On Fong and Sucheston's mixing property of operators in a Hilbert space

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1. Introduction

Let T be a bounded linear operator on a (real or complex) Hilbert space $\mathfrak{H}$. A matrix (a_{ni}) $(n, i=0, 1, 2, \dots)$ is said to be *uniformly regular* (U. R.) if

$$\sup_n \sum_i |a_{ni}| = a < \infty, \quad \sup_i |a_{ni}| = b_n \to 0 \quad (n \to \infty), \quad \text{and} \quad \lim_n \sum_i a_{ni} = 1.$$

In this article we consider the problem of the equivalence of assertions (a) and (b) below. h is an element of $\mathfrak{H}$.

(a) $T^n h$ converges weakly.

(b) For every U. R. matrix (a_{ni}), $\sum_i a_{ni} T^i h$ converges strongly (to the weak limit in (a)).

In the more general context of a Banach space $\mathfrak{B}$, (b)$\Rightarrow$(a) is always true ([8]), but (a)$\Rightarrow$(b) may fail even if T is a contraction and (a) holds for every $h \in \mathfrak{B}$ ([3]). This equivalence (in a weaker form) was first proved for the special case where $\mathfrak{H}=L_2$ of a probability space and $Th=h\circ\overline{T}$ for an invertible, measure preserving transformation $\overline{T}$ on that space ([4]). This was recently generalized to an arbitrary contraction on $\mathfrak{H}$ in [1], [13] and, in the form as stated above, FONG and SUCHESTON [8]. In this article we shall prove in Theorem 1 the equivalence for a much wider class of operators. This class contains all operators similar to contractions, and we shall give some sufficient conditions for such similarity to hold. By an application of the uniform boundedness principle, it is easy to show that conditions (2.0—1) in Theorem 1 imply that T is power-bounded, i.e. $\sup \|T^n\| < \infty$, provided that the operator B is bounded. Whether the equivalence is true for a general power-

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bounded operator is an open question. Note that if (a) holds, then $\{T^i h: i \geqq 0\}$ is bounded and the expression in (b) is meaningful. For (b)$\Rightarrow$(a) with T not power-bounded, we require such expressions to be finite sums. With this modification, (b) implies that $\{T^i h: i \geqq 0\}$ is bounded (see [1, p. 237]), and hence (a) ([8]).

2. Main Theorem

Theorem 1. *Let T be an operator on a Hilbert space $\mathfrak{H}$. Assume that there exist Hilbert spaces $\mathfrak{G}$, $\mathfrak{K}$, a contraction C on $\mathfrak{K}$, and operators $A: \mathfrak{K} \to \mathfrak{G}$, $R: \mathfrak{G} \to \mathfrak{H}$ and $B, S: \mathfrak{H} \to \mathfrak{G}$ which are bounded except possibly B such that*

$$RS = \text{identity operator on } \mathfrak{H} \tag{2.0}$$

and

$$\lim \|AC^n A^* Bh - ST^n h\| = 0 \quad \text{for all} \quad h \in \mathfrak{H}. \tag{2.1}$$

Then for any fixed $h \in \mathfrak{H}$, the following conditions (a) *and* (b) *are equivalent:*

(a) *$T^n h$ converges weakly.*

(b) *For every U. R. matrix (a_{ni}), $\sum_i a_{ni} T^i h$ converges strongly (to the weak limit in* (a)*).*

Proof. We only need to prove (a)$\Rightarrow$(b). In (2.1), C can be assumed to be an isometry. In fact, there exists an isometry U on a Hilbert space $\mathfrak{L} \supset \mathfrak{K}$ satisfying $C^n = PU^n | \mathfrak{K}$, $n \geqq 0$, where P is the orthoprojector from $\mathfrak{L}$ onto $\mathfrak{K}$ (see e.g. [18], p. 11), thus implying $AC^n A^* = (AP) U^n (AP)^*$. Henceforth we shall replace C by an isometry U.

