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Multiparameter strong laws of large numbers. II (Higher order moment restrictions)

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§ 1. Introduction

We use the notations introduced in [5] with the exceptions that at present

(i) it is more convenient to write $\zeta_{\mathbf{k}}$ into the form $\zeta_{\mathbf{k}}=a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)$, where $\{a_{\mathbf{k}}\}=\{a_{\mathbf{k}}: \mathbf{k}\in Z_+^d\}$ is a set of numbers (coefficients) and $\{\varphi_{\mathbf{k}}(x)=\{\varphi_{\mathbf{k}}(x): \mathbf{k}\in Z_+^d\}$ is a set of measurable functions defined on a positive measure space (X, A, μ);

(ii) by $\mathbf{m}=(m_1, \ldots, m_d)\to\infty$ we always mean that only $\max(m_1, \ldots, m_d)\to\infty$ (and $\min(m_1, \ldots, m_d)\not\to\infty$ may also occur).

We consider the d-multiple series

$$\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)=\sum_{j=1}^{d}\sum_{k_j=1}^{\infty} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x), \tag{1.1}$$

where the multiindex $\mathbf{k}=(k_1, \ldots, k_d)$ belongs to Z_+^d, the partially ordered set of the d-tuples of positive integers, d being a fixed positive integer. The set of d-tuples of non-negative integers is denoted by Z^d. For $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$ write

$$S(\mathbf{b},\mathbf{m};\, x)=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)=\sum_{j=1}^{d}\sum_{k_j=b_j+1}^{b_j+m_j} a_{k_1,\ldots,k_d}(x)\ \varphi_{k_1,\ldots,k_d}(x)$$

and

$$M(\mathbf{b},\mathbf{m};\, x)=\max_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}}|S(\mathbf{b},\mathbf{k};\, x)|=\max_{1\leqq j\leqq d}\ \max_{1\leqq k_j\leqq m_j}|S(b_1, \ldots, b_d;\, k_1, \ldots, k_d;\, x)|.$$

In case $\mathbf{b}=\mathbf{0}$ write $S(\mathbf{0},\mathbf{m};x)=S(\mathbf{m};x)$ (rectangular partial sums of (1.1)) and $M(\mathbf{0},\mathbf{m};x)=M(\mathbf{m};x)$.

Throughout the paper we assume that there exist a number $r>2$ and a constant C such that the inequality

$$\int|S(\mathbf{b},\mathbf{m};\, x)|^r\,d\mu(x)\leqq C\Big(\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} a_{\mathbf{k}}^2\Big)^{r/2} \tag{1.2}$$

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holds for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z^d_+$, and either for all sets $\{a_\mathbf{k}\}$ (in §§ 1—2) or for only the single set $\{a_\mathbf{k}\equiv 1\}$ of coefficients (in §§ 3—4).

Here and in the sequel the integrals, unless stated otherwise, are taken over X; $C, C_1, C_2, \ldots$ denote positive constants, not necessarily the same at different occurrences.

Example 1. Let r be an integer, $r\geqq 2$. The set $\{\varphi_\mathbf{k}(x)\}$ is said to be *multiplicative of order* r if for all systems of pairwise distinct $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_r$ from Z^d_+ we have

$$\int \left(\prod_{p=1}^{r} \varphi_{\mathbf{k}_p}(x) \right) d\mu(x) = 0. \tag{1.3}$$

This definition for $d=1$ is due to Alexits [1, p. 186].

The arguments of Gapoškin [2], Komlós and Révész [3] in the case $d=1$ obviously apply to the case $d\geqq 2$ and lead to the following result: *Let r be an even integer,* $r\geqq 4$. *If* $\{\varphi_\mathbf{k}(x)\}$ *is multiplicative of order r and*

$$\int \varphi^r_\mathbf{k}(x)\, d\mu(x) \leqq C \tag{1.4}$$

for all $\mathbf{k}\in Z^d_+$, *then we have* (1.2) *for all* $\{a_\mathbf{k}\}$.

Example 2. The vanishing of the integrals in (1.3) is of no relevance, only their "smallness" in a certain sense is needed.

