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On the tensor product of weights on W^*-algebras

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1. Let φ and ψ be normal semifinite weights on the W^*-algebras $\mathcal{M}$ and $\mathcal{N}$, respectively. Using the Tomita—Takesaki theory ([13]) and the Pedersen—Takesaki theorem on the equality of weights ([10]), CONNES ([3], 1.1.3) (see also [9]) proved that there exists a unique normal semifinite weight $\varphi\bar{\otimes}\psi$ on $\mathcal{M}\bar{\otimes}\mathcal{N}$ such that

(1) $\quad a\in\mathfrak{M}_\varphi^+,\ b\in\mathfrak{M}_\psi^+ \Rightarrow a\otimes b\in\mathfrak{M}_{\varphi\bar{\otimes}\psi}^+$ and $(\varphi\bar{\otimes}\psi)(a\otimes b)=\varphi(a)\psi(b)$,

(2) $$\mathrm{s}(\varphi\bar{\otimes}\psi)=\mathrm{s}(\varphi)\bar{\otimes}\mathrm{s}(\psi),$$

(3) $\quad \sigma_t^{\varphi\bar{\otimes}\psi}(x\bar{\otimes}y)=\sigma_t^\varphi(x)\bar{\otimes}\sigma_t^\psi(y)$ for $t\in\mathbf{R},\ x\in\mathrm{s}(\varphi)\mathcal{M}\mathrm{s}(\varphi),\ y\in\mathrm{s}(\psi)\mathcal{N}\mathrm{s}(\psi)$.

Here and in the sequel we use the standard notations in the Tomita—Takesaki theory ([12], [13]). In particular, $\mathrm{s}(\varphi)$ is the support projection of φ and $\mathfrak{M}_\varphi^+=\{x\in\mathcal{M}^+;\ \varphi(x)<+\infty\}$. If φ is not faithful, then $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$ means, of course, the modular automorphism group associated with the restriction of φ to $\mathrm{s}(\varphi)\mathcal{M}\mathrm{s}(\varphi)$.

If φ and ψ are normal positive functionals, then condition (1) alone is sufficient to insure the uniqueness in the definition of $\varphi\bar{\otimes}\psi$. However, in the general case it is often difficult to check condition (3) above for some candidates for $\varphi\bar{\otimes}\psi$.

The aim of this Note is to offer alternative equivalent definitions for $\varphi\bar{\otimes}\psi$ and to prove some very natural properties of the tensor product of weights.

2. From the works of COMBES ([1]), HAAGERUP ([7]) and PEDERSEN and TAKESAKI ([10]) (see also [6]) we know that for every normal weight φ on $\mathcal{M}$ there exists a family $\{\varphi_i\}_{i\in I}$ of normal positive functionals on $\mathcal{M}$ such that $\varphi=\sum_{i\in I}\varphi_i$, i.e.;

(4) $$\varphi(x)=\sum_{i\in I}\varphi_i(x)\quad \textit{for all}\quad x\in\mathcal{M}^+.$$

In particular, there is an increasing net $\{\varphi_i\}_{i\in I}$ of normal positive functionals on $\mathcal{M}$ such that $\varphi_i\uparrow\varphi$, i.e.:

(5) $$\varphi(x)=\sup_i\varphi_i(x)=\lim_i\varphi_i(x)\quad \textit{for all}\quad x\in\mathcal{M}^+.$$

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On the other hand, and this is the main technical tool we shall use, from the recent work of CONNES ([4]) it follows that

(6) *if φ is a normal semifinite weight on $\mathcal{M}$ and $\{\varphi_i\}_{i\in I}$ is an increasing net of normal weights on $\mathcal{M}$ such that $\varphi_i\uparrow\varphi$, then*

$$\sigma_t^{\varphi_i}(x)\xrightarrow{s}\sigma_t^{\varphi}(x)\quad(t\in\mathbf{R})$$

for every $x\in\bigcup_{i\in I}s(\varphi_i)\mathcal{M}s(\varphi_i)$.

Here $\xrightarrow{s}$ means convergence in the ultra-strong topology on $\mathcal{M}$ of some section $\{i\in I\colon i\geqq i_0\}$ of the net involved.

