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Hyperinvariant subspaces of weak contractions

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Introduction

The aim of this paper is to study Hyperlat T, the hyperinvariant subspace lattice, of a completely non-unitary (c.n.u.) weak contraction T with finite defect indices. The work here is a continuation of the investigations of Hyperlat T which we made in [14] and [15]. There we only considered c.n.u. C_{11} contractions with finite defect indices. Now we shall generalize the results of [14] and [15]. Among other things, we shall show that for the contractions considered, (i) if T_1 is quasi-similar to T_2, then Hyperlat T_1 is (lattice) isomorphic to Hyperlat T_2 (Corollary 3.4) and (ii) Hyperlat T is (lattice) generated by subspaces of the forms $\overline{\text{ran } S}$ and ker V where S, V are operators in $\{T\}''$, the double commutant of T (Theorem 3.8). We also give necessary and sufficient conditions, in terms of the characteristic function and the Jordan model of T, that Lat T, the invariant subspace lattice of T, be equal to Hyperlat T.

Preliminaries and results

We follow the notations and terminologies used in [14] and [15]. Only the concepts concerning weak contractions will be reviewed here.

A contraction T is called a *weak contraction* if (i) its spectrum $\sigma(T)$ does not fill the open unit disc, and (ii) $I-T^*T$ is of finite trace. Examples of weak contractions are $C_0(N)$ contractions and c.n.u. C_{11} contractions with finite defect indices. The characteristic function Θ_T of every weak contraction T admits a *scalar multiple*, that is, there exist a contractive analytic function Ω and a scalar valued analytic function $\delta\not\equiv 0$ such that $\Omega\Theta_T=\Theta_T\Omega=\delta$. For a c.n.u. weak contraction

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T on H we can consider its C_0-C_{11} decomposition. Let $H_0, H_1 \subseteq H$ be the invariant subspaces for T such that $T_0 \equiv T|H_0$ and $T_1 \equiv T|H_1$ are the C_0 and C_{11} parts of T, respectively. Indeed, T_0 and T_1 are equal to those appearing in the triangulations

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & Y \\ 0 & T_0' \end{bmatrix}$$

on $H = H_0 \oplus H_0^\perp$ and $H = H_1 \oplus H_1^\perp$ corresponding to the $*$-canonical factorization $\Theta_T = \Theta_{*e}\Theta_{*i}$ and the canonical factorization $\Theta_T = \Theta_i \Theta_e$, respectively. H_0 and H_1 are even hyperinvariant for T and satisfy $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$. For the details the readers are referred to [4], Chap. VIII.

* *

It was shown in [4], p. 334 that $H_0 = \ker m(T)$ and $H_1 = \overline{\operatorname{ran} m(T)}$, where m is the minimal function of T_0. Note that $m(T) \in \{T\}''$. Now we have the following supplementary result.

Theorem 1. *If T is a c.n.u. weak contraction on H and H_0, H_1 are subspaces of H such that $T_0 = T|H_0$ and $T_1 = T|H_1$ are the C_0 and C_{11} parts of T, respectively, then there exists an operator S in $\{T\}''$ such that $H_0 = \overline{\operatorname{ran} S}$ and $H_1 = \ker S$.*

