

On the concentration of distribution of additive functions

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1. We say that $g(n)$ is *additive* if $g(mn) = g(m) + g(n)$ holds for every coprime pairs m, n of positive integers. If, moreover, $g(p^a) = g(p)^a$ for every prime power p^a , then $g(n)$ is called *strongly additive*. By $p, p_1, p_2, \dots, q, q_1, q_2, \dots$ we denote prime numbers, c, c_1, c_2, \dots are suitable positive constants. $P(n)$ and $\kappa(n)$ denote the largest and the smallest prime factor of n . The symbol \ll is used instead of O ; $\# \{ \}$ is the counting function of the set indicated in brackets $\{ \}$. For a distribution function $H(x)$ let $\varphi_H(\tau)$ denote its characteristic function. Let

$$Q(h) = Q_H(h) = \sup_x (H(x+h) - H(x))$$

be the continuity module — concentration — of H . We say that H satisfies a Lipschitz condition if $Q(h) \ll h$ as $h \rightarrow 0$.

We assume that $g(n)$ is strongly additive and that

$$(1.1) \quad \sum_p \frac{g^2(p)}{p} < \infty.$$

The theorem of Erdős—Wintner [1] guarantees that the function $g(n) - A_n$, where

$$(1.3) \quad A_n = \sum_{p \leq n} \frac{g(p)}{p},$$

has a limit distribution, i.e. the relation

$$(1.4) \quad \frac{1}{N} \# \{n \leq N | g(n) - A_n < x\} \rightarrow F(x)$$

holds at every continuity point of $F(x)$, where $F(x)$ is a distribution function. If,

moreover, $\Sigma g(p)/p$ converges, then the values $g(n)$ have a limit distribution too, i.e.

$$(1.5) \quad \frac{1}{N} \# \{n \leq N | g(n) < x\} \rightarrow G(x),$$

at every continuity point of the distribution function $G(x)$.

We have the relations

$$(1.6) \quad \varphi_F(\tau) = \prod_p \left(\left(1 - \frac{1}{p}\right) e^{-i\tau \frac{g(p)}{p}} + \frac{1}{p} e^{i\tau \left(1 - \frac{1}{p}\right) g(p)} \right),$$

$$(1.7) \quad \varphi_G(\tau) = \prod_p \left(1 - \frac{1}{p} + \frac{e^{i\tau g(p)}}{p} \right).$$

From these forms we can see that both F and G can be represented as the distribution of the sum of infinitely many mutually independent random variables having purely discrete distributions. By the well-known theorem of P. LÉVY [2] G and F are continuous if

$$(1.8) \quad \sum_{p \in Z_g} 1/p = \infty, \quad \text{where } Z_g = \{p | g(p) \neq 0\}.$$

Furthermore, assuming the validity of (1.3) we have that F and G are of pure type, either absolutely continuous or singular (see E. LUKÁCS [3]). To decide the question if a distribution function were absolutely continuous or singular seems to be quite difficult. The first result upon this has been achieved by P. ERDŐS [4]; namely it was proved that if $g(p) = O(p^{-\delta})$, δ being any positive constant, then $G(x)$ is singular. Recently JOGESH BABU [5] has proved that $G(x)$ is absolutely continuous if $g(n)$ is generated by $g(p) = (\log p)^{-a}$ ($0 < a < 2$). The main idea of the proof is that $\varphi_G(\tau)$ is square-integrable in $(-\infty, \infty)$, and so by using Plancherel's theory of Fourier integrals it must have an inverse in $L^2(-\infty, \infty)$ that is the density function of $G(x)$.

It is known that a distribution function H satisfies Lipschitz condition if $|\varphi_H(\tau)|$ is integrable in $(-\infty, \infty)$, and so it is absolutely continuous. The method of Jogesh Babu gives that G satisfies Lipschitz condition if $g(p) = (\log p)^{-a}$ ($0 < a < 1$).

The aim of this paper is to investigate the singularity or absolute continuity of distribution functions for some classes of additive functions.

We shall prove the following theorems.