Finally, from the proof of ([10], Lemma 5.2) it is easy to infer the following improvement of ([10], Lemma 5.2):

(7) *if φ_1, φ_2 are normal semifinite weights on $\mathcal{M}$ such that $s(\varphi_1)\leqq s(\varphi_2)$ and there exists an s-dense σ^{φ_2}-invariant $*$-subalgebra $\mathcal{A}$ of $\mathfrak{M}_{\varphi_2}$ such that*

$$\varphi_1(a^*a)\leqq\varphi_2(a^*a)\quad\textit{for all}\quad a\in\mathcal{A},$$

then $\varphi_1\leqq\varphi_2$, i.e. $\varphi_1(x)\leqq\varphi_2(x)$ for all $x\in\mathcal{M}^+$.

In all this paper $\mathcal{M}$ and $\mathcal{N}$ will denote two W^*-algebras.

3. Lemma. *Let φ_1, φ_2 be normal semifinite weights on $\mathcal{M}$ and ψ a normal semifinite weight on $\mathcal{N}$. If $\varphi_1\leqq\varphi_2$, then $\varphi_1\bar{\otimes}\psi\leqq\varphi_2\bar{\otimes}\psi$.*

Proof. If $\varphi_1\leqq\varphi_2$, then $s(\varphi_1)\leqq s(\varphi_2)$, whence, by (2), $s(\varphi_1\bar{\otimes}\psi)=s(\varphi_1)\bar{\otimes}s(\psi)\leqq\leqq s(\varphi_2)\bar{\otimes}s(\psi)=s(\varphi_2\bar{\otimes}\psi)$. Moreover, by (1) and (3), the algebraic tensor product $\mathcal{A}=\mathfrak{M}_{\varphi_2}\otimes\mathfrak{M}_{\psi}$ is an s-dense $\sigma^{\varphi_2\bar{\otimes}\psi}$-invariant $*$-subalgebra of $\mathfrak{M}_{\varphi_2\bar{\otimes}\psi}$. Since $\varphi_1\leqq\varphi_2$ are positive linear functionals on the $*$-algebra $\mathfrak{M}_{\varphi_2}$ and $\psi\geqq0$ on the $*$-algebra $\mathfrak{M}_{\psi}$, it follows that $\varphi_1\otimes\psi\leqq\varphi_2\otimes\psi$ on the $*$-algebra $\mathcal{A}$. Thus $\varphi_1\bar{\otimes}\psi\leqq\varphi_2\bar{\otimes}\psi$, by (7).

4. Theorem. *Let φ, ψ be normal semifinite weights and $\{\varphi_i\}_{i\in I}$, $\{\psi_j\}_{j\in J}$ be increasing nets of normal weights on $\mathcal{M}$, $\mathcal{N}$, respectively. If $\varphi_i\uparrow\varphi$ and $\psi_j\uparrow\psi$, then $\varphi_i\bar{\otimes}\psi_j\uparrow\varphi\bar{\otimes}\psi$.*

Proof. By Lemma 3, $\{\varphi_i\bar{\otimes}\psi_j\}_{i\in I,j\in J}$ is an increasing net of normal weights on $\mathcal{M}\bar{\otimes}\mathcal{N}$ and $\varphi_i\bar{\otimes}\psi_j\leqq\varphi\bar{\otimes}\psi$ for all $i\in I$, $j\in J$. Consequently, the formula

$$\omega(z)=\sup_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)=\lim_{ij}(\varphi_i\bar{\otimes}\psi_j)(z),\quad(z\in(\mathcal{M}\bar{\otimes}\mathcal{N})^+)$$

defines a normal semifinite weight ω on $\mathcal{M}\bar{\otimes}\mathcal{N}$. For $a\in\mathfrak{M}_{\varphi}^+$, $b\in\mathfrak{M}_{\psi}^+$, we have

$$\omega(a\otimes b)=\sup_{ij}\varphi_i(a)\psi_j(b)=\sup_i\varphi_i(a)\sup_j\psi_j(b)=\varphi(a)\psi(b).$$

On the other hand, it is easy to see that $\mathbf{s}(\varphi_i)\uparrow\mathbf{s}(\varphi)$, $\mathbf{s}(\psi_j)\uparrow\mathbf{s}(\psi)$ and $\mathbf{s}(\varphi_i\bar{\otimes}\psi_j)\uparrow\mathbf{s}(\omega)$, hence

$$\mathbf{s}(\omega)=\mathbf{s}(\varphi)\bar{\otimes}\,\mathbf{s}(\psi).$$

Finally, by the result (6), for $t\in\mathbf{R}$, $x\in\bigcup_{i\in I}\mathbf{s}(\varphi_i)\mathscr{M}\mathbf{s}(\varphi_i)$, $y\in\bigcup_{j\in J}\mathbf{s}(\psi_j)\mathscr{N}\mathbf{s}(\psi_j)$, we have

$$\sigma_t^{\varphi_i}(x)\xrightarrow{s}\sigma_t^{\varphi}(x),\quad \sigma_t^{\psi_j}(y)\xrightarrow{s}\sigma_t^{\psi}(y)$$

and

$$\sigma_t^{\varphi_i\bar{\otimes}\psi_j}(x\otimes y)\xrightarrow{s}\sigma_t^{\omega}(x\otimes y).$$