Suppose (a) holds. Since the limit is a fixed point of T, we can and do assume that it is 0. Given $\varepsilon > 0$, there exists an integer N such that for all $m \geqq N$, $\|AU^m A^* Bh - ST^m h\| \leqq \varepsilon$. Hence

$$\left\| \sum_i a_{ni} T^i h \right\| \leqq b_n \sum_{i=0}^{N-1} \|T^i h\| + \|R\| \left\| \sum_{i=N}^{\infty} a_{ni} S T^i h \right\|, \tag{2.2}$$

$$\left\| \sum_{i=N}^{\infty} a_{ni} S T^i h \right\| \leqq a\varepsilon + \left\| \sum_{i=N}^{\infty} a_{ni} A U^i A^* Bh \right\| \leqq a\varepsilon + \|A\| \left\| \sum_{i=N}^{\infty} a_{ni} U^i A^* Bh \right\|. \tag{2.3}$$

By the assumption, there exists a positive integer $M \geqq N$ such that for all $m \geqq M$,

$$|\langle ST^m h, Bh \rangle| = |\langle Bh, ST^m h \rangle| = |\langle S^* Bh, T^m h \rangle| \leqq \varepsilon. \tag{2.4}$$

Hence for all $m \geqq M$ and $i, j \geqq 0$, we have

$$(2.5)\qquad |\langle U^i A^* Bh, U^{i+m} A^* Bh\rangle| = |\langle Bh, AU^m A^* Bh\rangle| \leqq |\langle Bh, ST^m h\rangle| + \\ + |\langle Bh, AU^m A^* Bh - ST^m h\rangle| \leqq \varepsilon + \varepsilon \|Bh\|;$$

and similarly,

$$(2.6)\qquad |\langle U^{j+m} A^* Bh, U^j A^* Bh\rangle| \leqq \varepsilon + \varepsilon \|Bh\|.$$

Hence

$$(2.7)\qquad \Big\| \sum_{i=N}^{\infty} a_{ni} U^i A^* Bh \Big\|^2 = \sum_{i=N}^{\infty} \sum_{j=N}^{\infty} a_{ni} \bar{a}_{nj} \langle U^i A^* Bh, U^j A^* Bh\rangle \leqq \\ \leqq (2M+1) a b_n \|A^* Bh\|^2 + a^2 \varepsilon (1 + \|Bh\|),$$

as can be seen by dividing the double sum into parts where $|i-j| \leqq M$ and $|i-j| > M$, respectively, and using (2.5) and (2.6). Finally (2.2), (2.3) and (2.7) imply $\lim \|\sum_i a_{ni} T^i h\| = 0$.

Remarks. (1) The proof actually shows that (c) $\langle T^n h, S^* Bh\rangle \to 0$ implies (b). This together with (b)$\Rightarrow$(a) shows that (c) is equivalent to $T^n h \to 0$ weakly. In fact, we have for all k, $z \in \mathfrak{K}$, $\limsup |\langle C^n k, z\rangle| \leqq \|z\| \cdot \limsup |\langle C^n k, k\rangle|^{1/2}$ ([6], Lemma 2.1). Applying this to $k = A^* Bh$, $z = A^* R^* y$ for any $y \in \mathfrak{H}$ and utilizing (2.0) and (2.1), it is not hard to show that $\limsup |\langle T^n h, y\rangle| \leqq \|A^* R^* y\| \cdot \limsup |\langle T^n h, S^* Bh\rangle|^{1/2}$. In the case of T being a contraction, (c)$\Rightarrow$(b) was implicitly proved in [8] by a somewhat different method. We can also prove the general case from this by observing that (c) implies, by (2.1), $\langle C^n A^* Bh, A^* Bh\rangle \to 0$, and hence, applying the contraction case and using (2.1) again and (2.0), (b).