Theorem 1. *Let $g(n)$ be a strongly additive function,*

$$(1.9) \quad D(y) = \sum_{p > y} \frac{|g(p)|}{p},$$

and suppose that the inequalities

$$(1.10) \quad D(t^A) < 1/t,$$

$$(1.11) \quad |g(p_1) - g(p_2)| > 1/t \quad \text{if } p_1 \neq p_2 < t^\delta$$

hold, with suitable positive constants A and δ , for every large t . Then

$$(1.12) \quad (\log t)^{-1} \ll Q_G(1/t) \ll (\log t)^{-1} \quad (t \rightarrow \infty),$$

where the constants involved by \ll may depend on g .

This result was achieved by TJAN [7] and P. ERDŐS [8] for $\log \frac{\varphi(n)}{n}$, and for $\log \frac{\sigma(n)}{n}$, resp.

Theorem 2. Let $g(n)$ be strongly additive satisfying (1.1). Then for the concentration $Q(h)$ of $F(x)$ or $G(x)$ (if it exists) we have

$$(1.13) \quad Q(4D_R) \cong \frac{c}{\log R} \quad (R \cong 2),$$

c being an absolute positive constant, and

$$(1.14) \quad D_R = \left(\sum_{p > R} \frac{g^2(p)}{p} \right)^{1/2}.$$

Remarks.

- 1) This assertion is non-trivial only if $D_R \log R \rightarrow 0$ ($R \rightarrow \infty$), since $Q_H(1/t) \gg 1/t$ ($t \rightarrow \infty$) for every $H(x)$.
- 2) If $g(p) = (\log p)^{-\gamma}$ ($\gamma \geq 1$ constant), then $D_R = (1 + o(1)) \frac{(\log R)^{-\gamma}}{\sqrt{2\gamma}}$ and so $Q_G(1/t) \gg \frac{1}{t^{1/\gamma}}$.

Theorem 3. If the strongly additive $g(n)$ is generated by $g(p) = (\log p)^{-\gamma}$, then

$$(1.15) \quad \frac{1}{t^{1/\gamma}} \ll Q_G(1/t) \ll \frac{(\log \log t)^2}{t^{1/\gamma}}$$

if $\gamma > 1$, while for $\gamma = 1$

$$(1.16) \quad \frac{1}{t} \ll Q_G(1/t) \ll \frac{(\log \log t)^2 \log t}{t}.$$

Remarks.

- 1) We guess that $Q_G(1/t) \ll \frac{1}{t^{1/\gamma}}$ for $\gamma > 1$ but we are unable to prove it.
- 2) We also guess that $G(x)$ is singular if $0 \leq g(p) \leq (\log p)^{-\gamma}$, $\gamma > 2$. This seems not to be known even if $g(p) = (\log p)^{-\gamma}$.
- 3) By our method we could estimate the concentration for other functions if $g(p)$ is monotonic. The following assertion holds. Let $t(u) > 0$ to monotonically decreasing in $(1, \infty)$, $g(p) = t(p)$ for primes p . Let $y(\tau), z(\tau)$ be defined by the

relations $t(y(\tau)) = \frac{y(\tau)^{1/4}}{\tau}$; $t(z(\tau)) = 1/\tau$. Suppose that for large τ , $y(\tau) < \tau^c$, $z(\tau) > e^{\tau^{1+\varepsilon}}$ ($\varepsilon > 0$ constant), and that the integral

$$\int_{y(\tau)}^{z(\tau)} \frac{\cos \tau t(u)}{u \log u} du$$

is bounded as $\tau \rightarrow \infty$. Then $Q_F(h) \ll 1/h$. These conditions hold if $g(p)$ decreases regularly and

$$\sum \frac{g^2(p)}{p} < \infty; \quad \sum \frac{g(p)}{p} = \infty.$$

Theorem 4. *There exists a monotonically decreasing function $t(u)$ satisfying the conditions*

$$\sum \frac{t(p)}{p} = \infty, \quad \sum \frac{t^2(p)}{p} < \infty,$$

for which the distribution function $F(x)$ of the strongly additive $g(n)$ defined by $g(p) = t(p)$ is singular.

