0) - f^{(r)}\left(\frac{\pi}{2n}\right) = \sum_{m=n}^{\infty} \frac{1}{m} c_m - \sum_{m=n}^{\infty} \frac{1}{m} c_m \cos m \frac{\pi}{2n},$$

and from the monotonicity of $\{c_m\}$ it follows that

$$\sum_{m=n}^{\infty} \frac{c_m}{m} \cos m \frac{\pi}{2n} \leqq 0,$$

and so

$$f^{(r)}(0) - f^{(r)}\left(\frac{\pi}{2n}\right) \geqq \frac{1}{2} \sum_{k=0}^{\infty} \bar{\Omega}\left(\frac{\varepsilon_k}{2^k n}\right) \sum_{m=2^k n}^{2^{k+1} n - 1} \frac{1}{m} \geqq \frac{1}{4} \sum_{k=0}^{\infty} \bar{\Omega}\left(\frac{\varepsilon_k}{2^k n}\right).$$

Consequently, by a suitable choice of $\{\varepsilon_k\}$ one can attain that the above defined function $f_{\{\varepsilon_k\},n}(x) = f_n(x)$ satisfies

$$(2.21)\quad f_n^{(r)}(0) - f_n^{(r)}\left(\frac{\pi}{2n}\right) \geqq \frac{1}{10} \bar{\Omega}^*\left(\frac{1}{n}\right) \geqq \frac{1}{20} \bar{\Omega}^*\left(\frac{\pi}{2n}\right),$$

for in the definition of $\bar{\Omega}^*$ we could have supposed that $\{\varepsilon_k\}$ is monotone.

Let now

$$f(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} f_{n_m}(x).$$

This function f satisfies (18); indeed, using the convexity of Ω, we get from (2.20)

$$\sum_{k=0}^{\infty} \Omega(k^r |s_k(f;\, x)-f(x)|) \leqq \sum_{k=0}^{\infty} \Omega\left(k^r \sum_{m=1}^{\infty} \frac{1}{2^m} |s_k(f_{n_m};\, x)-f_{n_m}(x)|\right) \leqq$$

$$\leqq \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^m} \Omega(k^r |s_k(f_{n_m};\, x)-f_{n_m}(x)|) =$$

$$= \sum_{m=1}^{\infty} \frac{1}{2^m} \sum_{k=0}^{\infty} \Omega(k^r |s_k(f_{n_m};\, x)-f_{n_m}(x)|) \leqq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

From the remark made after the definition of the functions $f_{\{\varepsilon_k\},n}(x)$, it follows that $\tilde{f}^{(r)}$ exists; and using (2.19), (2.21) we obtain

$$\tilde{f}^{(r)}(0)-\tilde{f}^{(r)}\left(\frac{\pi}{2n_m}\right) \geqq \frac{1}{2^m}\left(\tilde{f}_{n_m}^{(r)}(0)-\tilde{f}_{n_m}^{(r)}\left(\frac{\pi}{2n_m}\right)\right) \geqq \frac{1}{2^m}\frac{1}{20}\bar{\Omega}^*\left(\frac{\pi}{2n_m}\right) \geqq \frac{2^m}{20}\,\omega\left(\frac{\pi}{2n_m}\right),$$

so

$$\omega(\tilde{f}^{(r)};\, \delta) \neq O(\omega(\delta)),$$

which proves our Remark.

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