(2) An operator T on $\mathfrak{H}$ is said to be *similar* to a contraction C on $\mathfrak{K}$ if there exists a (boundedly) invertible operator $A\colon \mathfrak{K} \to \mathfrak{H}$ such that $T = ACA^{-1}$. Then $T^n = AC^n A^{-1} = AC^n A^* (A^{*-1} A^{-1})$, $n \geqq 0$, and the condition (2.1) is satisfied.

(3) Theorem 1 applies to operators of the C_A classes of H. Langer (see [18], p. 55) and the now classical $C_\varrho = C_{\varrho I}$ classes, $\varrho > 0$, of Sz.-Nagy and Foiaş ([17]). They are those operators T on $\mathfrak{H}$ satisfying $T^n = A^{1/2} P_{\mathfrak{H}} U^n A^{1/2}$, $n \geqq 1$, for a positive and (boundedly) invertible operator A on $\mathfrak{H}$ and a unitary operator U on a Hilbert space $\mathfrak{K} \supset \mathfrak{H}$. Note also that $C_A \subset C_{\|A\|}$ ([12]) and that the union of all C_ϱ classes is dense (in the norm topology) in the set of power-bounded operators ([10]). (b) is valid for all $h \in \mathfrak{H}$ in case of operators with their spectra lying inside the open unit disc. This follows from the fact that $\lim \|T^n\|^{1/n} < 1$ implies $\lim \|T^n h\| = 0$ for all $h \in \mathfrak{H}$. We should also mention that the operators considered here are all similar to contractions (see a general theorem in [11]), and that some power-bounded operators are not similar to any contraction ([7]).

3. Similarity to contractions

We shall give three sufficient conditions for T on $\mathfrak{H}$ to be similar to a contraction. The Corollary below generalizes a result of Sz.-Nagy [16]. The special case where T is power-bounded and $\limsup \|T^n h\| \geqq m\|h\|$, $h \in \mathfrak{H}$, is tacitly contained in [1, p. 238]. In [1] and [16] Banach limits are used as the main tool. Our proof is of a more constructive nature. Theorem 2 will also be used in the proof of Theorem 3.

Theorem 2. *Let T be an operator on $\mathfrak{H}$ satisfying, for a positive number M,*

$$n^{-1} \sum_{i=0}^{n-1} \|T^i h\|^2 \leqq M^2 \|h\|^2 \quad (n \geqq 1,\ h \in \mathfrak{H}). \tag{3.1}$$

Then there exists a positive operator R on $\mathfrak{H}$ such that

$$T^* R T = R \quad \text{and} \quad R \leqq M^2 I. \tag{3.2}$$

If, in addition, there exists a positive number m such that

$$m^2 \|h\|^2 \leqq n^{-1} \sum_{i=0}^{n-1} \|T^i h\|^2 \quad (n \geqq 1,\ h \in \mathfrak{H}) \tag{3.3}$$

then R and its positive square root P are invertible and

$$PTP^{-1} \text{ is an isometry and } mI \leqq P \leqq MI. \tag{3.4}$$

Corollary. *If T is an operator on $\mathfrak{H}$ and there exist positive numbers m, M, p such that*

$$m^p \|h\|^p \leqq \limsup n^{-1} \sum_{i=0}^{n-1} \|T^i h\|^p \leqq M^p \|h\|^p \quad (h \in \mathfrak{H}), \tag{3.5}$$

then

$$(m/M)\|h\| \leqq \|T^n h\| \leqq (M/m)\|h\| \quad (n \geqq 0,\ h \in \mathfrak{H}), \tag{3.6}$$

and T is similar to a contraction.

The same conclusion holds if we replace in the middle term of (3.5) lim sup *by* lim inf *and even if we replace this middle term by* $\limsup \|T^n h\|^p$ *or by* $\liminf \|T^n h\|^p$.