Proof. We consider T being defined on $H \equiv [H_{\mathfrak{D}}^2 \oplus \overline{\Delta L_{\mathfrak{D}}^2}] \ominus \{\Theta_T w \oplus \Delta w \colon w \in H_{\mathfrak{D}}^2\}$ by $T(f \oplus g) = P(e^{it} f \oplus e^{it} g)$ for $f \oplus g \in H$, where Θ_T is the characteristic function of T, $\Delta(t) = (I_{\mathfrak{D}} - \Theta_T(t)^* \Theta_T(t))^{1/2}$ and P denotes the (orthogonal) projection onto H. Since Θ_T admits a scalar multiple, the same is true for its outer factor Θ_e and inner factor Θ_i (cf. [4], p. 217). Let $\delta_1 \not\equiv 0$ and $\delta_2 \not\equiv 0$ be their respective scalar multiples, and let Ω_1 and Ω_2 be contractive analytic functions such that $\Omega_1 \Theta_e = \Theta_e \Omega_1 = \delta_1 I_{\mathfrak{D}}$ and $\Omega_2 \Theta_i = \Theta_i \Omega_2 = \delta_2 I_{\mathfrak{D}}$. We may assume that δ_1 is outer and δ_2 is inner (cf. [4], p. 217). Let $\delta = \delta_1 \delta_2$ and $\Omega = \Omega_1 \Omega_2$. Then $\Omega \Theta_T = \Theta_T \Omega = \delta I_{\mathfrak{D}}$. Consider the operator $S = P \begin{bmatrix} \delta_1 & 0 \\ \bar{\delta}_2 \Delta \Omega & 0 \end{bmatrix}$. We prove $H_1 = \ker S$ and $H_0 = \overline{\operatorname{ran} S}$ in the following steps. In each step the first statement is proved.

(1) $S \in \{T\}''$. Let $V = P \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ be an operator in $\{T\}'$, where A is a bounded analytic function while B and C are bounded measurable functions satisfying the conditions $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + C\Delta = \Delta A_0$ a.e., where A_0 is another bounded analytic function (cf. [5]). An easy calculation shows that

$$SV = P \begin{bmatrix} \delta_1 A & 0 \\ \bar{\delta}_2 \Delta \Omega A & 0 \end{bmatrix} \quad \text{and} \quad VS = P \begin{bmatrix} A\delta_1 & 0 \\ B\delta_1 + C\bar{\delta}_2 \Delta \Omega & 0 \end{bmatrix}.$$

We have $\bar{\delta}_2\Delta\Omega A\delta = \bar{\delta}_2\Delta\Omega A\Theta_T\Omega = \bar{\delta}_2\Delta\Omega\Theta_T A_0\Omega = \bar{\delta}_2\Delta\delta A_0\Omega = \delta_1\Delta A_0\Omega = \delta_1(B\Theta_T + C\Delta)\Omega = B\delta_1\delta + C\bar{\delta}_2\Delta\Omega\delta = (B\delta_1 + C\bar{\delta}_2\Delta\Omega)\delta$. Since $\delta \not\equiv 0$, we conclude that $\bar{\delta}_2\Delta\Omega A = B\delta_1 + C\bar{\delta}_2\Delta\Omega$. Hence $SV = VS$ and we have $S\in\{T\}''$.

(2) $H_1 \subseteq \ker S$. It was shown in [6] that $H_1 = \{f\oplus g\in H\colon f\in\Theta_i H^2_{\mathfrak{D}}\}$. For $\Theta_i u\oplus g\in H_1$, $S(\Theta_i u\oplus g) = P(\delta_1\Theta_i u\oplus\bar{\delta}_2\Delta\Omega\Theta_i u) = P(\Theta_i\Theta_e\Omega_1 u\oplus\bar{\delta}_2\Delta\Omega_1\Omega_2\Theta_i u) = P(\Theta\Omega_1 u\oplus\Delta\Omega_1 u) = 0$, which shows that $H_1\subseteq\ker S$.

(3) $\ker S\subseteq H_1$. For $f\oplus g\in\ker S$, $S(f\oplus g) = P(\delta_1 f\oplus\bar{\delta}_2\Delta\Omega f) = (\delta_1 f - \Theta_T w)\oplus(\bar{\delta}_2\Delta\Omega f - \Delta w) = 0$ for some $w\in H^2_{\mathfrak{D}}$. Hence $\delta_1 f = \Theta_T w$. Note that $\frac{1}{\delta_1}\Theta_e w = \Theta_i^* f$ is an element of $L^2_{\mathfrak{D}}$. However $\frac{1}{\delta_1}\Theta_e w$ is also analytic in the open unit disc, and therefore belongs to $H^2_{\mathfrak{D}}$. We conclude that $f = \Theta_i w'$, where $w' = \frac{1}{\delta_1}\Theta_e w\in H^2_{\mathfrak{D}}$. This shows that $f\oplus g\in H_1$, and hence $\ker S\subseteq H_1$.

