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A short proof of the fact that biholomorphic automorphisms of the unit ball in certain L^p spaces are linear

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1. As a consequence of his investigations on the Carathéodory and Kobayashi distances on domains in locally convex vector spaces, E. VESENTINI [1] proved that biholomorphic automorphisms of the unit ball*) of $L^1(\Omega, \mu)$ are all linear, whenever the underlying measure space (Ω, μ) is not a unique atom. In this paper we shall provide a quite different approach to the problem which applies to $L^p(\Omega, \mu)$ as well, for every $p\in[1, \infty)$.

Theorem. *Let (Ω, μ) be a measure space having two disjoint subsets Ω', Ω'' such that $0<\mu(\Omega')$, $\mu(\Omega'')<\infty$. Then for any $p\in[1,\infty)\setminus\{2\}$, all biholomorphic automorphisms of the unit ball of $L^p(\Omega, \mu)$ are linear.*

Our method is based on a result of W. KAUP and H. UPMEIER [2] concerning Aut $B(E)$ for general Banach spaces E. Here we present a direct proof of the theorem, which may have interest because of its extreme brevity. However, we remark that one can also determine the general algebraic form of an element from Aut $B(L^2(\Omega, \mu))$ in a similar way.

2. First we prove a lemma. To this end, let E denote an arbitrarily fixed Banach space with norm $\|.\|$, E^* the dual of E endowed with the norm $\|.\|_*$.

Lemma. Aut $B(E)$ *contains only linear mappings if and only if the relation*

$$(1) \quad \langle q(x, x), \varphi\rangle = -\overline{\langle c, \varphi\rangle} \quad \text{for all} \quad x\in E, \varphi\in E^* \quad \text{with} \quad \|x\| = \|\varphi\|_* = 1 = \langle x, \varphi\rangle$$

entails $c=0$ whenever $c\in E$ and q is a bilinear form from $E\times E$ into E.

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*) In general, if $B(E)$ denotes the open unit ball of a Banach space E then the biholomorphic automorphisms of $B(E)$ are defined as those one-to-one mappings of $B(E)$ onto itself whose Fréchet derivative exists at every point $x\in B(E)$ as an invertible operator. We shall denote the group formed by the biholomorphic automorphisms of $B(E)$ by Aut $B(E)$.

Proof. According to [2, p. 131], there can be found a subspace V in E and a conjugate-linear mapping $v\mapsto q_v$ from V into the space of the (continuous) E-bilinear forms such that $\operatorname{Aut}(D)$ is generated by the group G_0 of the surjective linear isometries of E onto itself any by the images under the exponential map of the vector fields $\big(v+q_v(z,z)\big)\dfrac{\partial}{\partial z}$ $(v\in V)$. Thus, for $\operatorname{Aut} B(E)=G_0$ it is necessary and sufficient that there exist a $c\in E\setminus\{0\}$ and a bilinear form $q\colon E\times E\to E$ such that the vector field $\big(c+q(z,z)\big)\dfrac{\partial}{\partial z}$ be tangent to $\partial B(E)$ (the boundary of $B(E)$), i.e.

$$\operatorname{Re}\langle c+q(z,z),\psi\rangle=0 \quad\text{whenever}\quad \|z\|=\|\psi\|_*=1=\langle z,\psi\rangle. \tag{2}$$

Suppose now that the vectors $c, x\in E$, $\varphi\in E^*$ and the E-bilinear form q satisfy $\|x\|=\|\varphi\|_*=1=\langle x,\varphi\rangle$ and (2). Then for all $\lambda\in\mathbf{C}$ with $|\lambda|=1$ we have $\|\lambda x\|=$ $=\|\bar\lambda\varphi\|_*=1=\langle\lambda x,\bar\lambda\varphi\rangle$ whence $0=\operatorname{Re}\langle c+q(\lambda x,\lambda x),\bar\lambda\varphi\rangle=\operatorname{Re}[\lambda(\overline{\langle c,\varphi\rangle}+\langle q(x,x),\varphi\rangle)]$. Therefore $\overline{\langle c,\varphi\rangle}+\langle q(x,x),\varphi\rangle=0$ which completes the proof of the Lemma.

3. Now we shall proceed to the proof of the Theorem. Henceforth let $p\in[1,\infty)$ be arbitrarily fixed and set $E\equiv L^p(\Omega,\mu)$. As usual we shall identity E^* with $L^{p/(p-1)}(\Omega,\mu)$ and the pairing operation with $\langle x,\varphi\rangle\equiv\int_\Omega x(\xi)\cdot\varphi(\xi)\,d\mu(\xi)$ (for all $x\in E$ and $\varphi\in E^*$), respectively.