Proof of Corollary. The middle term in (3.5) is unchanged if we change h to $T^j h$, for any $j \geqq 0$. Hence $m^p \|T^i h\|^p \leqq M^p \|T^j h\|^p$, for any $i, j \geqq 0$. The first conclusion then follows. Theorem 2 applies now to give the second conclusion.

Proof of Theorem 2. Consider first the separable case. So assume that there is a countable dense subset $\{h_1, h_2, \ldots\}$ of $\mathfrak{H}$. Let $R_n = n^{-1}\sum_{i=0}^{n-1} T^{*i}T^i$, $n \geqq 1$. R_n is positive and (3.1) implies $R_n \leqq M^2 I$, $n \geqq 1$. Hence for each $j \geqq 1$, $\{R_n h_j \colon n \geqq 1\}$ is bounded and so weakly sequentially compact ([5, II.3.28]). Using the diagonal process, we can extract a subsequence $\{R_n'\}$ such that $R_n' h_j$ converges weakly for each j. It follows that R_n' converges in the weak operator topology to a positive operator $R \leqq M^2 I$. $T^* R_n' T$ converges to T^*RT. On the other hand, $T^* R_n T - R_n = n^{-1}(T^{*n}T^n - I)$, $n \geqq 1$. We claim that $n^{-1}T^{*n}T^n$ converges weakly to 0. This then implies that $T^* R_n' T$ has to converge to R, and thus $T^*RT = R$. For the claim, observe that for each $h \in \mathfrak{H}$ and each positive integer n,

$$n^{-1}\|T^n h\|^2 \sum_{j=1}^{n} j^{-1} = n^{-1} \sum_{j=1}^{n} (j^{-1}\|T^{j-1}T^{n-j+1}h\|^2) \leqq$$

$$\leqq n^{-1} \sum_{j=1}^{n} (M^2\|T^{n-j+1}h\|^2) = M^2 n^{-1} \sum_{i=0}^{n-1} \|T^i T h\|^2 \leqq M^4 \|Th\|^2,$$

by applying (3.1) twice. But $\sum_{j=1}^{n} j^{-1}$ diverges, and hence $n^{-1}\|T^n h\|^2 \to 0$. Now for any $h, k \in \mathfrak{H}$, $|n^{-1}\langle T^{*n}T^n h, k\rangle| \leqq (n^{-1}\|T^n h\|^2)^{1/2}(n^{-1}\|T^n k\|^2)^{1/2} \to 0$, proving the claim. Thus (3.2) is proved.

If in addition (3.3) is assumed, then $m^2 I \leqq R_n$, $n \geqq 1$. In particular $m^2 I \leqq R_n'$, $n \geqq 1$, whence $m^2 I \leqq R$. Thus $m^2 I \leqq R \leqq M^2 I$ and so $mI \leqq P \leqq MI$, and R and P are invertible. From $T^* P^2 T = P^2$, we get $(PTP^{-1})^*(PTP^{-1}) = I$, showing that PTP^{-1} is an isometry.

When $\mathfrak{H}$ is not separable, we proceed as follows. Given any $h \in \mathfrak{H}$, the closed subspace generated by $\{h\} \cup \{S_1 \ldots S_n h \colon n \geqq 1,\ S_i = T \text{ or } T^*,\ 1 \leqq i \leqq n\}$ is separable, contains h, and reduces T. Utilising this construction and employing transfinite induction, $\mathfrak{H}$ can be decomposed into a direct sum of a family of mutually orthogonal, separable, closed subspaces, each reducing T. The construction for the separable case applies to each of these subspaces, and we get a positive operator R on $\mathfrak{H}$ satisfying (3.2) and, if (3.3) is assumed, $m^2 I \leqq R$. The rest of the proof is as before.

Theorem 3 below generalizes the result of G.-C. Rota [15, Th. 2] that every operator T with spectral radius $r < 1$ is similar to a proper contraction (one of norm < 1). This is because $r = \lim \|T^n\|^{1/n}$ and so by the root test for series, $\sum_{n=0}^{\infty} \|T^n\|^2 < \infty$, implying the case $s = 0$ of Theorem 3. Another case, $s = 1$, was treated by Holbrook ([11]) under the assumption that T is power-bounded.