2. Proof of Theorems 2 and 4. We shall prove Theorem 2 for $F(x)$ only. The proof is almost the same for $G(x)$.

$F(x)$ can be represented as the distribution function of θ_1 ; $\theta_R = \sum_{p \leq R} \xi_p$, where ξ_p are mutually independent random variables with the distribution

$$P\left(\xi_p = g(p) \left(1 - \frac{1}{p}\right)\right) = \frac{1}{p}, \quad P(\xi_p = -g(p)/p) = 1 - \frac{1}{p},$$

for the mean value $M\theta_R$ and variance $D\theta_R$ we have $M\theta_R = 0$, $D\theta_R = D_R$. Consequently, by the Chebyshev inequality,

$$P(|\theta_R| < \Lambda D_R) \geq 1 - \frac{1}{\Lambda^2}.$$

So by

$$d = \sum_{p \leq R} \frac{g(p)}{p}$$

we have

$$\begin{aligned} F(-d + \Lambda D_R) - F(-d - \Lambda D_R) &\geq P\left(\xi_p = \frac{-g(p)}{p} \ (\forall p \leq R) \wedge |\theta_R| \leq \Lambda D_R\right) \\ &\geq \left(1 - \frac{1}{\Lambda^2}\right) \prod_{p \leq R} (1 - 1/p) \gg (1 - 1/\Lambda^2) \cdot \frac{1}{\log R} \quad (R \geq 2). \end{aligned}$$

By putting $\Lambda = 2$ our assertion follows immediately.

To prove Theorem 4 we define our $g(p)$ as follows. Let $R_1=1$, R_{l+1} be defined by $R_l = \log \log \log \log \log R_{l+1}$, $\lambda_l = \exp(\exp(\exp R_l))$, $g(p) = \frac{1}{\lambda_l}$ if $p \in [R_l, R_{l+1})$.

Then

$$\sum_{p > R_l} \frac{g(p)}{p} = \infty, \quad \sum_{p > R_l} \frac{g^2(p)}{p} \ll \frac{1}{\lambda_l^2} \log \log R_{l+1} \ll \frac{1}{\lambda_l}.$$

Let m run over the square-free integers all prime factor of which is less than R . By Theorem 2, for fixed m , the number of integers n with

$$n = mv \leq N, \quad \kappa(m) \geq R_l, \quad g(v) - (A_N - A_{R_l}) \in \left[-\frac{c}{\lambda_l}, \frac{c}{\lambda_l}\right]$$

is greater than a constant time of

$$\frac{N}{m} \prod_{p < R_l} \left(1 - \frac{1}{p}\right).$$

Summing up for m we have

$$\begin{aligned} \# \left\{ n = mv \leq N \mid g(n) \in \bigcup_m \left[g(m) - A_{R_l} - \frac{c}{\lambda_l}, g(m) - A_{R_l} + \frac{c}{\lambda_l} \right] \right\} \\ \gg N \prod_{p < R_l} (1 - 1/p) \sum_{p(m) \leq R_l} \frac{1}{m} \gg N. \end{aligned}$$

So the intervals

$$\bigcup_m \left[g(m) - A_{R_l} - \frac{c}{\lambda_l}, g(m) - A_{R_l} + \frac{c}{\lambda_l} \right]$$

cover a positive percentage of integers. The whole length of these intervals is less than $c2^{\pi(R_l)}/\lambda_l$. This quantity tends to zero as $l \rightarrow \infty$. By this the theorem is proved.

3. Lemmas. Let $\mathcal{S}(A)$ be an arbitrary set of distinct square free integers m having the following properties:

$$(1) \quad A \leq \kappa(m),$$

$$(2) \quad \text{if } p_1 | m_1, p_2 | m_2, m_1 \neq m_2 \in \mathcal{S}(A), \text{ then } \frac{m_1}{p_1} \neq \frac{m_2}{p_2}.$$

Let $\varrho(n)$ be a multiplicative function such that $0 \leq \varrho(p) \leq 1 + O(1/p^\delta)$ ($\delta > 0$ constant). Moreover, let

$$(3.1) \quad T(A) = \sum_{m \in \mathcal{S}(A)} \frac{\varrho(m)}{m}.$$

Lemma 1. For $2 \leq A$ we have

$$(3.2) \quad T(A) \leq \frac{c_1}{A \log A},$$

c_1 being an absolute constant.

