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On a set-mapping problem of Hajnal and Máté

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In the course of a wide-ranging survey of combinatorial set theory, A. Hajnal and A. Máté prove by a forcing argument the consistency of the following combinatorial principle with the Generalized Continuum Hypothesis GCH, and ask whether if follows from the Axiom of Constructibility $V=L$ (see [4], Thm. 5.4 and Problem 8).

(HM) There is a function f: $\{(\alpha, \beta, \gamma)\colon \alpha<\beta<\gamma<\omega_2\}\to\omega_2$ such that for any uncountable $A\subseteq\omega_2$ there exist $\alpha<\beta<\gamma$ in A with $f(\alpha, \beta, \gamma)\in A$.

(We are using the same standard set-theoretic notation as [4], except that we use ω_α rather than $\aleph_\alpha$ for the αth transfinite cardinal.) We present here a proof that $V=L$ implies HM by a metamathematical method which we feel has interest beyond this particular problem.

1. Jensen's Absoluteness Principle. The language $L[Q_1, Q_2]$ is just like ordinary first order logic, except for the presence of two generalized quantifiers:

$Q_1 x\varphi(x)$ meaning: There exist uncountably many x such that $\varphi(x)$.

$Q_2 x\varphi(x)$ meaning: There exist at least ω_2 many x such that $\varphi(x)$.

As is explained in some detail in the final paragraphs of [3], R. B. JENSEN's work on model theory establishes the following principle:

($*$) Let φ be a sentence of $L[Q_1, Q_2]$. Suppose there is a Boolean-valued extension $V^{\mathcal{B}}$ of the universe of set theory in which GCH holds, such that in $V^{\mathcal{B}}$ it is true that φ has a model. Then already in the constructible universe L it is true that φ has a model.

This principle provides a method for turning a consistency proof for a combinatorial principle ψ into a derivation of ψ from $V=L$. Namely, it suffices to find a sentence φ of $L[Q_1, Q_2]$ for which we can prove, using GCH if needs be, that φ has a model if and only if ψ holds. Unfortunately this method does not seem to

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apply directly to the principle HM. What we will show here is that it applies to a certain principle which implies HM.

2. Quagmires. The principle we have in mind is just a bit complicated. A *tree* is a partial order $\mathcal{T}=(T, <)$ in which the predecessors of any element are well ordered. The order type of the predecessors of $t \in T$ is called the *rank* $|t|$ of t. The αth *level* T_α of the tree is the set of t with $|t|=\alpha$, and its *height* the least α with $T_\alpha=\emptyset$. For present purposes a *Kurepa tree* may be defined as a tree of height ω_1+1 in which T_{ω_1} has cardinality ω_2, distinct elements of T_{ω_1} have distinct sets of predecessors, and T_α is countable for $\alpha<\omega_1$.

A *quagmire* $(T, <, \lhd, Q)$ is a Kurepa tree $(T, <)$ equipped with a binary relation $\lhd$ and a trinary function Q such that:

(1) $\lhd$ holds only between elements of equal rank, and linearly orders each level T_α of the tree.

(2) Q is defined on those triples (y', x', x) with $y' \lhd x' < x$, and for any such, $y' < Q(y', x', x) \lhd x$.

(3) *(Commutativity)* If $y'' \lhd x'' < x' < x$, then $Q(Q(y'', x'', x'), x', x) = Q(y'', x'', x)$.

(4) *(Coherence)* If $z' \lhd y' \lhd x' < x$, then $Q(z', y', Q(y', x', x)) = Q(z', x', x)$.

(5) *(Completeness)* If $y \lhd x \in T_{\omega_1}$, then for some $\alpha<\omega_1$, $Q(P_\alpha(y), P_\alpha(x), x)=y$.

Here P_α is the *projection* function which assigns to any t with $|t| \geqq \alpha$ the unique $u<t$ with $|u|=\alpha$. Note that the condition $Q(P_\alpha(y), P_\alpha(x), x)=y$ implies $P_\alpha(y) \lhd P_\alpha(x)$, else Q would not be defined on this triple.

What we are going to show, assuming GCH, is that:

(A) The existence of a quagmire implies HM.

(B) There is a sentence of $L[Q_1, Q_2]$ which has a model if and only if there exists a quagmire.

(C) There is a Boolean-valued extension $V^{\mathcal{B}}$ of the universe of set theory in which GCH holds and there exists a quagmire.

3. Proof of (A). We will show, assuming CH, that *if* there exists a quagmire $(T, <, \lhd, Q)$, then HM holds. We begin by deriving from these assumptions the following combinatorial principle, due to Silver. (For its consequences, cf. [5].)