Theorem 3. *Let T be an operator on $\mathfrak{H}$, $0 \leqq s \leqq 1$ a fixed number, and $Q=Q(T,s)=|I-sT^*T|^{1/2}$ (by symbolic calculus). Assume that there exist positive numbers M, N such that* (3.1) *is satisfied and*

$$\sum_{n=0}^{\infty} \|QT^n h\|^2 \leqq N^2 \|h\|^2 \quad (h \in \mathfrak{H}). \tag{3.7}$$

Then there exists a positive operator P on $\mathfrak{H}$ satisfying

$$I \leqq P \leqq (N^2+sM^2)^{1/2} I \tag{3.8}$$

such that PTP^{-1} is a contraction, and a proper one in the case $s=0$.

Condition (3.1) is redundant in the case $s=0$, i.e., $Q=I$.

Proof. Condition (3.7) implies that the increasing sequence of positive operators $S_n=\sum_{i=0}^{n-1} T^{*i}Q^2T^i$, $n \geqq 1$, converges in the weak operator topology to a positive operator $S \leqq N^2 I$. In fact for any $h, k \in \mathfrak{H}$, and any $n > m \geqq 0$,

$$|\langle (S_n-S_m)h, k\rangle| = \left| \sum_{i=m}^{n-1} \langle QT^i h, QT^i k\rangle \right| \leqq$$

$$\leqq \sum_{i=m}^{n-1} \|QT^i h\| \cdot \|QT^i k\| \leqq \left(\sum_{i=m}^{n-1} \|QT^i h\|^2 \right)^{1/2} \left(\sum_{i=m}^{n-1} \|QT^i k\|^2 \right)^{1/2},$$

whence the assertion follows. From the identities $Q^2+T^*S_nT=S_{n+1}$, $n \geqq 1$, we get $Q^2+T^*ST=S$.

For each positive integer n,

$$S+sR_n \geqq \sum_{i=0}^{n-2} s^i T^{*i}Q^2T^i + n^{-1} \sum_{j=1}^{n-1} s^j T^{*j}T^j \geqq n^{-1} \sum_{j=1}^{n-1} \left(\sum_{i=0}^{j-1} s^i T^{*i}Q^2T^i + s^j T^{*j}T^j \right).$$

Since $Q^2+sT^*T=|I-sT^*T|+sT^*T \geqq I$, it follows by easy induction that the terms in the first summation form an increasing sequence of positive operators, each $\geqq I$. Hence $S+sR_n \geqq (1-1/n)I$. By Theorem 2, there exists a positive operator $R \leqq M^2 I$ on $\mathfrak{H}$ with $T^*RT=R$, and by the above inequalities, and considerations as in the proof of Theorem 2, $S+sR \geqq I$. Summing up the results in this and the last paragraphs, we get $Q^2+T^*P^2T=P^2$ and $I \leqq P^2 \leqq (N^2+sM^2)I$, where P is the positive square root of $S+sR$. Hence (3.8) follows and P is invertible. With $C=PTP^{-1}$,

$$(QP^{-1})^*(QP^{-1})+C^*C = P^{-1}(Q^2+T^*P^2T)P^{-1} = P^{-1}P^2P^{-1} = I.$$

This shows that $C^*C \leqq I$ and C is a contraction. In the case $s=0$, we have $Q=I$, and the above equality becomes $P^{-2}+C^*C=I$. Hence for each $h\in\mathfrak{H}$,

$$\|Ch\|^2=\langle h, C^*Ch\rangle=\langle h, h\rangle-\langle h, P^{-2}h\rangle=\|h\|^2-\|P^{-1}h\|^2\leqq\|h\|^2(1-\|P\|^{-2}).$$

Thus C is a proper contraction.