(2) and (3) imply that $H_1 = \ker S$. Next we prove that $H_0 = \overline{SH}$.

(4) $\overline{SH}\subseteq H_0$. It was shown in [6] that $H_0 = \{f\oplus g\in H\colon \Theta_T g = \Delta_* f\}$, where $\Delta_* = (I_{\mathfrak{D}} - \Theta_T\Theta_T^*)^{1/2}$. For any $f\oplus g\in H$, $S(f\oplus g) = (\delta_1 f - \Theta_T w)\oplus(\bar{\delta}_2\Delta\Omega f - \Delta w)$ for some $w\in H^2_{\mathfrak{D}}$. Note that $(I_{\mathfrak{D}} - \Theta_T^*\Theta_T)\Omega = \Omega - \Theta_T^*\delta = \Omega(I_{\mathfrak{D}} - \Theta_T\Theta_T^*)$, whence $\Delta\Omega = \Omega\Delta_*$. Similarly, $\Theta_T\Delta = \Delta_*\Theta_T$. Thus $\Theta_T(\bar{\delta}_2\Delta\Omega f - \Delta w) = \bar{\delta}_2\Delta_*\Theta_T\Omega f - \Delta_*\Theta_T w = \bar{\delta}_2\Delta_*\delta f - \Delta_*\Theta_T w = \Delta_*(\delta_1 f - \Theta_T w)$, which shows that $S(f\oplus g)\in H_0$, and hence $\overline{SH}\subseteq H_0$.

(5) $S|H_0 = \delta_1(T_0)$. Since H_0 is the invariant subspace corresponding to $\Theta_T = \Theta_{*e}\Theta_{*i}$ and Θ_{*i} is inner from both sides, $H_0 = \{\Theta_{*e}u\oplus Z^{-1}(\Delta_2 u)\colon u\in H^2_{\mathfrak{D}}\}\ominus\{\Theta_T w\oplus\Delta w\colon w\in H^2_{\mathfrak{D}}\}$, where $\Delta_2 = (I_{\mathfrak{D}} - \Theta_{*e} * \Theta_{*e})^{1/2}$ and Z is the unitary operator from $\overline{\Delta L^2_{\mathfrak{D}}}$ onto $\overline{\Delta_2 L^2_{\mathfrak{D}}}$ such that $Z(\Delta v) = \Delta_2\Theta_{*i}v$ for $v\in L^2_{\mathfrak{D}}$ (cf. [4], p. 288). For any $\Theta_{*e}u\oplus Z^{-1}(\Delta_2 u)\in H_0$, we have $S(\Theta_{*e}u\oplus Z^{-1}(\Delta_2 u)) = (\delta_1\Theta_{*e}u - \Theta_T w)\oplus(\bar{\delta}_2\Delta\Omega\Theta_{*e}u - \Delta w)$ for some $w\in H^2_{\mathfrak{D}}$. Since Θ_T, along with Θ_{*e} and Θ_{*i}, admits a scalar multiple, $\Theta_T(t)^{-1} = \Theta_{*i}(t)^{-1}\Theta_{*e}(t)^{-1}$ exists for almost all t. Therefore, $\Omega = \delta\Theta_T^{-1} = \delta\Theta_{*i}^{-1}\Theta_{*e}^{-1}$ a.e. We have $Z(\bar{\delta}_2\Delta\Omega\Theta_{*e}u) = \Delta_2\Theta_{*i}\bar{\delta}_2\Omega\Theta_{*e}u = \Delta_2\Theta_{*i}\bar{\delta}_2\delta\Theta_{*i}^{-1}\Theta_{*e}^{-1}\Theta_{*e}u = \delta_1\Delta_2 u$, and it follows that $S(\Theta_{*e}u\oplus Z^{-1}(\Delta_2 u)) = (\delta_1\Theta_{*e}u - \Theta_T w)\oplus(\delta_1 Z^{-1}(\Delta_2 u) - \Delta w)$. This shows that $S|H_0 = \delta_1(T_0)$.