In case $d=1$, according to Gapoškin [2], a sequence $\{\varphi_i(x)\}_{i=1}^\infty$ is said to be *weakly multiplicative of order r*, where r is an even positive integer, if there exists a non-negative function $h(l)$ such that for every $1\leqq i_1<i_2<\ldots<i_r$ we have

$$\left| \int \left(\prod_{p=1}^{r} \varphi_{i_p}(x) \right) d\mu(x) \right| \leqq h(l)$$

with $l=\min(i_2-i_1, i_4-i_3, \ldots, i_r-i_{r-1})$ and

$$\sum_{l=1}^{\infty} l^{(r-2)/2} h(l) < \infty.$$

Now it is proved in [2] (and announced in [3]) that *if* $r\geqq 4$, $\{\varphi_i(x)\}_{i=1}^\infty$ *is a weakly multiplicative sequence of order r, which satisfies* (1.4), *then we have* (1.2) *for all* $\{a_i\}_{i=1}^\infty$.

In case $d\geqq 2$, let (X_j, A_j, μ_j) be a positive measure space, $\{\varphi_i^{(j)}(x_j)\}_{i=1}^\infty$ a sequence of measurable functions on X_j for each $j=1, 2, \ldots, d$. Let

$$(X, A, \mu) = \mathop{\times}_{j=1}^{d} (X_j, A_j, \mu_j)$$

be the product measure space and let

$$\varphi_{\mathbf{k}}(x)=\prod_{j=1}^{d}\varphi_{k_j}^{(j)}(x_j),\quad \text{where}\quad \mathbf{k}=(k_1,\ldots,k_d)\quad\text{and}\quad x=(x_1,\ldots,x_d).$$

The following statement holds: *If for some* $r\geqq 2$ *each sequence* $\{\varphi_i^{(j)}(x_j)\}_{i=1}^{\infty}$ $(j=1,2,\ldots,d)$ *satisfies the inequality*

$$\int_{X_j}\Big|\sum_{i=b+1}^{b+m}a_i\varphi_i^{(j)}(x_j)\Big|^r d\mu_j(x_j)\leqq C_j\Big(\sum_{i=b+1}^{b+m}a_i^2\Big)^{r/2}\tag{1.5}$$

for all $\{a_i\}_{i=1}^{\infty}$, $b\geqq 0$ *and* $m\geqq 1$, *then* $\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in Z_+^d\}$ *satisfies inequality* (1.2) *for all* $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$, $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$ *with* $C=\prod_{j=1}^{d}C_j$.

For simplicity, assume that $d=2$. Then by (1.5), Fubini's theorem, and Minkowski's inequality we get that

$$\int_{X_1}\int_{X_2}\Big|\sum_{i=b+1}^{b+m}\sum_{k=c+1}^{c+n}a_{ik}\varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2)\Big|^r d\mu_1(x_1)\,d\mu_2(x_2)=$$

$$=\int_{X_2}\Big\{\int_{X_1}\Big|\sum_{i=b+1}^{b+m}\Big(\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\Big)\varphi_i^{(1)}(x_1)\Big|^r d\mu_1(x_1)\Big\}d\mu_2(x_2)\leqq$$

$$\leqq C_1\int_{X_2}\Big\{\sum_{i=b+1}^{b+m}\Big(\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\Big)^2\Big\}^{r/2}d\mu_2(x_2)\leqq$$

$$\leqq C_1\Big\{\sum_{i=b+1}^{b+m}\Big(\int_{X_2}\Big|\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\Big|^r d\mu_2(x_2)\Big)^{2/r}\Big\}^{r/2}\leqq$$

$$\leqq C_1C_2\Big(\sum_{i=b+1}^{b+m}\sum_{k=c+1}^{c+n}a_{ik}^2\Big)^{r/2}.$$

This is the wanted inequality (1.2).

The results below will be obtained by adaptation of more or less standard arguments well-known in probability theory concerning limit theorems, and by making use of a recent maximal inequality of the author [4, Theorem 7]. It is worth stating the special case of this inequality for $\alpha=r/2$, $\gamma=r$ and $f(\mathbf{b},\mathbf{m})=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}a_{\mathbf{k}}^2$ in the form of a separate lemma.

