Acta Sci. Math., 41 (1979), 289—293

Scalar central elements in an algebra over a principal ideal domain

L. O. CHUNG and JIANG LUH

1. Introduction. Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar commutative if, for each $x, y \in A$, there exists $\alpha \in R$ depending on x, y such that $xy = \alpha yx$. RICH [3] proves that if A is scalar commutative and if R is a field then A is either commutative or anticommutative. KOH, LUH, and PUTCHA [1] prove that if A is scalar commutative with 1 and if R is a principal ideal domain then A is commutative. Recently, LUH and PUTCHA [2] generalized these results by proving that if A is an algebra with 1 over a principal ideal domain R such that for each $x, y \in A$ there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha xy = \beta yx$, then A is commutative.

In this paper a "local" scalar commutativity will be studied. We shall call an element $x \in A$ *scalar central* if for each $y \in A$, there exist $\alpha, \beta \in R$ depending on y such that $(\alpha, \beta) = 1$ and $\alpha xy = \beta yx$. We shall prove that if A is an associative algebra over a principal ideal domain R and if $x \in A$ is scalar central then there exists a positive integer n such that $x^n y = x^{n-1} yx = x^{n-2} yx^2 = \ldots = yx^n$ for *all* $y \in A$. If, in addition, A has 1 then $x^2 y = xyx = yx^2$. Therefore the results of Rich, Koh, Luh and Putcha for associative algebras immediately follow.

Throughout this paper A will denote an associative algebra over a principal ideal domain R, C will denote the center of A, $\mathbf{Z}^+$ the set of all positive integers and $\mathbf{N}$ the set of natural numbers. If $a, b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$ then (α, β) denotes the greatest common divisor of α and β. If $a \in A$ then the order of a, denoted by $o(a)$, is the generator of the ideal $I = \{\alpha | \alpha \in R, \alpha a = 0\}$ of R. $o(a)$ is unique up to associates.

2. Main results. Throughout this section x will denote a scalar central element in A. Let y be an arbitrary element in A. We assume $\alpha, \beta, \alpha_1, \beta_1 \in R$ to be such that

Received April 17, 1978, in revised form July 20, 1978.

$(\alpha, \beta)=(\alpha_1, \beta_1)=1$ and

(1) $$\alpha xy = \beta yx,$$

(2) $$\alpha_1 x(x+y) = \beta_1(x+y)x.$$

From (1) and (2), we obtain

(3) $$(\alpha_1\beta-\alpha\beta_1)xy = \beta(\beta_1-\alpha_1)x^2,$$

(4) $$(\alpha_1\beta-\alpha\beta_1)yx = \alpha(\beta_1-\alpha_1)x^2.$$

We begin with

Lemma 2.1. *If* $(\alpha_1-\beta_1)qx^k=0$, *where* $k\in\mathbf{Z}^+$, $k\geqq 2$ *and* $q\in R$, *then* $q[x^iy, x^{k-i}]=0$ *for* $i=0, 1, 2, \ldots, k-1$.

Proof. By (2), $\alpha_1 qx^{i+1}(x+y)x^{k-i-1}=\beta_1 qx^i(x+y)x^{k-i}$ which is reduced to

(5) $$\alpha_1 qx^{i+1}yx^{k-i+1} = \beta_1 qx^i yx^{k-i}.$$

In particular, $\beta_1^k qx^ky=\alpha_1^k qx^ky=\beta_1^k qyx^k=\alpha_1^k qyx^k$. Since $(\alpha_1, \beta_1)=1$, $qx^ky=qyx^k$. Thus, by (5), $\alpha_1^i qx^iyx^{k-i}=\beta_1^i qyx^k=\beta_1^i qx^ky=\alpha_1^i qx^ky$, and $\beta_1^i qx^iyx^{k-i}=\alpha_1^i qx^ky=$ $=\beta_1^i qx^ky$. Consequently, $\alpha_1^i q(x^iyx^{k-i}-x^ky)=\beta_1^i q(x^iyx^{k-i}-x^ky)=0$. Since $(\alpha_1^i, \beta_1^i)=1$, $q(x^iyx^{k-i}-x^ky)=0$. That is, $q[x^iy, x^{k-i}]=0$ as required.

It is clear that there exists an integer $n\geqq 3$ such that $o(x^n)=o(x^{n+1})$.

Lemma 2.2. *Suppose* $o(x^n)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{Z}^+$. *If* $p^lx^ny=0$ *for some* $l\in\mathbf{N}$, $l<m$, *then* $[x^iy, x^{n-i}]=0$ *for* $i=0, 1, 2, \ldots, n-1$.