We now present a similarity theorem in a measure-theoretic setting. Let $(X, \mathfrak{F}, \mu)=(X, \mu)$ be a σ-finite measure space, and $L_p=L_p(X, \mathfrak{F}, \mu)$, $1\leqq p<\infty$, the usual Banach spaces of functions. Let $\overline{M}^+$ be the set of extended-valued nonnegative measurable functions (modulo μ-null functions) on (X, μ). A linear operator τ on $\overline{M}^+$ is *monotone* if f_n, $f\in\overline{M}^+$, $f_n\uparrow f$ a.e. implies $\tau f_n\uparrow\tau f$ a.e. (cf. [2], p. 389). For such a τ, its adjoint is uniquely defined as a (linear) operator τ^* on $\overline{M}^+$ satisfying $\int f\cdot\tau^*g\,d\mu=\int g\cdot\tau f\,d\mu$, for all $f, g\in\overline{M}^+$. It is easy to show that τ^* is also monotone and that $\tau^{**}=\tau$. If for a fixed $1\leqq p<\infty$, T is a positive (in the sense that $TL_p^+\subset L_p^+$), bounded linear operator on L_p, then it extends uniquely to a monotone operator τ on $\overline{M}^+$, according to the definition: $\tau f=\lim Tf_n$ a.e., where $f\in\overline{M}^+$, $f_n\in L_p^+$, and $f_n\uparrow f$. For each $f\in\overline{M}^+$, such a sequence f_n always exists and the definition of τf is unambiguous. We shall simply write T for the extended τ.

Theorem 4. *Let τ be a monotone operator on $\overline{M}^+$ and $1\leqq p<\infty$ a fixed number. Assume that, for $p=1$, $\tau^*k\leqq k$; and for $p>1$,*

$$\tau^*\big(k(\tau h)^{p-1}\big)\leqq kh^{p-1} \text{ for some functions } 0<h, k<\infty. \tag{3.9}$$

Then σ, defined on $\overline{M}^+$ as $\sigma f=k^{1/p}\tau(fk^{-1/p})$, is a positive L_p contraction. Further, (3.9) *is equivalent to*

$$\tau\big(k_1(\tau^*h_1)^{p'-1}\big)\leqq k_1h_1^{p'-1} \text{ for some functions } 0<h_1, k_1<\infty, \text{ with} \tag{3.10}$$
$$k_1=k^{-p'+1} \text{ and } 1/p+1/p'=1.$$

Corollary. *Suppose T is a positive (in the sense that $TL_2^+\subset L_2^+$), bounded operator on L_2, and $T^*(kTh)\leqq kh$ or $T(kT^*h)\leqq kh$ for some functions $0<h<\infty$ $m\leqq k\leqq M$, where m, M are positive constants. Then T is similar to a positive contraction on L_2.*

Proof of Theorem 4. The case $p=1$ is easy. Consider the case $p>1$. First we show (3.9)$\Rightarrow$(3.10). Suppose (3.9) holds. Then $\tau h<\infty$. For if $\tau h=\infty$ on a set E of positive measure, then for all positive numbers N,

$$N\tau^*(k1_E)=\tau^*(Nk1_E)\leqq\tau^*\big(k(\tau h)^{p-1}\big)\leqq kh^{p-1}<\infty,$$

implying $\tau^*(k1_E)=0$. So $0=\int h\cdot\tau^*(k1_E)d\mu=\int_E k\cdot\tau h\,d\mu=\infty$, a contradiction. Let