Lemma 1. *Let* $r>2$ *and* $\{a_{\mathbf{k}}\}$ *be given. If inequality* (1.2) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$, *then*

$$\int M^r(\mathbf{b},\mathbf{m};\,x)\,d\mu(x)\leqq C_1\Big(\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}a_{\mathbf{k}}^2\Big)^{r/2}\tag{1.6}$$

also holds for all $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$.

2§. A.e. convergence of series (1.1)

Theorem 1. *Let $r>2$ and let $\{a_{\mathbf{k}}\}$ be such that*

$$\sum_{\mathbf{k}\geqq\mathbf{1}} a_{\mathbf{k}}^2<\infty. \tag{2.1}$$

If inequality (1.2) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$, *then*

$$S(\mathbf{b},\mathbf{m};\,x)\to 0 \quad a.e.\ as\quad \mathbf{b}\to\infty \quad and\quad \mathbf{m}\in Z_+^d; \tag{2.2}$$

furthermore,

$$\int\big(\sup_{\mathbf{b}\geqq\mathbf{0}}\sup_{\mathbf{m}\geqq\mathbf{1}} |S(\mathbf{b},\mathbf{m};\,x)|\big)^r\,d\mu(x)\leqq C_2\big(\sum_{\mathbf{k}\geqq\mathbf{1}} a_{\mathbf{k}}^2\big)^{r/2}. \tag{2.3}$$

In particular, from (2.2) it follows that the d-multiple series (1.1) converges a.e. in the sense that its rectangular partial sums $S(\mathbf{m};x)$ converge a.e. as $\min(m_1,\dots,m_d)\to\infty$. (See more detailed in [6].)

Lemma 2 ([6, Lemma 1]). *For all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$

$$\max_{\mathbf{1}\leqq\mathbf{p}\leqq\mathbf{q}\leqq\mathbf{m}}\Big|\sum_{\mathbf{b}+\mathbf{p}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{q}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)\Big|\leqq 2^d M(\mathbf{b},\mathbf{m};x).$$

Proof of Theorem 1. Condition (2.1) implies the existence of a sequence $\{\mathbf{m}_l=(m_{1l},\dots,m_{dl})\}_{l=1}^\infty$ in Z_+^d for which

(i) $1\leqq m_{j1}<m_{j2}<\dots$ for each $j=1,2,\dots,d$;

(ii) $\Big\{\sum_{\mathbf{k}\geqq\mathbf{1}}-\sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}_l}\Big\}a_{\mathbf{k}}^2\leqq(l+1)^{-2(r+2)/r}\sum_{\mathbf{k}\geqq\mathbf{1}}a_{\mathbf{k}}^2\quad(l=1,2,\dots)$.

It follows from (i) that $\min(m_{1l},\dots,m_{dl})\to\infty$ as $l\to\infty$, and from (ii) that

$$\sum_{l=0}^\infty (l+1)^r\Big\{\sum_{\mathbf{k}\geqq\mathbf{1}}-\sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}_l}\Big\}a_{\mathbf{k}}^2\leqq 2\big(\sum_{\mathbf{k}\geqq\mathbf{1}}a_{\mathbf{k}}^2\big)^{r/2}\quad(\mathbf{m}_0=\mathbf{0}). \tag{2.4}$$

Motivating by the representation

$$S(\mathbf{m}_{l+1};\,x)-S(\mathbf{m}_l;\,x)=\sum_{\boldsymbol{\varepsilon}} S(\boldsymbol{\varepsilon}\mathbf{m}_l,\,\boldsymbol{\varepsilon}(\mathbf{m}_{l+1}-\mathbf{m}_l)+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{m}_l;\,x),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is extended over all 2^d-1 choices of $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_d)$ with coordinates $\varepsilon_j=0$ or 1, the case $\varepsilon_1=\dots=\varepsilon_d=0$ excluded, we introduce the following maxima:

$$M_{t,l}(x)=M(\boldsymbol{\varepsilon}\mathbf{m}_l,\,\boldsymbol{\varepsilon}(\mathbf{m}_{l+1}-\mathbf{m}_l)+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{m}_l;\,x),$$

where $t=\varepsilon_1+2\varepsilon_2+\dots+2^{d-1}\varepsilon_d$. It is clear that $1\leqq t\leqq 2^d-1$.