(6) $\overline{SH} = H_0$. Since δ_1 is outer, $\delta_1(T_0)$ is a quasi-affinity (cf. [4], p. 118). Hence $\overline{\delta_1(T_0)H_0} = H_0$. By (4) and (5), this implies that $\overline{SH} = H_0$.

The next lemma is needed in the proof of Theorem 3.3.

Lemma 2. *Let T be a c.n.u. weak contraction on H and let H_0, H_1 be subspaces of H such that $T_0 = T|H_0$ and $T_1 = T|H_1$ are the C_0 and C_{11} parts of T, respectively. If H_0', $H_0'\subseteq H$ are invariants subspaces for T such that $H_0'\vee H_1' = H$ and $T|H_0'\in C_0$, $T|H_1'\in C_{11}$, then $H_0 = H_0'$ and $H_1 = H_1'$.*

Proof. The maximality property of H_0 and H_1 implies that $H_0' \subseteq H_0$ and $H_1' \subseteq H_1$ (cf. [4], p. 331). Now we show that $H_0 \subseteq H_0'$. Since $H_0 = \overline{\operatorname{ran} S}$ where S is the operator defined in Theorem 1, for any $h \in H_0$ and $\varepsilon > 0$ there exists some k in H such that $\|h - Sk\| < \varepsilon$. The hypothesis $H = H_0' \vee H_1'$ implies that $\|k - k_0 - k_1\| < \varepsilon$ holds for some $k_0 \in H_0'$ and $k_1 \in H_1'$. Hence $\|Sk - Sk_0 - Sk_1\| = \|Sk - Sk_0\| < \|S\| \varepsilon$, and it follows that $\|h - Sk_0\| < (1 + \|S\|)\varepsilon$. Since $Sk_0 = \delta_1(T_0)k_0 = \delta_1(T)k_0 \in H_0'$ and ε is arbitrary, we conclude that $h \in H_0'$ and hence $H_0' = H_0$. $H_1' = H_1$ can be proved in a similar fashion by noting that $H_1 = \overline{\operatorname{ran} m(T)}$ and $H_0 = \ker m(T)$, where m denotes the minimal function of T_0.

Now we have the following main theorem.

Theorem 3. *Let T be a c.n.u. weak contraction on H and let H_0, H_1 be subspaces of H such that $T_0 = T|H_0$ and $T_1 = T|H_1$ are the C_0 and C_{11} parts of T, respectively. Then the following lattices are isomorphic:*

$$\operatorname{Hyperlat} T, \quad \operatorname{Hyperlat} T_0 \oplus \operatorname{Hyperlat} T_1, \quad \textit{and} \quad \operatorname{Hyperlat}(T_0 \oplus T_1).$$

Proof. Since T_0 and T_1 are of class C_{00} and of class C_{11}, respectively, Hyperlat $T_0 \oplus$ Hyperlat $T_1 \cong$ Hyperlat$(T_0 \oplus T_1)$ follows from Prop. 3 and Lemma 4 of [2].