Proof. We split the elements of $\mathcal{S}(A)$ according to $P(m) \in [A^{2^h}, A^{2^{h+1}})$. Let $T_h(A)$ denote the part of the sum (3.1) corresponding to this interval. From (2) we have

$$T_h(A) \leq \frac{1}{A^{2^h}} \sum \frac{\varrho(n)}{n}$$

where the sum extends over the square free n with $A \leq \kappa(n) < P(n) \leq A^{2^{h+1}}$. So

$$\sum \frac{\varrho(m)}{m} \ll \prod_{A < p \leq A^{2^{h+1}}} \left(1 + \frac{\varrho(p)}{p}\right) \ll \frac{\log A^{2^{h+1}}}{\log A}.$$

Using this inequality for every $h \geq 0$ we have (3.2).

Remark. Since $T(1) \leq 1 + T(2)$, therefore by Lemma 1, $T(1)$ is bounded.

We shall use the following Esseen type inequality due to A. S. FAIBLEIB [6] which we quote as

Lemma 2. For an arbitrary distribution function $H(x)$ we have

$$(3.3) \quad Q_H(h) \leq C \sup_{t \geq 1/h} \frac{1}{t} \int_0^t |\varphi_H(\tau)| d\tau.$$

Lemma 3. Let $\gamma > 0$ be fixed,

$$(3.4) \quad S = \sum_{\tau^{10} < p < e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log p)^{-\gamma}}{p}.$$

Then S is bounded as $\tau \rightarrow \infty$.

Proof. First of all we shall prove that

$$E = \sum_{\tau^{10} \leq n \leq e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log n)^{-\gamma}}{n \log n}$$

is bounded as $\tau \rightarrow \infty$. Indeed,

$$\begin{aligned} \left| E - \int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u} du \right| &\ll \sum_{\tau^{10} \leq n} \frac{1}{n \log n} \left(\frac{\tau}{(\log n)^\gamma} - \frac{\tau}{(\log(n+1))^\gamma} \right) \\ &\ll \frac{\tau}{\tau^{10} (\log \tau)^{1+\gamma}}. \end{aligned}$$

To estimate the integral we substitute $y = \tau / (\log u)^\gamma$, and we get immediately that

$$\int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u} du = \frac{1}{\gamma} \int_1^{\tau/(10 \log \tau)^\gamma} \frac{\cos y}{y^{1/\gamma}} dy = O(1).$$

So it is enough to prove that $S - E = O(1)$ as $\tau \rightarrow \infty$.

Let $\tau^{10} \leq M \leq e^{\tau^{1/7}}$; $N_1 = M + jN^{3/4}$ ($j=0, 1, \dots, [M^{1/4}]$), $N = M^{3/4}$, $N_2 = N_1 + N$, and consider the quantity

$$S(N_1, N_2) = \sum_{N_1 \leq p < N_2} \frac{\cos \tau (\log p)^{-\gamma}}{p} - \sum_{N_1 \leq n < N_2} \frac{\cos \tau (\log n)^{-\gamma}}{n \log n}.$$

To estimate it we use the prime number theorem for short intervals in the form

$$(3.5) \quad \Delta_{N_1}(u) = \sum_{n=N_1}^u (\Lambda(n) - 1) \ll \frac{N}{(\log N_1)^{10}} \quad (N_1 \leq u \leq N_2).$$

Since

$$\frac{1}{N_1 \log N_1} - \frac{1}{n \log n} = - \int_{N_1}^n \frac{\log x + 1}{x^2 (\log x)^2} dx \leq \frac{2(n - N_1)}{N_1^2 \log N_1}$$

for $N_1 \leq n \leq N_2$, therefore

$$(3.6) \quad S(N_1, N_2) \ll \sum_{N_1 \leq p < N_2} 1/p^2 + \frac{N^2}{N_1^2} + \frac{|L(N_1, N_2)|}{N_1 \log N_1},$$

where

$$L(N_1, N_2) = \sum_{n=N_1}^{N_2} (\Lambda(n) - 1) \cos \tau (\log n)^{-\gamma}.$$

By using partial summation,

$$L(N_1, N_2) = \Delta_{N_1}(N_2) \cos \tau (\log N_2)^{-\gamma} + \sum_{n=N_1}^{N_2-1} \Delta_{N_1}(n) \left(\cos \frac{\tau}{(\log n)^\gamma} - \cos \frac{\tau}{(\log(n+1))^\gamma} \right).$$