We are going to show that

$$\sum_{l=0}^\infty (l+1)^r\Big(\sum_{t=1}^{2^d-1} M_{t,l}^r(x)\Big)<\infty\quad\text{a.e.} \tag{2.5}$$

Inequality (1.2), via Lemma 1, yields

$$\sum_{l=0}^{\infty}(l+1)^r\left(\sum_{t=1}^{2^d-1}\int M_{t,l}^r(x)\,d\mu(x)\right)\leqq \tag{2.6}$$

$$\leqq C_1\sum_{l=0}^{\infty}(l+1)^r\{(\sum_{1\leqq\mathbf{k}\leqq\mathbf{m}_{l+1}}-\sum_{1\leqq\mathbf{k}\leqq\mathbf{m}_l})a_{\mathbf{k}}^2\}^{r/2}\leqq 2C_1(\sum_{\mathbf{k}\geqq 1}a_{\mathbf{k}}^2)^{r/2},$$

the last inequality is owing to (2.4). Hence B. Levi's theorem implies (2.5).

Let us now estimate $S(\mathbf{b},\mathbf{m};x)$ with arbitrary $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$. Recall that $\mathbf{b}\not\leqq\mathbf{m}_l$ iff $b_j>m_{jl}$ for at least one j $(1\leqq j\leqq d)$. In the special case when there exists a non-negative integer l such that $\mathbf{b}\not\leqq\mathbf{m}_l$ and $\mathbf{b}+\mathbf{m}\leqq\mathbf{m}_{l+1}$, by Lemma 2 we obviously have

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\sum_{t=1}^{2^d-1}M_{t,l}(x).$$

In the general case let us determine non-negative integers u and v such that

$$\mathbf{b}\not\leqq\mathbf{m}_u\quad\text{and}\quad\mathbf{b}\leqq\mathbf{m}_{u+1}\ ;\quad\mathbf{b}+\mathbf{m}\not\leqq\mathbf{m}_v\quad\text{and}\quad\mathbf{b}+\mathbf{m}\leqq\mathbf{m}_{v+1}.$$

It is clear that such u and v (uniquely) exist, and $0\leqq u\leqq v$. Again by virtue of Lemma 2 we have

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\sum_{l=u}^{v}\left(\sum_{t=1}^{2^d-1}M_{t,l}(x)\right),$$

whence, using Hölder's inequality,

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\left\{\sum_{l=u}^{v}(l+1)^r\left(\sum_{t=1}^{2^d-1}M_{t,l}^r(x)\right)\right\}^{1/r}\times \tag{2.7}$$

$$\times\left\{\sum_{l=u}^{v}\frac{2^d-1}{(l+1)^{r'}}\right\}^{1/r'}\quad\text{with}\quad r'=\frac{r}{r-1}>1.$$

By (2.5) we can conclude that $|S(\mathbf{b},\mathbf{m};x)|$ is a.e. as small as required if $\max(b_1,\ldots,b_d)$, and consequently u is large enough. This proves (2.2).

From (2.7) we obtain that

$$(\sup_{\mathbf{b}\geqq 0}\sup_{\mathbf{m}\geqq 1}|S(\mathbf{b},\mathbf{m};\,x)|)^r\leqq C_3\sum_{l=0}^{\infty}(l+1)^r\left(\sum_{t=1}^{2^d-1}M_{t,l}^r(x)\right).$$

Integrating both sides over X, the wanted inequality (2.3) comes from (2.6). This completes the proof of Theorem 1.

§ 3. Multiparameter SLLN

In the sequel we assume that all $a_{\mathbf{k}}=1$ in (1.1), i.e. from now on

$$S(\mathbf{b}, \mathbf{m};\ x) = \sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} \varphi_{\mathbf{k}}(x)$$

and

$$M(\mathbf{b}, \mathbf{m};\ x) = \max_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}} \Big| \sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{l}\leqq\mathbf{b}+\mathbf{k}} \varphi_{\mathbf{l}}(x) \Big| \quad (\mathbf{b}\in Z^d \text{ and } \mathbf{m}\in Z_+^d),$$

although our results remain valid in the more general setting when $\sum_{\mathbf{k}\geqq\mathbf{1}} a_{\mathbf{k}}^2=\infty$ and $\{a_{\mathbf{k}}\}$ behaves sufficiently "regularly".