Proof. We proceed by induction on l. Suppose $l=0$. Then $x^ny=0$. By (3) and (4), we get $0=(\alpha_1\beta-\alpha\beta_1)x^ny=\beta(\beta_1-\alpha_1)x^{n+1}=(\alpha_1\beta-\alpha\beta_1)xyx^{n-1}$, and $0=(\alpha_1\beta-\alpha\beta_1)xyx^{n-1}=\alpha(\beta_1-\alpha_1)x^{n+1}$. Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$ and $p^m|(\beta_1-\alpha_1)$. So $(\beta_1-\alpha_1)x^n=0$. Thus, by Lemma 2.1, $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Now we assume $l>0$ and $\alpha_1\beta-\beta\alpha_1=p^t\delta$, where $(p, \delta)=1$, $t\in\mathbf{N}$.

Suppose $t\geqq l$. Then, by (3), $0=p^t\delta x^ny=\beta(\beta_1-\alpha_1)x^{n+1}=p^t\delta xyx^n$, and hence by (4), $0=p^t\delta xyx^n=\alpha(\beta_1-\alpha_1)x^{n+1}$. Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$. Again by Lemma 2.1, $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Suppose $t<l$. Then, by (3), $0=p^l\delta x^ny=p^{l-t}\beta(\beta_1-\alpha_1)x^{n+1}$. So $p^m|p^{l-t}\beta(\beta_1-\alpha_1)$. By (3), $p^{l-t}p^t\delta xyx^{n-1}=0$ and, by (4), $p^{l-t}\alpha(\beta_1-\alpha_1)x^{n+1}=p^l\delta xyx^{n-1}=0$. Hence, we have $p^{l-t}(\beta_1-\alpha_1)x^{n+1}=0$ and $p^m|p^{l-t}(\beta_1-\alpha_1)$. Since $l<m$, $p^t|(\beta_1-\alpha_1)$ and $\beta_1-\alpha_1=p^t\gamma$, where $\gamma\in R$. Thus, by (3), $p^t\delta x^ny=\beta p^t\gamma x^{n+1}$, i.e. $p^tx^n(\delta y-\beta\gamma x)=0$. Since $t<l$, $[x^i(\delta y-\beta\gamma x), x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$, by the induction hypothesis. This implies that $\delta[x^iy, x^{n-i}]=0$. On the other hand, since $(\alpha_1-\beta_1)p^mx^{n+1}=0$, $p^m[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$, by Lemma 2.1. Since $(p^m, \delta)=1$, we obtain $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$. This completes the proof.

Lemma 2.3. *Suppose* $o(x^n)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{N}$. *Then* $[x^i y, x^{n-i}]=0$ *for* $i=0, 1, 2, \ldots, n-1$.

Proof. Again we let $\alpha_1\beta-\alpha\beta_1=p^t\delta$. Suppose $t\geqq m$. Then, by (3) and (4) respectively, we have

$$0=p^t\delta x^n y=\beta(\beta_1-\alpha_1)x^{n+1} \quad \text{and} \quad 0=p^t\delta y x^n=\alpha(\beta_1-\alpha_1)x^{n+1}.$$

Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$. By Lemma 2.1, $[x^i y, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Now suppose $t<m$. Then by (3), $0=p^m\delta x^n y=p^{m-t}\beta(\beta_1-\alpha_1)x^{n+1}$. So $p^t|\beta(\beta_1-\alpha_1)$. Let $\beta(\beta_1-\alpha_1)=p^t\gamma$, where $\gamma\in R$. Then, by (3), $p^t x^n(\delta y-\gamma x)=0$. By Lemma 2.2, $[x^i(\delta y-\gamma x), x^{n-i}]=0$. So $\delta[x^i y, x^{n-i}]=0$. On the other hand, since $(\alpha_1-\beta_1)p^m x^{n+1}=0$, $p^m[x^i y, x^{n-i}]=0$ by Lemma 2.1. Thus, $[x^i y, x^{n-i}]=0$ since $(p^m, \delta)=1$.

Lemma 2.4. *Suppose* $o(x^n)=p_1^{m_1}p_2^{m_2}\ldots p_s^{m_s}$, *where* $p_1, p_2, \ldots, p_s$ *are non-associate primes in* R, *and* $m_1, m_2, \ldots, m_s\in\mathbf{Z}^+$. *Then* $[x^i y, x^{n-i}]=0$ *for* $i=0, 1, 2, \ldots, n-1$.