(W) There exists a Kurepa tree $(T, <)$ equipped with a function W defined on ω_1, such that:
For $\alpha<\omega_1$, $W(\alpha)$ is a countable family of subsets of the level T_α. For any countable $S \subseteq T_{\omega_1}$ there exists $\alpha<\omega_1$ such that for any $\alpha \leqq \beta < \omega_1$, $\{P_\beta(x) \colon x \in S\} \in W(\beta)$.

Indeed, to derive W given CH and a quagmire, note that for each $\alpha<\omega_1$, the αth level T_α of the quagmire is countable, so its power set can be enumerated in an ω_1-sequence $X_{\alpha,\beta}$ for $\beta<\omega_1$. For $x \in T$ and $\alpha, \beta < |x|$ let $S(\alpha, \beta, x)$ be the image $\{Q(y', P_\alpha(x), x)\colon y' \lhd P_\alpha(x) \;\&\; y' \in X_{\alpha,\beta}\}$ of the βth subset of T_α under the map $Q(\cdot, P_\alpha(x), x)$. For $\gamma < \omega_1$, let $W(\gamma) = \{S(\alpha, \beta, x)\colon \alpha, \beta < \gamma \;\&\; x \in T_\gamma\}$, a countable family of subsets of T_γ.

Now it follows by the Completeness condition in the definition of quagmire that any $x \in T_{\omega_1}$ has at most ω_1 $\lhd$-predecessors. Hence given a countable $S \subseteqq T_{\omega_1}$, there must exist an x with $y \lhd x$ for all $y \in S$. Again by Completeness, for each $y \in S$ there is then an $\alpha(y) < \omega_1$ with $y = Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), x)$. Let $\alpha = \sup\{\alpha(y)\colon y \in S\}$. By Commutativity, for any $\alpha \leqq \delta < \omega_1$ and $y \in S$, the element $y' = Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), P_\delta(x))$ satisfies $P_{\alpha(y)}(y) < y' \lhd P_\delta(x)$ and $y' < Q(y', P_\delta(x), x) = Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), x) = y$. Hence $y' = P_\delta(y)$ and $Q(P_\delta(y), P_\delta(x), x) = y$.

If now we fix a β such that $\{P_\alpha(y)\colon y \in S\} = X_{\alpha,\beta}$ and let γ be $> \alpha$ and β, then for any $\gamma \leqq \delta < \omega_1$ it is readily verified that $\{P_\delta(y)\colon y \in S\} = S(\alpha, \beta, P_\delta(x)) \in W(\delta)$, which suffices to prove Silver's principle W above. This established, we go on, still assuming CH and the existence of a quagmire, to derive the following combinatorial principle, due to Hajnal and Máté:

(HM′) There exists a sequence of functions $H_\alpha\colon \omega_2 \to \omega_2$, for $\alpha < \omega_1$, such that for any infinite $S \subseteqq \omega_2$ there exists a $\gamma < \omega_1$ such that for any $\gamma \leqq \delta < \omega_1$ there exists an $x \in S$ with $H_\delta(x) \in S$.

Towards proving this, we first note that we may assume without loss of generality that in our quagmire no level T_α has a $\lhd$-least element. (Otherwise we can construct a new quagmire with this property by taking:

$$T' = \omega \times T,$$

$$(m, x) <' (n, y) \;\leftrightarrow\; m = n \;\&\; x < y,$$

$$(m, x) \lhd' (n, y) \;\leftrightarrow\; x \lhd y \;\text{ or }\; (x = y \;\&\; m > n),$$

$$Q'((m, y'), (n, x'), (n, x)) = (m, Q(y', x', x)),$$

i.e. by replacing each element x of the original quagmire by a sequence $\ldots (2, x), (1, x), (0, x)$.)

This settled, we go on to construct for each $\alpha < \omega_1$ a map $h_\alpha\colon T_\alpha \to T_\alpha$ such that $h_\alpha(x) \lhd x$ for each $x \in T_\alpha$, and for any infinite $S \in W(\alpha)$ there exists a $y \in S$ with $h_\alpha(y) \in S$. Since $W(\alpha)$ is countable, this can be accomplished by a simple diagonal construction in ω stages, whose details are left to the reader. Having the h_α, we define maps $H_\alpha\colon T_{\omega_1} \to T_{\omega_1}$ by $H_\alpha(x) = Q(h_\alpha(P_\alpha(x)), P_\alpha(x), x)$.