Hence,

$$\sigma_t^{\omega}(x\otimes y)=\sigma_t^{\varphi}(x)\otimes\sigma_t^{\psi}(y).$$

Since $\mathbf{s}(\varphi_i)\uparrow\mathbf{s}(\varphi)$, $\mathbf{s}(\psi_j)\uparrow\mathbf{s}(\psi)$, the above equality still holds for $x\in\mathbf{s}(\varphi)\mathscr{M}\mathbf{s}(\varphi)$, $y\in\mathbf{s}(\psi)\mathscr{N}\mathbf{s}(\psi)$.

Thus, ω satisfies all conditions (1), (2), (3) which define $\varphi\bar{\otimes}\psi$. Consequently $\omega=\varphi\bar{\otimes}\psi$, i.e. $\varphi_i\bar{\otimes}\psi_j\uparrow\varphi\bar{\otimes}\psi$.

5. In particular if the φ_i's and the ψ_j's are normal positive functionals such that $\varphi_i\uparrow\varphi$, $\psi_j\uparrow\psi$, then

$$(\varphi\bar{\otimes}\psi)(z)=\sup_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)\quad \big(z\in(\mathscr{M}\bar{\otimes}\mathscr{N})^+\big) \tag{8}$$

is an alternative equivalent definition of the weight $\varphi\bar{\otimes}\psi$, independent of the choice of the families $\{\varphi_i\}$, $\{\psi_j\}$, whose existence is guaranteed by (5).

6. As a first application we obtain the distributivity of the tensor product with respect to addition:

Corollary. *Let φ_1, φ_2 be normal semifinite weights on $\mathscr{M}$ such that $\varphi_1+\varphi_2$ is semifinite and ψ is a normal semifinite weight on $\mathscr{N}$. Then*

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi=\varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

Proof. Let $\{\psi_j\}$ be an increasing net of normal positive functionals on $\mathscr{N}$ such that $\psi_j\uparrow\psi$.

Assume that φ_1, φ_2 are normal positive functionals. Since the distributivity property is obvious for normal positive functionals, by Theorem 4 we obtain

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi=\sup_j(\varphi_1+\varphi_2)\bar{\otimes}\psi_j=\sup_j\varphi_1\bar{\otimes}\psi_j+\sup_j\varphi_2\bar{\otimes}\psi_j=\varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

Now, in the general case, let $\{\varphi_{1i}\}$, $\{\varphi_{2k}\}$ be increasing nets of normal positive functionals on $\mathscr{M}$ such that $\varphi_{1i}\uparrow\varphi_1$, $\varphi_{2k}\uparrow\varphi_2$. It is then obvious that $\varphi_{1i}+\varphi_{2k}\uparrow\varphi_1+\varphi_2$. Using Theorem 4 and the first part of the proof, we obtain

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi=\sup_{ik}(\varphi_{1i}+\varphi_{2k})\bar{\otimes}\psi=\sup_{ik}(\varphi_{1i}\bar{\otimes}\psi+\varphi_{2k}\bar{\otimes}\psi)=$$
$$=\sup_i\varphi_{1i}\bar{\otimes}\psi+\sup_k\varphi_{2k}\bar{\otimes}\psi=\varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

7. If $\varphi=\sum_i \varphi_i$ and $\psi=\sum_j \psi_j$, then from Corollary 6 and Theorem 4 it follows that

$$(\varphi\bar{\otimes}\psi)(z)=\sum_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)\quad (z\in(\mathcal{M}\bar{\otimes}\mathcal{N})^+). \tag{9}$$

In particular if the φ_i's and the ψ_j's are normal positive functionals, then the above relation gives another alternative equivalent definition of $\varphi\bar{\otimes}\psi$, independent of the choice of the families $\{\varphi_i\}$ and $\{\psi_j\}$, whose existence is guaranteed by (4).

The weight φ is called strictly semifinite ([2]) if there exists a family $\{\varphi_i\}$ of normal positive functionals with mutually orthogonal supports such that $\varphi=\sum_i \varphi_i$. If both φ and ψ are strictly semifinite, then, by (9), $\varphi\bar{\otimes}\psi$ is again strictly semifinite. This result is originally due to COMBES ([2]).