Hence, by (3.6) we get

$$L(N_1, N_2) \ll \frac{N}{(\log N_1)^{10}} \left(1 + \sum_{n=N_1}^{N_2-1} \left| \cos \frac{\tau}{(\log n)^\gamma} - \cos \frac{\tau}{(\log(n+1))^\gamma} \right| \right).$$

Since $\tau/(\log n)^\gamma$ is monotonic and cosine satisfies Lipschitz condition, the last sum is majorated by

$$\frac{\tau}{\log N_1} - \frac{\tau}{\log N_2}.$$

Consequently,

$$\begin{aligned} \sum_{0 \leq j \leq M^{1/4}} S(N_1, N_2) &\ll \sum_{M \leq p \leq 2M} \frac{1}{p^2} + \frac{M^{3/2} M^{1/4}}{M^2} + \frac{1}{(\log M)^{11}} + \\ &+ \frac{M^{-1/4}}{(\log M)^{10}} \left(\frac{\tau}{\log M} - \frac{\tau}{\log 2M} \right). \end{aligned}$$

By putting $M = 2^h \tau^{10}$, $h=0, 1, 2, \dots$, up to $M \leq e^{\tau^{1/7}}$ we have $S - E = O(1)$.

By this Lemma 3 has been proved.

4. Proof of Theorem 3. Let

$$\varphi(\tau) = \prod_p \left(1 + \frac{e^{i\tau(\log p)^{-\gamma}} - 1}{p} \right)$$

be the characteristic function of the limit distribution of $g(n)$ defined by $g(p) = (\log p)^{-\gamma}$. First we observe that

$$(4.1) \quad \log |\varphi(\tau)| \leq \operatorname{Re} \sum_{p \leq e^{\tau^{1/\gamma}}} \frac{e^{i\tau(\log p)^{-\gamma}} - 1}{p} + O(1).$$

Lemma 3 and the relation

$$\sum_{p \leq y} \frac{1}{p} = \log \log y + O(1)$$

gives that

$$(4.2) \quad \log |\varphi(\tau)| \leq -\frac{1}{\gamma} \log \tau + O(1) + \operatorname{Re} \sum_{p \leq \tau^{1/\gamma}} \frac{e^{i\tau(\log p)^{-\gamma}}}{p}.$$

Consequently, $\int_0^\infty |\varphi(\tau)| < \infty$ for $\gamma < 1$. Let $\gamma \geq 1$. From (4.2) we have

$$(4.3) \quad |\varphi(\tau)| \ll \tau^{-1/\gamma} |\psi(\tau)|,$$

where

$$(4.4) \quad \psi(\tau) = \prod_{p \leq R^{1/2}} \left(1 + \frac{e^{i\tau(\log p)^{-\gamma}}}{p} \right), \quad R \leq \tau \leq 2R.$$

Let $\psi(\tau) = \psi_1(\tau) \cdot \psi_2(\tau)$, where

$$\psi_1 = \prod_{p \leq (\log R)^4}, \quad \psi_2 = \prod_{(\log R)^4 < p \leq R^{1/2}}.$$

So we have

$$(4.5) \quad \int_R^{2R} |\varphi(\tau)| d\tau \ll \frac{1}{R^{1/\gamma}} (B_1(R) + B_2(R)),$$

where

$$(4.6) \quad B_j(R) = \int_R^{2R} |\psi_j(\tau)|^2 d\tau \quad (j = 1, 2).$$

First we estimate $B_2(R)$. We have

$$\psi_2(\tau) = 1 + \sum \frac{e^{i\tau g(m)}}{m},$$

where the summation is extended for the square-free m 's satisfying $(\log R)^4 \leq \kappa(m) \leq P(m) \leq R^{1/4}$. We have

$$B_2(R) \ll R + \sum \frac{1}{m} \min \left(R, \frac{1}{|g(m)|} \right) + \sum_{m,n} \frac{1}{mn} \min \left(R, \frac{1}{|g(m) - g(n)|} \right),$$

n runs over the same set as m .