Our permanent assumption is now that inequality (1.2) holds true only in this special $a_{\mathbf{k}}\equiv 1$ case, i.e. there exists a number $r>2$ such that

$$\int |S(\mathbf{b}, \mathbf{m};\ x)|^r\, d\mu(x) \leqq C|\mathbf{m}|^{r/2} \tag{3.1}$$

holds for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$, where $|\mathbf{m}|$ stands for $\prod_{j=1}^{d} m_j$. Hence Lemma 1 implies

$$\int M^r(\mathbf{b}, \mathbf{m};\ x)\, d\mu(x) \leqq C_1|\mathbf{m}|^{r/2}. \tag{3.2}$$

Theorem 2. *If inequality* (3.1) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$ *with an* $r>2$, *then for any* $\delta>0$ *we have*

$$\lim_{\mathbf{m}\to\infty} \frac{S(\mathbf{m};\ x)}{|\mathbf{m}|^{1/2}\left(\sum_{j=1}^{d}\log 2m_j\right)^{1/r}(\log\log 4|\mathbf{m}|)^{(d+\delta)/r}} = 0 \quad a.e. \tag{3.3}$$

and

$$\lim_{\mathbf{m}\to\infty} \frac{S(\mathbf{m};\ x)}{|\mathbf{m}|^{1/2}(\log 2|\mathbf{m}|)^{d/r}(\log\log 4|\mathbf{m}|)^{(1+\delta)/r}} = 0 \quad a.e. \tag{3.4}$$

Here and in the sequel all logarithms are of base 2. Further, $S(\mathbf{m};\ x)=S(\mathbf{0}, \mathbf{m};\ x)$.

This result for $d=1$ (in a slightly weaker form) was proved by Serfling [7, Theorem 3.1].

Remark 1. For $d=1$ relations (3.3) and (3.4) coincide. For $d\geqq 2$, if $\mathbf{m}\to\infty$ is such a way that $m_1=m_2=\ldots=m_d$ then (3.3) is stronger than (3.4), while if $\mathbf{m}\to\infty$ in such a way that e.g. $m_2=\ldots=m_d=1$ then the situation is converse: (3.4) is stronger than (3.3).

Both (3.3) and (3.4) improve as r increases. By letting $r\to\infty$ we find, for any $\delta>0$,

$$\lim_{\mathbf{m}\to\infty}\frac{S(\mathbf{m};\,x)}{|\mathbf{m}|^{1/2}(\log 2|\mathbf{m}|)^{\delta}}=0\quad\text{a.e.}$$

This result is not far from the "$\leqq$" part of the law of the iterated logarithm.

Lemma 3. *For any* $\delta>0$, *we have*

$$\sum_{\mathbf{k}\geqq\mathbf{0}}|\mathbf{1}+\mathbf{k}|^{-1}\Big\{\log\Big(2+\sum_{j=1}^{d}k_j\Big)\Big\}^{-d-\delta}<\infty$$

and

$$\sum_{\mathbf{k}\geqq\mathbf{0}}\Big(1+\sum_{j=1}^{d}k_j\Big)^{-d}\Big\{\log\Big(2+\sum_{j=1}^{d}k_j\Big)\Big\}^{-1-\delta}<\infty.$$

Proof of Lemma 3. For simplicity, we only prove in the case $d=2$. Then the first series can be rewritten and estimated as follows

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ik(\log(i+k))^{2+\delta}}\leqq\sum_{i=1}^{\infty}\frac{1}{i}\Big\{\sum_{k=1}^{i}\frac{1}{k(\log(1+i))^{2+\delta}}+$$

$$+\sum_{k=i+1}^{\infty}\frac{1}{k(\log(1+k))^{2+\delta}}\Big\}\leqq C_4\sum_{i=1}^{\infty}\frac{1}{i(\log(1+i))^{1+\delta}}<\infty.$$

The convergence of the second series can be verified similarly.