Now for any denumerably infinite $S \subseteqq T_{\omega_1}$, our arguments above establish two things. First, there is an $x \in T_{\omega_1}$ and an $\alpha < \omega_1$ such that for all $y \in S$ and $\alpha \leqq \delta < \omega_1$, $Q(P_\delta(y), P_\delta(x), x) = y \lhd x$. Second, there is a $\beta < \omega_1$ such that for all $\beta \leqq \delta < \omega_1$, $\{P_\delta(y)\colon y \in S\} \in W(\delta)$. If $\gamma = \max(\alpha, \beta)$, then for any $\gamma \leqq \delta < \omega_1$, by construction there exist $y, z \in S$ with $h_\delta(P_\delta(y)) = P_\delta(z)$. Now by Coherence $H_\delta(y) =$

$=Q(h_\delta(P_\delta(y)), P_\delta(y), y)=Q(P_\delta(z), P_\delta(y), y)=Q(P_\delta(z), P_\delta(x), x)=z$, i.e. there is a $y\in S$ with $H_\delta(y)\in S$.

If we assume, as we may without loss of generality, that T_{ω_1} consists precisely of the ordinals $<\omega_2$, then this is precisely what is required to establish the principle HM′ above. Now as HAJNAL and MÁTÉ [4] show that HM′ and the existence of a Kurepa tree imply HM, our proof that CH and the existence of a quagmire imply HM is complete.

4. Proof of (B). VAUGHT [6] long ago proved that the existence of a Kurepa tree is equivalent to the existence of a model for a certain sentence φ of $L[Q_1, Q_2]$. For completeness we recall his argument here: φ will involve two singulary predicates T, O, plus two binary predicates $<_T$, $<_O$, plus a singulary function symbol r, plus a constant w. φ is the conjunction of the sentences (whose precise formalization we leave to the reader) expressing:

(1) $<_T$ partially orders T in such a way that the predecessors of any element are linearly ordered.

(2) $<_O$ linearly orders O, with last element w.

(3) r maps T onto O in such a way that for any $t\in T$ and $u\in O$, $u<_O r(t)$ if and only if there exists $t'<_T t$ with $u=r(t')$.

(4) $Q_1 u O(u)$ & $\forall u(u<_O w\to\neg Q_1 u'(u'<_O u))$

(5) $Q_2 t(T(t)$ & $r(t)=w)$ & distinct t with $r(t)=w$ have distinct sets of $<_T$-predecessors.

(6) $\forall u(u<_O w\to\neg Q_1 t(T(t)$ & $r(t)=u))$

If $(T, <_T)$ is a Kurepa tree, we get a model of this sentence φ by interpreting O as the set of ordinals $\leqq\omega_1$, $<_O$ as the usual order on this set, w as ω_1, and r as the rank function. Conversely, if $(T, <_T, O, <_O, w, r)$ is a model of φ, then using (4) above one easily sees that there is a $<_O$-cofinal subset Z of $\{u\in O\colon u<_O w\}$ which is well ordered by $<_O$ in order type ω_1. Then restricting $<_T$ to $\{t\in T\colon r(t)=w$ or $r(t)\in Z\}$ we get a Kurepa tree.

To get a formula φ' which has a model if and only if there exists a quagmire, simply take new symbols $\lhd$ and Q and conjoin the above φ with the sentences expressing conditions (1)—(5) in the definition of quagmire in § 2 above.

5. Proof of (C). It remains only to prove, assuming GCH, that some suitable set of forcing conditions gives rise to a Boolean-valued extension of the universe of set theory in which GCH holds and there exists a quagmire. The proof is so similar to the proof of the consistency of HM′ in [4] and the proof of the consistency of Silver's W in [2], that we leave most details to the reader.

As our forcing conditions we take the set $\mathscr{P}$ of all sixtuples $p=(\alpha_p, T_p, <_p, \lhd_p, Q_p, \Lambda_p)$ such that:

(0) T_p is a countable subset of ω_1.

(1) $(T_p, <_p)$ is a tree of height $\alpha_p+1<\omega_1$.

(2)—(5) in the definition of quagmire in § 2 above hold for $\lhd_p$ and Q_p.

(6) Λ_p maps a subset of ω_2 onto the α_pth level of the tree $(T_p, <_p)$, and is order preserving in the sense that for $\xi<\eta$ in dom Λ_p, we have $\Lambda_p(\xi)\lhd_p\Lambda_p(\eta)$. Note that the requirement that Λ_p be order preserving means that Λ_p is completely determined by its domain.