Other particular cases of (8) and (9) are mentioned in ([11], 0.1.2).

8. Another application concerns the relation between the tensor product and the balanced weight. Let us recall ([3], 1.2.2) that if φ_1, φ_2 are normal semifinite weights on $\mathcal{M}$, then the balanced weight $\theta(\varphi_1,\varphi_2)$ on the W^*-algebra $\mathrm{Mat}_2(\mathcal{M})\cong\cong\mathcal{M}\otimes\mathrm{Mat}_2(\mathbf{C})$ of 2 by 2 matrices over $\mathcal{M}$ is defined by

$$\theta(\varphi_1,\varphi_2)\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}=\varphi_1(x_{11})+\varphi_2(x_{22}).$$

Now let ψ be a normal semifinite weight on $\mathcal{N}$. Then $\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi$ and $\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)$ are both normal semifinite weights on the W^*-algebra

$$\mathrm{Mat}_2(\mathcal{M})\bar{\otimes}\mathcal{N}\cong\mathcal{M}\bar{\otimes}\mathcal{N}\otimes\mathrm{Mat}_2(\mathbf{C})\cong\mathrm{Mat}_2(\mathcal{M}\bar{\otimes}\mathcal{N})$$

and we have the following

Corollary. $\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)=\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi$.

Proof. It is obvious that if $\varphi_{1i}\uparrow\varphi_1$ and $\varphi_{2k}\uparrow\varphi_2$, then $\theta(\varphi_{1i},\varphi_{2k})\uparrow\theta(\varphi_1,\varphi_2)$. Also, the stated equality is obvious for normal positive functionals. Thus the corollary follows using (5) and Theorem 4.

9. Consider again the balanced weight $\theta(\varphi_1,\varphi_2)$ and assume that $\mathrm{s}(\varphi_2)\leqq\mathrm{s}(\varphi_1)$. Then the Connes cocycle ([3], 1.2.2) $u_t=[D\varphi_2:D\varphi_1]_t$, $(t\in\mathbf{R})$, is defined by the equality

$$\sigma_t^{\theta(\varphi_1,\varphi_2)}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2) & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ u_t & 0\end{pmatrix}\quad (t\in\mathbf{R}).$$

Thus, using Corollary 8, for $v_t=[D(\varphi_2\bar{\otimes}\psi):D(\varphi_1\bar{\otimes}\psi)]_t$ we get

$$\begin{pmatrix}0 & 0\\ v_t & 0\end{pmatrix}=\sigma_t^{\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2\bar{\otimes}\psi) & 0\end{pmatrix}=\sigma_t^{\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2)\bar{\otimes}\mathrm{s}(\psi) & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ u_t\bar{\otimes}\mathrm{s}(\psi) & 0\end{pmatrix}.$$

Consequently,

$$[D(\varphi_2\bar{\otimes}\psi):D(\varphi_1\bar{\otimes}\psi)]_t=[D\varphi_2:D\varphi_1]_t\bar{\otimes}\mathrm{s}(\psi).$$

Using this equality and the chain rule for the Connes cocycle ([3], 1.2.3), we obtain the following

Corollary. *Let φ_1, φ_2 be normal semifinite weights on $\mathcal{M}$ with $\mathbf{s}(\varphi_2)\leqq \mathbf{s}(\varphi_1)$ and let ψ_1, ψ_2 be normal semifinite weights on $\mathcal{N}$ with $\mathbf{s}(\psi_2)\leqq \mathbf{s}(\psi_1)$. Then*

$$[D(\varphi_2\bar{\otimes}\psi_2)\colon D(\varphi_1\bar{\otimes}\psi_1)]_t = [D\varphi_2\colon D\varphi_1]_t\bar{\otimes}[D\psi_2\colon D\psi_1]_t \quad (t\in\mathbf{R}).$$

This result is stated by DIGERNES ([5], 2.4), where the proposed proof consists of checking the *KMS* conditions insuring the uniqueness of the Connes' cocycle ([5], 2.2), but only for decomposable elements

$$z_1\in(\mathfrak{N}^*_{\varphi_2}\cap\mathfrak{N}_{\varphi_1})\otimes(\mathfrak{N}^*_{\psi_2}\cap\mathfrak{N}_{\psi_1}),\quad z_2\in(\mathfrak{N}^*_{\varphi_1}\cap\mathfrak{N}_{\varphi_2})\otimes(\mathfrak{N}^*_{\psi_1}\cap\mathfrak{N}_{\psi_2}).$$

However, it is not obvious *à priori* that this entails the *KMS* condition for all

$$z_1\in\mathfrak{N}^*_{\varphi_2\bar{\otimes}\psi_2}\cap\mathfrak{N}_{\varphi_1\bar{\otimes}\psi_1},\quad z_2\in\mathfrak{N}^*_{\varphi_1\bar{\otimes}\psi_1}\cap\mathfrak{N}_{\varphi_2\bar{\otimes}\psi_2},$$

which is the real requirement for the uniqueness.