Next we show that a subspace $K \subseteq H$ is hyperinvariant for T if and only if $K = K_0 \vee K_1$ where $K_0 \subseteq H_0$ and $K_1 \subseteq H_1$ are hyperinvariant for T_0 and T_1, respectively. To prove one direction, let $K \subseteq H$ be hyperinvariant for T and let $K_0 = K \cap H_0$, $K_1 = K \cap H_1$. Note that $\sigma(T|K) \subseteq \sigma(T)$ [1] and hence $T|K$ is also a weak contraction. Thus K_0 and K_1 are subspaces of K on which the C_0 and C_{11} parts of $T|K$ act (cf. [4], p. 332). We have $K = K_0 \vee K_1$. Now we show the hyperinvariance of K_0 and K_1. Note that $H_0 = \overline{SH}$, where S is the operator defined in Theorem 1. For any $S_0 \in \{T_0\}'$, consider the operator $S_0 S$ on H. It is easily seen that $S_0 S \in \{T\}'$. Since $K_0 = K \cap H_0$ is hyperinvariant for T, $S_0 S K_0 \subseteq K_0$. As proved in Theorem 1, $S|H_0 = \delta_1(T_0)$ for some outer function δ_1. Thus $\overline{SK_0} = \overline{\delta_1(T|K_0)K_0} = K_0$. It follows that $S_0 K_0 \subseteq K_0$ and hence K_0 is hyperinvariant for T_0. That K_1 is hyperinvariant for T_1 can be proved similarly by noting that $H_1 = \overline{m(T)H}$ where m is the minimal function of T_0 and $m(T|K_1)$, being an analytic function of a c.n.u. C_{11} contraction, is a quasi-affinity (cf. [4], p. 123).

To prove the converse, let $S \in \{T\}'$ and $S_0 = S|H_0$, $S_1 = S|H_1$. It is obvious that $S_0 \in \{T_0\}'$ and $S_1 \in \{T_1\}'$. If $K_0 \subseteq H_0$ and $K_1 \subseteq H_1$ are hyperinvariant for T_0 and T_1, respectively, then $S_0 K_0 \subseteq K_0$ and $S_1 K_1 \subseteq K_1$. Hence $S(K_0 \vee K_1) \subseteq K_0 \vee K_1$, which shows that $K_0 \vee K_1$ is hyperinvariant for T and proves our assertion.

That K_0 and K_1 are uniquely determined by K follows from Lemma 2, and it is easily seen that Hyperlat $T \cong$ Hyperlat $T_0 \oplus$ Hyperlat T_1.

In [11] a specific description of the elements in Hyperlat T for a special class of c.n.u. weak contractions is given.

Corollary 4. *Let* T_1, T_2 *be c.n.u. weak contractions with finite defect indices. If* T_1 *is quasi-similar to* T_2, *then* Hyperlat T_1 *is isomorphic to* Hyperlat T_2.

Proof. Let T_{10}, T_{20} be the C_0 parts of T_1, T_2 and T_{11}, T_{21} be their C_{11} parts, respectively. If T_1 is quasi-similar to T_2, then T_{10}, T_{11} are quasi-similar to T_{20}, T_{21}, respectively (cf. [10]). Since T_1, T_2 have finite defect indices, T_{10}, T_{20} are of class $C_0(N)$ and the defect indices of T_{11}, T_{21} are also finite. Thus Hyperlat $T_{10} \cong$ Hyperlat T_{20} and Hyperlat $T_{11} \cong$ Hyperlat T_{21} (cf. [7] and [14], resp.). Now Hyperlat $T_1 \cong$ $\cong$ Hyperlat T_2 follows from Theorem 3.

Recall that a c.n.u. weak contraction T is *multiplicity-free* if T admits a cyclic vector and that T is multiplicity-free if and only if its C_0 part and C_{11} part are (cf. [12]).

Corollary 5. *Let* T *be a c.n.u. multiplicity-free weak contraction on* H *with defect indices* $n<+\infty$. *Let* $K \subseteq H$ *be an invariant subspace for* T *with the corresponding regular factorization* $\Theta_T = \Theta_2 \Theta_1$. *Then the following are equivalent to each other:*

(1) $K \in$ Hyperlat T;

(2) *the intermediate space of* $\Theta_T = \Theta_2 \Theta_1$ *is of dimension* n.

Proof. (1)$\Rightarrow$(2). If $K \in$ Hyperlat T, then, as proved before, $T|K$ is a weak contraction. Hence its characteristic function admits a scalar multiple, which implies that the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension n.