Proof of Theorem 2. Lemma 1 constitutes the basis of the proof. Applying Chebyshev's inequality to (3.2) we obtain that

$$\mu\{M(\mathbf{b},\mathbf{m};\,x)\geqq\lambda\}\leqq\frac{C_1|\mathbf{m}|^{r/2}}{\lambda^r}\quad(\mathbf{b}\in Z^d,\,\mathbf{m}\in Z_+^d\ \text{and}\ \lambda>0).\tag{3.5}$$

Substituting here

$$\lambda(\mathbf{m})=|\mathbf{m}|^{1/2}\Big(\prod_{j=1}^{d}\log 2m_j\Big)^{1/r}(\log\log 4|\mathbf{m}|)^{(d+\delta)/r}\quad\text{or}$$

$$|\mathbf{m}|^{1/2}(\log 2|\mathbf{m}|)^{d/r}(\log\log 4|\mathbf{m}|)^{(1+\delta)/r}$$

for λ, we get that

$$\mu\{M(\mathbf{m};\,x)\geqq\lambda(\mathbf{m})\}\leqq C_1\Big(\prod_{j=1}^{d}\log 2m_j\Big)^{-1}(\log\log 4|\mathbf{m}|)^{-d-\delta}\quad\text{or}$$

$$C_1(\log 2|\mathbf{m}|)^{-d}(\log\log 4|\mathbf{m}|)^{-1-\delta}.$$

Let $\mathbf{m}=2^{\mathbf{k}}$ where $\mathbf{k}$ runs over Z^d. Then, by Lemma 3,

$$\sum_{\mathbf{k}\geqq 0} \mu\{M(2^{\mathbf{k}};\, x) \geqq \lambda(2^{\mathbf{k}})\} < \infty.$$

Hence, via the Borel—Cantelli lemma, we have

$$M(2^{\mathbf{k}};\, x) < \lambda(2^{\mathbf{k}}) \quad \text{a.e.,}$$

with the exception of a finite number (depending on x) of $\mathbf{k}$.

It is obvious that if $2^{\mathbf{k}} \leqq \mathbf{m} \leqq 2^{\mathbf{k}+1}$ with some $\mathbf{k}\geqq \mathbf{0}$, then we have

$$\lambda(\mathbf{m}) \geqq \lambda(2^{\mathbf{k}}) \quad \text{and} \quad |S(\mathbf{m};\, x)| \leqq M(2^{\mathbf{k}+1};\, x).$$

Consequently,

$$\frac{|S(\mathbf{m};\, x)|}{\lambda(\mathbf{m})} \leqq \frac{M(2^{\mathbf{k}+1};\, x)}{\lambda(2^{\mathbf{k}})} < \frac{\lambda(2^{\mathbf{k}+1})}{\lambda(2^{\mathbf{k}})} \quad \text{a.e.,} \tag{3.6}$$

provided $\max(k_1, \ldots, k_d)$ is large enough. Since the right-most member in (3.6) is bounded as $\mathbf{k}\to\infty$, it follows that

$$S(\mathbf{m};\, x) = O\{\lambda(\mathbf{m})\} \quad \text{a.e.} \tag{3.7}$$

Taking into consideration that δ may be chosen arbitrarily small (but positive), we can change "O" to "o" in (3.7), as a result of which we get the wanted (3.3) and (3.4).

§ 4. Rates of convergence

Turning to the rate of convergence in (3.3) and (3.4), we can state

Theorem 3. *If inequality* (3.1) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z^d_+$ *with an* $r>2$, *then for any choices of* α *and* β *satisfying*

$$0 \leqq \beta < \alpha r - 1 \tag{4.1}$$

and for any $\varepsilon>0$ *we have*

$$\sum_{\mathbf{m}\geqq 1} \frac{1}{|\mathbf{m}|\,(\log 2|\mathbf{m}|)^{d-\beta}}\, \mu\left\{ \sup_{\substack{k_j\geqq m_j \\ \text{for at least one } j,\\ 1\leqq j\leqq d}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|^{1/2}\left(\prod_{j=1}^{d} \log 2k_j\right)^{\alpha}} \geqq \varepsilon \right\} < \infty \tag{4.2}$$

and

$$\sum_{\mathbf{m}\geqq 1} \frac{1}{|\mathbf{m}|\left(\prod_{j=1}^{d} \log 2m_j\right)^{1-\beta}}\, \mu\left\{ \sup_{\mathbf{k}\geqq \mathbf{m}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|^{1/2}(\log 2|\mathbf{k}|)^{\alpha d}} \geqq \varepsilon \right\} < \infty. \tag{4.3}$$

This result for $d=1$ was also established by SERFLING [7, Theorem 5.3].