Let

$$(4.7) \quad K(1/R) = \sup_x \sum_{g(m) \in [x, x+1/R)} 1/m.$$

Let x be fixed. We observe that the set of m 's standing in the right hand side satisfies

the conditions of Lemma 1 with $A = (\log R)^4$, $\varrho \equiv 1$. Indeed, if $|g(m_1) - g(m_2)| \leq 1/R$, $p_1/m_1, p_2/m_2$, then

$$\begin{aligned} \left| g\left(\frac{m_1}{p_1}\right) - g\left(\frac{m_2}{p_2}\right) \right| &\geq |g(p_1) - g(p_2)| - |g(m_1) - g(m_2)| \geq \\ &\geq \left| \frac{1}{(\log p_1)^\gamma} - \frac{1}{(\log p_2)^\gamma} \right| - 1/R > 0, \end{aligned}$$

and so $\frac{m_1}{p_1} \neq \frac{m_2}{p_2}$. So we have

$$K(1/R) \ll \frac{\log \log R}{(\log R)^4}.$$

Furthermore, the contribution of the pairs m, n for which $|g(m) - g(n)| \geq R^2$ is majorated by

$$\frac{1}{R^2} \prod_{p \leq R^{1/2}} (1 + 1/p)^2 \ll \frac{(\log R)^2}{R^2}.$$

Consequently

$$\begin{aligned} (4.8) \quad B_2(R) &\ll R + \sum_n \frac{1}{n} \left(\sum_{0 \leq j \leq R^3} \frac{R}{j+1} \left\{ \sum_{|g(m) - g(n)| \in \left[j/R, \frac{j+1}{R} \right]} 1/m \right\} \right) + \\ &+ \sum_{0 \leq j \leq R^3} \frac{R}{j+1} \left(\sum_{|g(m)| \in \left[j/R, \frac{j+1}{R} \right]} 1/m \right) \ll R. \end{aligned}$$

Since $|\psi_1(\tau)| \leq \prod_{p \leq (\log R)^4} (1 + 1/p) \ll \log \log R$, therefore $B_1(R) \ll (\log \log R)^2 R$. So we have

$$\int_R^{2R} |\varphi(\tau)| d\tau \ll R^{1-1/\gamma} (\log \log R)^2.$$

Applying this inequality for $R = T/2^h$ ($h = 1, 2, \dots$) we get

$$\frac{1}{T} \int_1^T |\varphi(\tau)| d\tau \ll \begin{cases} \frac{(\log \log T)^2}{T^{1/\gamma}}, & \text{if } \gamma > 1, \\ \frac{(\log \log T)^2 \log T}{T}, & \text{if } \gamma = 1. \end{cases}$$

From Lemma 2 our theorem immediately follows.

5. Proof of Theorem 1. First we prove the second inequality in (1.12). Let

$$g(n; y) = \sum_{p|n, p \equiv y} g(p).$$

Since from (1.10)

$$\sum_{n \leq N} |g(n; t^{2A})| \leq ND(t^{2A}) \leq \frac{N}{t^2},$$

we have

$$(5.1) \quad \# \{n \leq N; |g(n; t^{2A})| \geq 1/t\} \leq \frac{N}{t}.$$

For a natural number n let $e(n)$ denote the product of those prime factors of n that are less than t^{2A} ; let $f(n) = n/e(n)$. From (5.1) we get that with the exception of at most N/t integers if $n \leq N$ and $g(n) \in [x, x+1/t]$, then $g(e(n)) \in [x-1/t, x+1/t]$. Let x and t be fixed, and $a_1 < a_2 < \dots < a_R$ be the sequence of those square-free integers all prime divisors of which is less than t^{2A} and $g(a_j) \in [x-1/t, x+1/t]$. Let $E(a_j)$ be the number of those $n \leq N$ for which $a_j | e(n)$ and $(a_j, e(n)) = a_j$ holds. By using the Eratosthenian sieve we have

$$(5.2) \quad E(a_j) \leq 1 + O(1) \frac{N \varrho(a_j)}{a_j} \prod_{p < t^{2A}} \left(1 - \frac{1}{p}\right) \quad (N \rightarrow \infty),$$

where $\varrho(m) = \prod_{p|m} \frac{1}{1-1/p}$. Since $\prod_{p < t^{2A}} (1-1/p) \ll (\log t)^{-1}$,

we have

$$(5.3) \quad Q_G(1/t) \ll \frac{1}{t} + \frac{1}{\log t} \sup_x \sum_{g(a_j) \in [x-1/t, x+1/t]} \frac{\varrho(a_j)}{a_j}.$$