For any $x\in E$, let x denote the function $\xi\mapsto x(\xi)\cdot|x(\xi)|^{p-2}$ (with the convention $0\cdot 0^{p-2}\equiv 0$). Observe that here

$$x^*\in E^*,\quad \|x^*\|_*=\|x\|^{p-1},\quad \langle x,x^*\rangle=\|x\|^p \quad\text{for all}\quad x\in E. \tag{3}$$

Then assume that the function $x\in E$ and the E-bilinear form q satisfy (1). Applying (3) we see that

$$\langle q(x/\|x\|,x/\|x\|),(x/\|x\|)^*\rangle=-\overline{\langle c,(x/\|x\|)^*\rangle}\quad\text{for all}\quad x\in E\setminus\{0\},$$

that is

$$\langle q(x,x),x^*\rangle=-\|x\|^2\overline{\langle c,x^*\rangle}\quad\text{for all}\quad x\in E. \tag{1'}$$

In particular, if F and G are any two disjoint subsets of Ω such that $0<\mu(F)$, $\mu(G)<\infty$ then

$$\int_\Omega q(1_F+\lambda\cdot 1_G,1_F+\lambda\cdot 1_G)(1_F+\bar\lambda|\lambda|^{p-2}1_G)\,d\mu=$$
$$=-\big(\mu(F)+|\lambda|^p\cdot\mu(G)\big)^{2/p}\int_\Omega\bar c(1_F+\lambda\cdot|\lambda|^{p-2}1_G)\,d\mu$$

for all $\lambda\in\mathbf{C}$. (For any μ-measurable subset $H\subset\Omega$ of finite μ-measure, 1_H denotes the characteristic function of H, considered as an element in E.)

Thus, by setting

$$\alpha_0 \equiv \int_F q(1_F, 1_F)\, d\mu, \quad \alpha_1 \equiv \int_F [q(1_F, 1_G) + q(1_G, 1_F)]\, d\mu, \quad \alpha_2 \equiv \int_F q(1_G, 1_G)\, d\mu,$$

$$\beta_0 \equiv \int_G q(1_F, 1_F)\, d\mu, \quad \beta_1 \equiv \int_G [q(1_F, 1_G) + q(1_G, 1_F)]\, d\mu, \quad \beta_2 \equiv \int_G q(1_G, 1_G)\, d\mu,$$

$$\mu_1 \equiv \mu(F), \quad \mu_2 \equiv \mu(G), \quad \gamma_1 \equiv \int_F \bar{c}\, d\mu, \quad \gamma_2 \equiv \int_G \bar{c}\, d\mu$$

we obtain

$$\sum_{k=0}^{2} \alpha_k \lambda^k + \bar{\lambda} |\lambda|^{p-2} \cdot \sum_{k=0}^{2} \beta_k \lambda^k = -(\mu_1 + |\lambda|^p \mu_2)^{2/p} (\gamma_1 + \lambda \cdot |\lambda|^{p-2} \gamma_2)$$

for all $\lambda \in \mathbf{C}$. Therefore for any $\varrho > 0$ and $\vartheta \in \mathbf{C}$ with $|\vartheta| = 1$,

$$(\beta_0 \cdot \varrho^{p-1}) \vartheta^{-1} + (\alpha_0 + \beta_1 \cdot \varrho^p) + (\alpha_1 \cdot \varrho + \beta_2 \cdot \varrho^{p+1}) \vartheta + (\alpha_2 \cdot \varrho^2) \vartheta^2 =$$
$$= -(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} [\gamma_1 + (\gamma_2 \cdot \varrho^{p-1}) \vartheta].$$

In particular, we have

$$\alpha_0 + \beta_1 \cdot \varrho^p = -(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} \gamma_1 \quad \text{for all} \quad \varrho > 0.$$

Hence $-\mu_2^{2/p} \cdot \gamma_1 = \lim_{\varrho \uparrow \infty} [-(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} \varrho^{-2}] = \lim_{\varrho \uparrow \infty} (\alpha_0 + \beta_1 \cdot \varrho^p) \cdot \varrho^{-2}$. This is possible only if $p = 2$ or $\gamma_1 = 0$. Thus if $p \neq 2$ then by definition of γ_1 we have

$$(4) \qquad \int_F \bar{c}\, d\mu = 0 \quad \text{whenever} \quad 0 < \mu(G) < \infty \quad \text{for some} \quad G \subset \Omega \setminus F.$$

But (4) immediately implies $c = 0$ because of our assumption on the measure space (Ω, μ). Thus, by the Lemma, $B(E)$ admits in case $p \neq 2$ only linear biholomorphic automorphisms. Q.E.D.

References

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