Remark 2. Observe that the more restrictive " $\sup\limits_{\substack{k_j \geqq m_j \\ \text{for at least one } j, \\ 1\leqq j\leqq d}}$ " in (4.2) is weakened into " $\sup\limits_{\substack{k_j \geqq m_j \\ \text{for every } j, \\ 1\leqq j\leqq d}}$ " in (4.3).

If inequality (3.1) is satisfied for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$ with arbitrarily large exponents r, then (4.2) and (4.3) hold for each choice of $\alpha>0$ and $\beta>0$.

The proof of Theorem 3 is based on (3.5) and on the following auxiliary result, which for the sake of brevity is stated only for $d=2$.

Lemma 4. *If* (4.1) *holds, then*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ik(\log 2ik)^{2-\beta}(\log 2i)^{\alpha r-1}}<\infty.$$

Proof of Lemma 4. An easy computation shows that the series in question can be estimated from above as follows

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{l=0}^{\infty}\sum_{i=2^l}^{2^{l+1}-1}\frac{1}{i(\log 2ik)^{2-\beta}(\log 2i)^{\alpha r-1}}\leqq\sum_{k=1}^{\infty}\frac{1}{k}\sum_{l=0}^{\infty}\frac{1}{(l+\log 2k)^{2-\beta}(l+1)^{\alpha r-1}}.$$

Now let us deal with the inner series:

$$\left\{\sum_{l=0}^{[\log 2k]}+\sum_{l=[\log 2k]+1}^{\infty}\right\}\frac{1}{(l+\log 2k)^{2-\beta}(l+1)^{\alpha r-1}}\leqq\frac{1}{(\log 2k)^{2-\beta}}\sum_{l=0}^{[\log 2k]}\frac{1}{(l+1)^{\alpha r-1}}+$$

$$+\sum_{l=[\log 2k]+1}^{\infty}\frac{1}{(l+1)^{\alpha r-\beta+1}}\leqq\frac{C_5}{(\log 2k)^{\alpha r-\beta}},$$

where $[.]$ denotes integral part. Taking into account that by (4.1) we have $\alpha r-\beta>1$, the proof is ready.

Proof of Theorem 3. We prove for $d=2$ only. The general case $d>2$ can be handled in the same way, merely the technical details become more complicated.

In virtue of Lemma 4, for (4.2) it is enough to demonstrate that

$$\mu(m,n)=\mu\left\{\sup_{i\geqq m \text{ or } k\geqq n}\frac{|S(i,k;\,x)|}{(ik)^{1/2}(\log 2i\log 2k)^{\alpha}}\geqq\varepsilon\right\}\leqq \tag{4.4}$$

$$\leqq C_6\left(\frac{1}{(\log 2i)^{\alpha r-1}}+\frac{1}{(\log 2k)^{\alpha r-1}}\right).$$

To this end, let the non-negative integers p and q be defined by

$$2^p \leqq m < 2^{p+1} \quad \text{and} \quad 2^q \leqq n < 2^{q+1}.$$

It is obvious that

$$(4.5) \qquad \mu(m, n) \leqq \left\{ \sum_{u=p}^{\infty} \sum_{v=q}^{\infty} + \sum_{u=p}^{\infty} \sum_{v=0}^{q-1} + \sum_{u=0}^{p-1} \sum_{v=q}^{\infty} \right\} \nu(u, v)$$

with

$$\nu(u, v) = \mu \left\{ \max_{2^u \leqq i < 2^{u+1}} \max_{2^v \leqq k < 2^{v+1}} \frac{|S(i, k; x)|}{(ik)^{1/2} (\log 2i \log 2k)^{\alpha}} \geqq \varepsilon \right\}.$$