Proof. Let $q_j=p_1^{m_1}\ldots p_{j-1}^{m_{j-1}}p_{j+1}^{m_{j+1}}\ldots p_s^{m_s}$, $j=1, 2, \ldots, s$. Then $q_j x$ is scalar central, $o((q_j x)^n)=o((q_j x)^{n+1})$, and hence, by Lemma 2.3, $q_j^n[x^i y, x^{n-i}]=[(q_j x)^i y, (q_j x)^{n-i}]=0$ for $j=1, 2, \ldots, s$; $i=0, 1, 2, \ldots, n-1$. Since the q_j's are relatively prime, we obtain $[x^i y, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Theorem 2.1. *Suppose* $x\in A$ *is scalar central and* $o(x^n)=o(x^{n+1})=0$, *where* $n\geqq 3$. *Then* $x\in C$.

Proof. Clearly $o(x^3)=0$. By (3), and (4) respectively, we obtain

$$(\alpha_1\beta-\alpha\beta_1)xyx=\beta(\beta_1-\alpha_1)x^3 \quad \text{and} \quad (\alpha_1\beta-\alpha\beta_1)xyx=\alpha(\beta_1-\alpha_1)x^3.$$

Hence $(\beta-\alpha)(\beta_1-\alpha_1)x^3=0$. This implies that $\beta=\alpha$ or $\beta_1=\alpha_1$. In either case, we have $xy=yx$. Since y is an arbitrary element in A, $x\in C$.

Theorem 2.2. *If* $x\in A$ *is scalar central then there exists* $n\in\mathbf{Z}^+$ *such that*

$$x^n y=x^{n-1}yx=x^{n-2}yx^2=\ldots=yx^n \quad \textit{for all} \quad y\in A.$$

Proof. This is an immediate consequence of Lemma 2.4 and Theorem 2.1.

3. Algebras with unity elements. We assume throughout this section that A is an algebra with 1 over a principal ideal domain R, and x is a scalar central element in A. Let y be an arbitrary element in A and $\alpha, \beta, \alpha_2, \beta_2\in R$ be such that $(\alpha, \beta)=(\alpha_2, \beta_2)=1$,

$$\alpha xy=\beta yx, \tag{1'}$$

$$\alpha_2 x(1+y)=\beta_2(1+y)x. \tag{2'}$$

Then

(3′) $$(\alpha_2\beta-\alpha\beta_2)xy = \beta(\beta_2-\alpha_2)x,$$

(4′) $$(\alpha_2\beta-\alpha\beta_2)yx = \alpha(\beta_2-\alpha_2)x.$$

Lemma 3.1. *If* $(\alpha_2-\beta_2)qx=0$, *where* $q\in R$, *then* $qxy=qyx$.

Proof. By (3′) and (4′), $(\alpha_2\beta-\alpha\beta_2)qxy=(\alpha_2\beta-\alpha\beta_2)qyx=0$. By (1′), $\alpha_2(\beta-\alpha)qxy=\beta_2(\beta-\alpha)qxy=0$. Since $(\alpha_2, \beta_2)=1$, $(\beta-\alpha)qxy=0$. So $\beta qxy=$ $=\alpha qxy=\beta qyx$. It follows that $\beta(qxy-qyx)=0$. Similarly, $\alpha(qxy-qyx)=0$. Thus, $qxy=qyx$.

Similarly to the arguments in Section 2 but using identities (1′), (2′), (3′), (4′) instead of (1), (2), (3), (4), we can readily prove the following

Lemma 3.2. *Suppose* $o(x^2)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{Z}^+$. *If* $p^l x^2y=0$ *for some* $l\in\mathbf{N}, l<m$, *then* $x^2y=xyx=yx^2$.

Lemma 3.3. *Suppose* $o(x^2)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{Z}^+$. *Then* $x^2y=xyx=yx^2$.

Lemma 3.4. *Suppose* $o(x^2)=p_1^{m_1}p_2^{m_2}\dots p_s^{m_s}$, *where* $p_1, p_2, \dots, p_s$ *are non-associate prime elements in* A *and* $m_1, m_2, \dots, m_s\in\mathbf{Z}^+$. *Then* $x^2y=xyx=yx^2$.

Theorem 3.1. *If* $x\in A$ *is scalar central and if* $o(x^2)=0$, *then* $x\in C$.

Theorem 3.2. *If* $x\in A$ *is scalar central then* $x^2y=xyx=yx^2$ *for all* $y\in A$.

We should note that under the hypothesis of Theorem 3.2, one could not expect $x\in C$.