We partially order $\mathcal{P}$ by setting $p<q$ if and only if:

(7) $\alpha_p>\alpha_q$ and $T_p\supseteqq T_q$ and $<_p, \lhd_p, Q_p$ extend $<_q, \lhd_q, Q_q$ respectively and dom $\Lambda_p\supseteqq$ dom Λ_q.

(8) For all $\xi\in$ dom Λ_q, $\Lambda_q(\xi)<_p\Lambda_p(\xi)$; and for $\xi<\eta$ in dom Λ_q, $Q_p(\Lambda_q(\xi), \Lambda_q(\eta), \Lambda_p(\eta))=\Lambda_p(\xi)$.

In order to show that $\mathcal{P}$ does what it should, we need the following:

Lemma. (a) *$\mathcal{P}$ is σ-closed; i.e. whenever $p_n\in\mathcal{P}$ for $n\in\omega$ and $p_{n+1}<p_n$ for all n, then there exists $p\in\mathcal{P}$ with $p<p_n$ for all n.*

(b) *$\mathcal{P}$ has the ω_2-chain condition; i.e. no set of pairwise incompatible elements of $\mathcal{P}$ has cardinality ω_2.*

(c) *For each $\alpha<\omega_1$ and $\xi<\omega_2$, $\{p$: $\alpha_p>\alpha$ & sup dom $\Lambda_p>\xi\}$ is dense in $\mathcal{P}$.*

The proof of the easy parts (a) and (c) will be left to the reader. As for part (b), let $A\subseteqq\mathcal{P}$ have cardinality ω_2. Assuming CH, there must exist an $A'\subseteqq A$ of cardinality ω_2 and fixed α, T, $<$, $\lhd$, and Q such that for all $p\in A'$, $\alpha_p=\alpha$, $T_p=T$, $<_p=<$, $\lhd_p=\lhd$, $Q_p=Q$. For assuming CH there are only ω_1 possibilities for these items.

{dom Λ_p: $p\in A'\}$ forms a set of ω_2 countable subsets of ω_2. By a well-known result of Erdős and Rado (cf. Thm. 2.3 of [4] or Lemma 3.6 of [2]) there exists a sequence p_ν, $\nu<\omega_2$ of elements of A' and a fixed $X\subseteqq\omega_2$ such that for any $\mu<\nu<\omega_2$, dom $\Lambda_{p_\mu}\cap$dom $\Lambda_{p_\nu}=X$ and sup dom $\Lambda_{p_\mu}<$inf (dom $\Lambda_{p_\nu}-X$).

Let $p=p_0$, $q=q_0$, $Y=$dom Λ_p, $Z=$dom Λ_q. Note $\Lambda_p|X=\Lambda_q|X$. To establish part (b) of the Lemma it will suffice to construct an $r\in\mathcal{P}$ with $r<p$ and $r<q$. This may be accomplished by taking:

$\alpha_r=\alpha+1$,

$T_r=T\cup\{t_\xi$: $\xi\in Y\cup Z\}$ where the t_ξ are distinct elements of ω_1-T,

$<_r=$the extension of $<$ defined so that $\Lambda_p(\eta)<_r t_\eta$ for $\eta\in Y$ and $\Lambda_q(\zeta)<_r t_\zeta$ for $\zeta\in Z$,

$\lhd_r=$the extension of $\lhd$ defined so that $t_\eta\lhd_r t_\zeta$ for $\eta<\zeta$ in $Y\cup Z$,

$Q_r=$the extension of Q defined so that $Q_r(\Lambda_p(\xi), \Lambda_p(\eta), t_\eta)=t_\xi$ for $\xi<\eta$ in Y, and $Q_r(\Lambda_q(\xi), \Lambda_q(\zeta), t_\zeta)=t_\xi$ for $\xi<\zeta$ in Z,

$\Lambda_r=$the function $\Lambda_r(\xi)=t_\xi$ for $\xi\in Y\cup Z$.

Details are left to the reader.

With the Lemma established, we let $\mathscr{B}$=the complete Boolean algebra of regular open subsets of $\mathscr{P}$. Parts (a) and (b) of the above Lemma and standard forcing lemmas (for which see e.g. [2]) imply that, assuming GCH, in the Boolean-valued extension $V^{\mathscr{B}}$ all cardinals are preserved and GCH holds.

Moreover if $G \in V^{\mathscr{B}}$ is a generic subset of $\mathscr{P}$, then the $p \in G$ can be fitted together to produce a quagmire. Again details are left to the reader. This completes the proof that $V=L$ implies HM.

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