By (3.5) it is not hard to check that

$$\nu(u, v) \leqq \mu\{M(2^{u+1}, 2^{v+1}; x) \geqq \varepsilon (u+1)^{\alpha} (v+1)^{\alpha} 2^{(u+v)/2}\} \leqq$$

$$\leqq C_1 2^r \varepsilon^{-r} ((u+1)(v+1))^{-\alpha r}.$$

Since

$$\sum_{u=p}^{\infty} \sum_{v=q}^{\infty} \nu(u, v) \leqq C_7 \varepsilon^{-r} ((p+1)(q+1))^{-\alpha r+1} \leqq C_8 \varepsilon^{-r} (\log 2m \log 2n)^{-\alpha r+1},$$

$$\sum_{u=p}^{\infty} \sum_{v=0}^{q-1} \nu(u, v) \leqq C_7 \varepsilon^{-r} (p+1)^{-\alpha r+1} \leqq C_8 \varepsilon^{-r} (\log 2m)^{-\alpha r+1},$$

and similarly,

$$\sum_{u=0}^{p-1} \sum_{v=q}^{\infty} \nu(u, v) \leqq C_8 \varepsilon^{-r} (\log 2m)^{-\alpha r+1},$$

from (4.5) we obtain the desired (4.4). This proves (4.2).

The proof of (4.3) can be executed in a similar manner as that of (4.2). The proof of Theorem 3 is complete.

It is clear that under the conditions of Theorem 2 we have $S(\mathbf{m}; x)/|\mathbf{m}| \to 0$ a.e. as $\mathbf{m} \to \infty$. For this SLLN we can prove essentially better convergence rates, however, now only with the weaker "$\sup\limits_{\substack{k_j \geqq m_j \\ \text{for every } j, \\ 1 \leqq j \leqq d}}$" instead of "$\sup\limits_{\substack{k_j \geqq m_j \\ \text{for at least one } j, \\ 1 \leqq j \leqq d}}$".

Theorem 4. *If inequality* (3.1) *holds for all* $\mathbf{b} \in Z^d$ *and* $\mathbf{m} \in Z_+^d$ *with an* $r > 2$, *then for any* $\delta > 0$ *and* $\varepsilon > 0$ *we have*

$$(4.6) \qquad \sum_{\mathbf{m} \geqq 1} \frac{|\mathbf{m}|^{(r-2)/2}}{\left(\prod_{j=1}^{d} \log 2m_j \right) (\log \log 4|\mathbf{m}|)^{d+\delta}} \mu \left\{ \sup_{\mathbf{k} \geqq \mathbf{m}} \frac{|S(\mathbf{k}; x)|}{|\mathbf{k}|} \geqq \varepsilon \right\} < \infty$$

and

$$\sum_{\mathbf{m}\geqq 1} \frac{|\mathbf{m}|^{(r-2)/2}}{(\log 2|\mathbf{m}|)^d (\log\log 4|\mathbf{m}|)^{1+\delta}} \mu\left\{\sup_{\mathbf{k}\geqq\mathbf{m}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|} \geqq \varepsilon\right\} < \infty. \tag{4.7}$$

Remark 3. Both convergence rates improve as r increases. Letting $r\to\infty$ results, for any $\alpha>0$ and $\varepsilon>0$,

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{\alpha} \mu\left\{\sup_{\mathbf{k}\geqq\mathbf{m}} |S(\mathbf{k};\, x)|/|\mathbf{k}| \geqq \varepsilon\right\} < \infty.$$

The proof of Theorem 4 runs along the same lines as that of Theorem 3. First we infer that

$$\mu\left\{\sup_{\mathbf{k}\geqq\mathbf{m}} |S(\mathbf{k};\, x)|/|\mathbf{k}| \geqq \varepsilon\right\} \leqq C_9 \varepsilon^{-r} |\mathbf{m}|^{-r/2}$$

(for $d=1$ see also in [7, Theorem 5.1]), then (4.6) and (4.7) follow from the fact that, for any $\delta>0$,

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{-1} \left\{\prod_{j=1}^{d} \log 2m_j\right\}^{-1} (\log\log 4|\mathbf{m}|)^{-d-\delta} < \infty$$

and

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{-1} (\log 2|\mathbf{m}|)^{-d} (\log\log 4|\mathbf{m}|)^{-1-\delta} < \infty.$$

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