(2)$\Rightarrow$(1). The hypothesis implies that $T|K$ has equal defect indices. It is easily seen that a c.n.u. contraction S with finite equal defect indices is a weak contraction if and only if $\det \Theta_S \not\equiv 0$. Since $\det \Theta_T \not\equiv 0$ implies that $\det \Theta_1 \not\equiv 0$, it follows that $T|K$ is a weak contraction. Let K_0, K_1 be subspaces of K on which the C_0 and C_{11} parts of $T|K$ act. We have $K = K_0 \vee K_1$. It follows from the proof of Theorem 3 that we have only to show that K_0 and K_1 are hyperinvariant for $T_0 = T|H_0$ and $T_1 = T|H_1$, the C_0 and C_{11} parts of T, respectively. Since $K_0 \subseteq H_0$ is invariant for the multiplicity-free $C_0(N)$ contraction T_0, it is hyperinvariant for it (cf. [8], Corollary 4.4). On the other hand, T_1 is a multiplicity-free C_{11} contraction on H_1 with finite defect indices and $K_1 \subseteq H_1$ is such that $T_1|K_1 \in C_{11}$. It follows easily from Theorem 1 of [14] that K_1 is hyperinvariant for T_1, completing the proof.

The next corollary gives necessary and sufficient conditions that Lat T be equal to Hyperlat T for the operators we considered.

Corollary 6. *Let T be a c.n.u. weak contraction on H with defect indices $n<+\infty$. Let $T_0=T|H_0$ and $T_1=T|H_1$ be its C_0 and C_{11} parts, respectively, and let Θ_e be the outer factor of the characteristic function Θ_T of T. Then the following conditions are equivalent:*

(1) *Lat* $T=$ Hyperlat T;

(2) *Lat* $T_0=$ Hyperlat T_0 *and Lat* $T_1=$ Hyperlat T_1;

(3) *T_0 and T_1 are multiplicity-free and $\Theta_e(t)$ is isometric on a set of positive Lebesgue measure;*

(4) *T is multiplicity-free and $\Theta_T(t)$ is isometric on a set of positive Lebesgue measure.*

Proof. (1)⇒(2). We only show that Lat $T_0=$ Hyperlat T_0; Lat $T_1=$ Hyperlat T_1 can be proved similarly. To this end, let $K_0\subseteqq H_0$ be an invariant subspace for T_0. It is obvious that $K_0\in$ Lat $T=$ Hyperlat T. Let S be the operator defined in Theorem 1. Then $H_0=\overline{SH}$ and $S|H_0=\delta_1(T_0)$ for some outer function δ_1. For any $S_0\in\{T_0\}'$, S_0S is an operator in $\{T\}'$. Hence $\overline{S_0SK_0}=\overline{S_0\delta_1(T|K_0)K_0}=$ $=\overline{S_0K_0}\subseteqq K_0$, which shows that $K_0\in$ Hyperlat T_0 and proves our assertion.

(2)⇒(3). This follows from Corollary 4.4 of [8] and Theorem 4.3 of [15].

(3)⇒(4). This follows from the remark before Corollary 5 and the fact that $\Theta_T(t)$ is isometric if and only if $\Theta_e(t)$ is.

(4)⇒(1). Let $K\in$ Lat T with the corresponding regular factorization $\Theta_T=\Theta_2\Theta_1$. In light of Corollary 5 it suffices to show that the intermediate space of $\Theta_T=$ $=\Theta_2\Theta_1$ is of dimension n. Note that rank $\Delta(t)=$ rank $\Delta_1(t)+$ rank $\Delta_2(t)$ a.e., where $\Delta(t)=(I-\Theta_T(t)^*\Theta_T(t))^{1/2}$ and $\Delta_j(t)=(I-\Theta_j(t)^*\Theta_j(t))^{1/2}$, $j=1,2$. The hypothesis implies that $\Delta(t)=0$ on a set of positive Lebesgue measure, say α. It follows that $\Delta_1(t)=\Delta_2(t)=0$ on α, and hence $\Theta_1(t)$ and $\Theta_2(t)$ are isometric for t in α. Therefore, the intermediate space of $\Theta_T=\Theta_2\Theta_1$ is of dimension n, as asserted.

