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Note on the convergence of Fourier series in the spaces Λ_ω^p

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In this note we consider the basis problem for the trigonometric system in the spaces Λ_ω^p defined as follows: Let ω be a modulus of continuity and let $1 \leqq p < \infty$ be a real number. The class Λ_ω^p consists of all functions $f \in L^p$ for which the norm

$$\|f\|_{p,\omega} = \|f\|_p + \|f\|_{p,\omega}^*$$

is finite, where

$$\|f\|_p = \left\{ \int_{-\pi}^{\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}}, \quad \|f\|_{p,\omega}^* = \sup_{0<\delta\leqq\pi} \frac{\omega_p(\delta, f)}{\omega(\delta)}.$$

(We refer to [1] for $\omega_p(f)$ and ω.) With respect to this norm Λ_ω^p is a nonseparable Banach space.

A sequence $\{f_n\}$ of elements in the Banach space B, which is a basis for its closed span $E(\{f_n\}, B) = E(B)$ is called a basic sequence.

Theorem 1. *For any ω and $1 < p < \infty$ the trigonometric system is a basic sequence in the space Λ_ω^p.*

If T_n is a trigonometric polynomial of degree n, then the inequality

$$\|T_n'\|_p \leqq n\omega_p\left(\frac{\pi}{n}, T_n\right) \tag{1}$$

holds [5].

For any $f \in L^p$ and $n \geqq 0$ the inequality[1])

$$\|S_n f\|_p \leqq C_p \|f\|_p \tag{2}$$

is true [4], where $S_n f$ denotes the n-th partial sum of the Fourier series of f.

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[1]) C_p will always denote positive constants depending only on p, not necessarily the same at each occurrence.

Proof of Theorem 1. Since for any absolutely continuous function f with $f' \in L^p$ the inequality

$$\omega_p(\delta,f) \leqq \delta \|f'\|_p \quad (0 < \delta \leqq \pi)$$

holds, by (1) we obtain

$$\omega_p(\delta, T_n) \leqq n\delta\omega_p\left(\frac{\pi}{n}, T_n\right) \tag{3}$$

for any trigonometric polynomial T_n. Furthermore, from (2) and a theorem of Jackson type in the space L^p (see [6]) for any $f \in L^p$ and $n \geqq 1$ the inequality

$$\|f - S_n f\|_p \leqq C_p \omega_p\left(\frac{\pi}{n}, f\right) \tag{4}$$

follows.

Using the inequality (3) we have

$$\omega_p(\delta, S_n f) \leqq n\delta\omega_p\left(\frac{\pi}{n}, S_n f\right) \leqq n\delta\left[\omega_p\left(\frac{\pi}{n}, f\right) + \omega_p\left(\frac{\pi}{n}, f - S_n f\right)\right] \leqq$$

$$\leqq n\delta\left[\omega_p\left(\frac{\pi}{n}, f\right) + 2\|f - S_n f\|_p\right]$$

so, by inequality (4),

$$\omega_p(\delta, S_n f) \leqq C_p n\delta\omega_p\left(\frac{\pi}{n}, f\right) \tag{5}$$

holds. From (5) and by a familiar inequality (see e.g. [8] p. 111)

$$\omega(\delta) \leqq 2\delta\eta^{-1}\omega(\eta) \quad (0 < \eta \leqq \delta \leqq \pi)$$

it follows that

$$\omega_p(\delta, S_n f) \leqq C_p \omega(\delta) \|f\|^*_{p,\omega} \quad \left(0 < \delta \leqq \frac{\pi}{n}\right).$$

If $\frac{\pi}{n} \leqq \delta \leqq \pi$, then by (4) we have

$$\omega_p(\delta, S_n f) \leqq \omega_p(\delta, f) + \omega_p(\delta, f - S_n f) \leqq$$

$$\leqq \omega_p(\delta, f) + 2\|f - S_n f\|_p \leqq C_p \omega_p(\delta, f).$$

From the last two inequalities we obtain

$$\|S_n f\|^*_{p,\omega} \leqq C_p \|f\|^*_{p,\omega}$$

and by (2)

$$\|S_n f\|_{p,\omega} \leqq C_p \|f\|_{p,\omega}.$$

Now our statement follows from a known theorem (see e.g. [7], p. 58). The proof is complete.