On the other hand, if $\psi_1=\psi_2=\psi$, then *using* Corollary 8 it is easy to show that for the S-operators ([5], (2.6)) we have

$$S_{\varphi_2\bar{\otimes}\psi,\varphi_1\bar{\otimes}\psi} = S_{\varphi_2,\varphi_1}\bar{\otimes}S_\psi.$$

Once this equality is obtained, the proof in ([5], 2.4) holds indeed.

10. For every normal semifinite weight φ on $\mathcal{M}$ and every positive self-adjoint operator A affiliated with the centralizer $\mathcal{M}_\varphi$ of φ there exists a unique normal semifinite weight φ_A on $\mathcal{M}$ such that $[D\varphi_A\colon D\varphi]_t=A^{it}$, $t\in\mathbf{R}$, ([10]). From Corollary 9 we infer the following result, originally obtained by KATAYAMA ([9]):

Corollary. *Let φ, ψ be normal semifinite weights on $\mathcal{M}$, $\mathcal{N}$, respectively, and let A, B be positive self-adjoint operators affiliated to $\mathcal{M}_\varphi$, $\mathcal{N}_\psi$, respectively. Then $A\bar{\otimes}B$ is a positive self-adjoint operator affiliated to $(\mathcal{M}\bar{\otimes}\mathcal{N})_{\varphi\bar{\otimes}\psi}$ and*

$$(\varphi\bar{\otimes}\psi)_{A\bar{\otimes}B} = \varphi_A\bar{\otimes}\psi_B.$$

11. Arguing as in the proof of Corollaries 6 and 8, with the help of (5) and Theorem 4 we obtain:

Corollary. *Let φ, ψ be normal semifinite weights on M, N, respectively, and let $\pi\colon\mathcal{M}_1\to\mathcal{M}$; $\varrho\colon\mathcal{N}_1\to\mathcal{N}$ be normal completely positive linear maps. If the weights $\varphi\circ\pi$, $\psi\circ\varrho$ are semifinite, then*

$$(\varphi\bar{\otimes}\psi)\circ(\pi\bar{\otimes}\varrho) = (\varphi\circ\pi)\bar{\otimes}(\psi\circ\varrho).$$

12. A final application concerns some operator valued weights ([8]) called Fubini mappings ([14]). For every normal semifinite weight ψ on $\mathcal{N}$ there is a unique normal semifinite operator valued weight $E^\psi_{\mathcal{M}}$ defined on $(\mathcal{M}\bar{\otimes}\mathcal{N})^+$ with values in the extended positive part ([8]) $\bar{\mathcal{M}}^+$ of $\mathcal{M}$, such that

$$\varphi(E^{\psi}_{\mathcal{M}}(z)) = (\varphi \bar{\otimes} \psi)(z) \quad (z \in (\mathcal{M} \bar{\otimes} \mathcal{N})^+) \tag{10}$$

for every normal positive functional φ on $\mathcal{M}$ (cf. also [11], 0.1.6). From Theorem 4 it follows that:

Corollary. *If ψ, ψ_j are normal semifinite weights on $\mathcal{N}$ and $\psi_j \uparrow \psi$, then*

$$E^{\psi}_{\mathcal{M}}(z) = \sup_j E^{\psi_j}_{\mathcal{M}}(z) \quad (z \in (\mathcal{M} \bar{\otimes} \mathcal{N})^+).$$

Also, the equality (10) extends to any normal semifinite weight φ on $\mathcal{M}$.

Actually, the operator valued weight $E^{\psi}_{\mathcal{M}}$ is nothing but the tensor product operator valued weight $\iota_{\mathcal{M}} \bar{\otimes} \psi$ ([8]), where $\iota_{\mathcal{M}}$ stands for the identity mapping on $\mathcal{M}$. We remark that Corollary 12 can be extended to an arbitrary normal semifinite operator valued weight instead of $\iota_{\mathcal{M}}$. Moreover, Theorem 4 can be extended to operator valued weights.

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