$F=\{\tau h=0\}$. Then $\int h\cdot\tau^* 1_F d\mu=\int_F \tau h d\mu=0$, and hence $\tau^* 1_F=0$. Define $h_1=$ $=k(\tau h)^{p-1}+1_F$. Then $0<h_1<\infty$, and (3.10) can be verified as follows: $\tau^* h_1=$ $=\tau^*(k(\tau h)^{p-1})+0\leqq kh^{p-1}$ by (3.9); consequently, $(\tau^* h_1)^{p'-1}=(\tau^* h_1)^{1/(p-1)}\leqq$ $\leqq(kh^{p-1})^{1/(p-1)}=k^{1/(p-1)}h=k_1^{-1}h$, and hence, $\tau(k_1(\tau^* h_1)^{p'-1})\leqq\tau h\leqq(k^{-1}h_1)^{1/(p-1)}=$ $=k_1 h_1^{1/(p-1)}=k_1 h_1^{p'-1}$, which is (3.10). Implication (3.10)$\Rightarrow$(3.9) can be proved similarly, by replacing (τ, h, k, p) by (τ^*, h_1, k_1, p'). From the definition of σ we can show that $\sigma^* f=k^{-1/p}\tau^*(k^{1/p}f), f\in\overline{M}^+$. Hence (3.9) transforms into $\sigma^*(\sigma u)^{p-1}\leqq$ $\leqq u^{p-1}$, where $u=hk^{1/p}$. This implies that σ is a contraction on L_p. In case of a Borel space, this implication follows from a dilation theorem in [2]. The general case is proved here by adapting the proof in [9] for the case $\sigma 1\leqq 1$, $\sigma^* 1\leqq 1$. In fact, we have $\sigma u<\infty$, just as $\tau h<\infty$. For $f\in\overline{M}^+$ and any $\lambda>0$,

$$\int 1_{\{\sigma f\geqq\lambda\sigma u>0\}}(\sigma f-\lambda\sigma u)\cdot(\sigma u)^{p-1}\,d\mu\leqq$$

$$\leqq\int\sigma(f-\lambda u)^+\cdot(\sigma u)^{p-1}\,d\mu=\int(f-\lambda u)^+\cdot\sigma^*(\sigma u)^{p-1}\,d\mu\leqq\int 1_{\{f\geqq\lambda u\}}(f-\lambda u)u^p\,d^{-1}\mu.$$

Multiplying both sides by λ^{p-2}, and integrating with respect to λ from 0 to ∞, we obtain, by the Fubini—Tonelli Theorem,

$$\left(\frac{1}{p-1}-\frac{1}{p}\right)\int(\sigma f)^p\,d\mu\leqq\left(\frac{1}{p-1}-\frac{1}{p}\right)\int f^p\,d\mu,$$

showing that σ is an L_p contraction.

Remarks (4). If $\sigma\colon\overline{M}^+(X,\mu)\to\overline{M}^+(Y,\nu)$ is monotone, $1\leqq p\leqq q<\infty$, $0<u\in L_q(X,\mu)$, and $\sigma^*(\sigma u)^{p-1}\leqq u^{q-1}$, then σ extends to a bounded, positive linear operator from $L_q(X,\mu)$ to $L_p(Y,\nu)$ with norm $\leqq\|u\|_q^{(q/p)-1}$. Indeed, by the method of the proof of Theorem 4, we have for all $f\in L_q^+(X,\mu)$, $\int(\sigma f)^p d\nu\leqq\int f^p u^{q-p}\,d\mu$. (This is trivial when $p=1$, for which case the condition on σ reads $\sigma^* 1\leqq u^{q-1}$.) By the Schwarz inequality, the last integral is $\leqq\left(\int f^q d\mu\right)^{p/q}\left(\int u^q d\mu\right)^{(q-p)/q}$. The conclusion follows. This generalizes a result in [14] for non-negative infinite matrices, as it can be easily shown that non-negative matrices are monotone. Analogous to Theorem 4, the inequality for σ is equivalent to $\sigma(\sigma^* v)^{q'-1}\leqq v^{p'-1}$ for some $0<v\in L_{p'}(Y,\nu)$ when $1<p$, where $1/p+1/p'=1$ and $1/q+1/q'=1$.

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