We remark that the preceding corollary generalizes part of the main result in [9].

Corollary 7. *Let T be a c.n.u. multiplicity-free weak contraction with finite defect indices. If $K_1, K_2\in$ Hyperlat T and $T|K_1$ is quasi-similar to $T|K_2$, then $K_1=K_2$.*

Proof. Since $K_1, K_2\in$ Hyperlat T, $T|K_1$, $T|K_2$ are weak contractions. Considering the C_0 and C_{11} parts of $T|K_1$ and $T|K_2$ and using the corresponding results for multiplicity-free $C_0(N)$ contractions and C_{11} contractions, we can deduce that $K_1=K_2$ (cf. [3], Theorem 2 and [14], Corollary 3). We leave the details to the interested readers.

The next theorem, being another application of Theorem 3, is interesting in itself.

Theorem 8. *Let T be a c.n.u. weak contraction on H with finite defect indices. Then* Hyperlat T *is (lattice) generated by subspaces of the forms* $\overline{\operatorname{ran} S}$ *and* $\ker V$, *where* $S, V \in \{T\}''$.

Proof. Let $T_0 = T|H_0$ and $T_1 = T|H_1$ be the C_0 and C_{11} parts of T, respectively, and let $K \in$ Hyperlat T. Since $T|K$ is a c.n.u. weak contraction, we may consider its C_0 part $T|K_0$ and C_{11} part $T|K_1$. By Theorem 1, $H_0 = \overline{SH}$ for some $S \in \{T\}''$. Since $K_0 \subseteq H_0$ is hyperinvariant for the $C_0(N)$ contraction T_0 (by Theorem 3), it follows from [13] that $K_0 = \bigvee_{i=1}^{n} [\ker \psi_i(T_0) \cap \overline{\xi_i(T_0) H_0}] = \bigvee_{i=1}^{n} [\ker \psi_i(T_0) \cap \overline{\xi_i(T) SH}]$, where ψ_i, ξ_i are inner functions, $i = 1, \dots, n$. On the other hand, since $K_1 \subseteq H_1$ is hyperinvariant for T_1 (by Theorem 3 again), Theorem 3.6 of [15] implies that $K_1 = \overline{VH_1}$ for some $V \in \{T_1\}''$. Hence $K_1 = \overline{Vm(T)H}$, where m denotes the minimal function of T_0. We claim that $K = \bigvee_{i=1}^{n} [\ker \psi_i(T) \cap \overline{\xi_i(T) SH}] \vee \overline{Vm(T)H}$. Indeed, this follows from $K = K_0 \vee K_1$ and the fact that $\ker \psi(T_0) = \ker \psi(T)$ for any $\psi \in H^\infty$. Since it is easily seen that $\psi_i(T)$, $\xi_i(T) S \in \{T\}''$ for all i and $Vm(T) \in \{T\}''$, the proof is complete.

Corollary 9. *Let T be a c.n.u. multiplicity-free weak contraction on H with finite defect indices and let K be a subspace of H. Then the following are equivalent:*

(1) $K \in$ Hyperlat T;

(2) $K = \overline{\operatorname{ran} S}$ *for some* $S \in \{T\}''$;

(3) $K = \ker V$ *for some* $V \in \{T\}''$.

Proof. The equivalence of (2) and (3) is easily established by considering T^* and $K^\perp$. (2)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(2) is proved by following the same line of arguments in the proof of Theorem 8 and noting that any hyperinvariant subspace for a multiplicity-free $C_0(N)$ contraction T is of the form $\overline{\operatorname{ran} \xi(T)}$ for some inner function ξ.

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