Example. Let $A=\left\{\begin{bmatrix} a & b \\ o & c\end{bmatrix} \middle| a, b, c\in\mathbf{Z}_2\right\}$ be the algebra of all upper triangular matrices over the ring $\mathbf{Z}_2$ of integers modulo 2. Let $x=\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$. Then A has a unity element, x is scalar central, but $x\notin C$.

4. Some special cases. We noted in passing that in an algebra over a principal ideal domain, scalar central elements need not lie in the centre of the algebra. However, we have the following

Theorem 4.1. *Suppose* A *is a semi-prime algebra (with or without* 1*) over a principal ideal domain* R. *Then all scalar central elements in* A *are in the centre* C *of* A.

Proof. Let x be a scalar central element. By Theorem 2.2, there is a *least* positive integer n such that $x^ny=x^{n-1}yx=x^{n-2}yx^2=\dots=yx^n$ for all $y\in A$.

Suppose $n>1$. For $y\in A$, let $\alpha, \beta\in R$ be such that $(\alpha, \beta)=1$ and $\alpha xy=\beta yx$. Noting that $\alpha x^{2n-2}y=\beta x^{2n-2}y$ and $\alpha yx^{2n-2}=\beta yx^{2n-2}$, we have for any $z\in A$ and $i=0, 1, 2, \ldots, n-2$,

$$\alpha^i(x^{n-1}y-x^iyx^{n-i-1})z\alpha^i(x^{n-1}y-x^iyx^{n-i-1})=$$
$$=\alpha^{2i}(x^{n-1}yzx^{n-1}y-x^iyx^{n-i-1}zx^{n-1}y-x^{n-1}yzx^iyx^{n-i-1}+x^iyx^{n-i-1}zx^iyx^{n-i-1})=$$
$$=\alpha^{2i}x^{2n-2}yzy-\alpha^i\beta^iyx^{n-1}zx^{n-1}y-\alpha^i\beta^ix^{n-1}yzyx^{n-1}+\beta^{2i}yx^{n-1}zyx^{n-1}=$$
$$=\alpha^{2i}x^{2n-2}yzy-\alpha^i\beta^iyx^{2n-2}zy-\alpha^i\beta^ix^{2n-2}yzy+\beta^{2i}yx^{2n-2}zy=0.$$

Thus, by the semiprimeness of A, $\alpha^i(x^{n-1}y-x^iyx^{n-i-1})=0$. Likewise, $\beta^{n-i-1}(x^{n-1}y-x^iyx^{n-i-1})=0$. Since $(\alpha^i, \beta^{n-i-1})=1$, $x^{n-1}y-x^iyx^{n-i-1}=0$ for $i=0, 1, 2, \ldots, n-2$. So $x^{n-1}y=x^{n-2}yx=x^{n-3}yx^2=\ldots=yx^{n-1}$ for all $y\in A$. This contradicts the minimality of n. Hence $n=1$ and $xy=yx$ for all $y\in A$.

Theorem 4.2. *Let A be an algebra with 1 over a principal ideal domain R. If x and $1+x$ are both scalar central then $x\in C$.*

Proof. By Theorem 3.2, for any $y\in A$, $xyx=x^2y$ and $(1+x)y(1+x)=(1+x)^2y$ which imply that $xy=yx$.

As a corollary we have the following result due to Luh and Putcha [2].

Corollary 4.1. *Let A be an algebra with 1 over a principal ideal domain R. If every element in R is scalar central then A is commutative.*

Remark. To generalize the concept of scalar central element one may call an element $x\in A$ scalar power central if for each $y\in A$ there exist $\alpha, \beta\in R$ and $n\in \mathbf{Z}^+$, depending on y, such that $\alpha x^ny=\beta yx^n$ and $(\alpha, \beta)=1$. It would be interesting to know whether analogous results remain true.

The authors would like to express their appreciation to M. Putcha for stimulating discussions. The first named author also would like to acknowledge the support by the Engineering Foundation of NCSU.

References

[1] K. Koh, J. Luh and M. S. Putcha, On the associativity and commutativity of algebras over commutative rings, *Pac. J. Math.*, **63** (1976), 423—430.

[2] J. Luh and M. S. Putcha, A commutativity theorem for non-associative algebras over a principal ideal domain, *Pac. J. Math.*, **68** (1977), 485—488.

[3] M. Rich, A commutativity theorem for algebras, *Amer. Math. Monthly*, **82** (1975), 377—379.

DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, N.C. 27650, U.S.A.