It has only remained to prove that

$$(5.4) \quad U_{x,t} = \sum_{g(a_j) \in [x, x+1/t]} \frac{\varrho(a_j)}{a_j} \ll 1$$

uniformly for $x \in (-\infty, \infty)$ as $t \rightarrow \infty$.

We write every a_j as mv where $P(m) < t^\delta$, $\chi(v) \geq t^\delta$, or $v=1$. So

$$U_{x,t} = \sum_v \frac{\varrho(v)}{v} \left\{ \sum_{g(m) \in [x-g(v), x+1/t-g(v)]} \frac{\varrho(m)}{m} \right\}.$$

The set of m 's satisfies the conditions of Lemma 1 (see (1.11)) so the inner sum is bounded, and we have

$$U_{x,t} \ll \prod_{t^\delta \leq p \leq t^{2A}} \left(1 + \frac{\varrho(p)}{p}\right) \ll 1.$$

We shall prove that

$$G(1/t) - G(-1/t) \geq \frac{c}{\log t} \quad (t \rightarrow \infty),$$

and by this the proof will be finished.

Let $P = \prod_{p < t^{c_1}} p$. It is obvious that

$$(5.5) \quad \sum_{n \leq N, (n, P)=1} 1 = ((1+o(1))N \prod_{p < t^{c_1}} (1-1/p) \cong \frac{c_2 N}{c_1 \log t} \quad (N \rightarrow \infty),$$

c_2 is an absolute constant. Furthermore,

$$\sum_{n \leq N, (n, P)=1} |g(n)| \leq \sum_{q > t^{c_1}} |g(q)| \sum_{qm \leq N, (m, P)=1} 1 \leq c_3 N \prod_{p \leq t^{c_1}} \left(1 - \frac{1}{p}\right) \sum_{q > t^{c_1}} \frac{|g(q)|}{q}.$$

By choosing $c_1 = 2A$, from (1.9) we have

$$\sum_{\substack{n \leq N, (n, P)=1 \\ |g(n)| \leq 1/t}} 1 \leq t \sum_{n \leq N, (n, P)=1} |g(n)| \leq tN \prod_{p|P} \left(1 - \frac{1}{p}\right) \sum_{p > t^{2A}} \frac{|g(p)|}{p} \leq \frac{c_4 N}{t \log t}.$$

This and (5.5) gives that

$$F(1/t) - F(-1/t) \cong \frac{c_2}{2A \log t} - \frac{c_4}{t \log t} \cong \frac{c_5}{\log t}.$$

By this the proof of our theorem is finished.

References

- [1] P. ERDŐS—A. WINTNER, Additive arithmetical functions and statistical independence, *Amer. J. Math.*, **61** (1939), 713—721.
- [2] P. LÉVY, Sur les séries dont les termes sont des variables éventuelles indépendantes, *Studia Math.*, **3** (1931), 119—155.
- [3] E. LUKÁCS, *Characteristic functions*, Griffin (London, 1960).
- [4] P. ERDŐS, On the smoothness of the asymptotic distribution of additive arithmetical functions, *Amer. J. Math.*, **61** (1939), 722—725.
- [5] G. JOGESH BABU, Absolutely continuous distribution functions $f(p) = (\log p)^{-a}$, $a > 0$, *Acta Arithm.*, **26** (1974/75), 401—403.
- [6] A. S. FAINLEIB, Generalization of Esseen inequality and its application in probabilistical number theory, *Izvest. Akad. Nauk.*, **32** (1968), 859—879. (Russian)
- [7] M. M. TJAN, On the question of the distribution of values of the Euler function $\phi(n)$, *Litovsk. Mat. Sb.*, **6** (1966), 105—119. (Russian)
- [8] P. ERDŐS, On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions, *Pacific J. Math.*, **52** (1974), 59—65.

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