In order to describe the subspaces $E(\Lambda_\omega^p)$ we consider the classes

$$\lambda_\omega^p = \left\{f\in \Lambda_\omega^p \colon \lim_{\delta\to 0} \frac{\omega_p(\delta, f)}{\omega(\delta)} = 0\right\}$$

which are closed subspaces in Λ_ω^p.

We show that *if the condition*

$$\lim_{\delta\to 0}\frac{\delta}{\omega(\delta)} = 0 \tag{6}$$

is fulfilled, then

$$E(\Lambda_\omega^p) = \lambda_\omega^p. \tag{7}$$

In fact, if the function $f\in\lambda_\omega^p$, then

$$\omega_p(\delta,f) \leqq \varepsilon_p(\delta,f)\,\omega(\delta),$$

where $\varepsilon_p(\delta,f)\downarrow 0$ as $\delta\downarrow 0$. We can take for example

$$\varepsilon_p(\delta,f) = \omega_\infty\left(\delta, \frac{\omega_p(f)}{\omega}\right).$$

For $\frac{\pi}{n}<\delta\leqq\pi$, by (4), we have

$$\frac{\omega_p(\delta,f-S_nf)}{\omega(\delta)} \leqq \frac{\omega_p(1,f-S_nf)}{\omega\left(\frac{\pi}{n}\right)} \leqq C_p\varepsilon_p\left(\frac{\pi}{n},f\right)$$

and for $0<\delta\leqq\frac{\pi}{n}$ from (5) the inequality

$$\frac{\omega_p(\delta,f-S_nf)}{\omega(\delta)} \leqq \frac{\omega_p(\delta,f)}{\omega(\delta)} + C_p\frac{n\delta}{\omega(\delta)}\omega_p\left(\frac{\pi}{n},f\right)$$

follows. As in the proof of Theorem 1, we obtain hence

$$\|f-S_nf\|_p \leqq C_{p,}\left[\omega_p\left(\frac{\pi}{n},f\right)+(1+\|f\|_{p,\omega}^*)\varepsilon_p\left(\frac{\pi}{n},f\right)\right]$$

and thus $\lambda_\omega^p\subset E(\Lambda_\omega^p)$.

Since by condition (6) $\sin nx$, $\cos nx\in\lambda_\omega^p$ $(n\geqq 0)$ and λ_ω^p is a closed subspace of Λ_ω^p, thus $E(\Lambda_\omega^p)\subset\lambda_\omega^p$ and (7) is proved.

If the condition (6) is not fulfilled, then Λ_ω^p contains only the functions which are equivalent to constants. Consequently, by Theorem 1 we have

Theorem 2. *The trigonometric system forms a basis in the space* λ_ω^p, $1<p<\infty$ *if and only if condition* (6) *holds.*

A system, which is a basic sequence for every permutation of its terms, is called an unconditional basic sequence.

For any $f \in L^2$ we have by the Parseval formula

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+h)-f(x-h)|^2 \, dx = 4 \sum_{n=1}^{\infty} (a_n^2+b_n^2) \sin^2 nh \quad (h>0),$$

where a_n and b_n are Fourier coefficients of n. Hence it is easy to obtain that *the trigonometric system is an unconditional basic sequence in* Λ_ω^p *for every* ω.

On the other hand, *if* $p \neq 2$, *then the trigonometric system does not form an unconditional basic sequence in the space* Λ_ω^p, *where* $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$. This statement follows from Konjushkov [2], Theorems 8 and 10.

In [3] we have given necessary and sufficient conditions that the Haar system should be a basic or unconditional basic sequence in the spaces Λ_ω^p, $1 \leqq p < \infty$.

Finally we remark that Theorem 1 is also true for other spaces. So we can consider the spaces defined by the modulus of smoothness of order k of functions; or, for example, we can take the spaces

$$W^r \Lambda_\omega^p = \{f\colon f^{(r-1)} \in AC, f^{(r)} \in \Lambda_\omega^p\}.$$

Since the proofs are the same as before, we omit them.

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