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The function model of a contraction and the space L^1/H_0^1

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Recently, new techniques were invented for obtaining invariant subspaces for rather general classes of operators on Hilbert space, see [2]—[5]. The present note constitutes a first step to exploit similar techniques in the understanding of the fine structure of the functional model, in the sense of [1], of completely non-unitary contractions.

1. Recalling the canonical model of a completely non-unitary contraction on a separable Hilbert space we consider a contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ on the unit disc $D=\{\lambda: |\lambda|<1\}$; $\mathfrak{E}$ and $\mathfrak{E}_*$ being separable Hilbert spaces. Setting $\Delta=\Delta(e^{it})=(I-\Theta(e^{it})^*\Theta(e^{it}))^{1/2}$ we define the Hilbert function spaces

$$\mathfrak{K}_+ = H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}, \quad \mathfrak{H} = \mathfrak{K}_+\ominus\{\Theta w\oplus\Delta w: w\in H^2(\mathfrak{E})\} \tag{1.1}$$

(see [1], Chapter VI). $P_{\mathfrak{H}}$ will denote orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{H}$.

We shall also have to do with spaces L^1, H^1, H_0^1, H^∞, all with respect to normalized Lebesgue measure $dm=dt/(2\pi)$ on the unit circle $\{e^{it}: 0\leq t<2\pi\}$. Recall that H^∞ is the Banach dual of the factor space L^1/H_0^1, through the bilinear form

$$\langle f^{\cdot}, u\rangle = \int fu\,dm \quad (f\in L^1, u\in H^\infty),$$

$f\mapsto f^{\cdot}$ denoting the natural map of L^1 onto L^1/H_0^1 (see e.g. [6]).

With any (ordered) pair $\{h, k\}$ of elements of H we associate the element hk^* of L^1 defined by

$$hk^*(e^{it}) = (h(e^{it}), k(e^{it}))_{\mathfrak{E}_*\oplus\mathfrak{E}} \quad (0\leq t<2\pi). \tag{1.2}$$

For sake of simplicity we shall also write, for any $f\in L^1$,

$$\|f\|_{L^1/H_0^1} \quad \text{instead of} \quad \|f^{\cdot}\|_{L^1/H_0^1},$$

and scalar product and norm of vectors without subscript will always mean those in the space $\mathfrak{H}$.

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2. With the operator valued function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ we associate the multiplication operator

$$(2.1) \qquad \Theta_\times : H^2(\mathfrak{E}) \to H^2(\mathfrak{E}_*) \text{ defined by } (\Theta_\times u)(e^{it}) = \Theta(e^{it})u(e^{it}) \quad (u \in H^2(\mathfrak{E}))$$

and its adjoint $\Theta_\times^*$ (i.e. the coanalytic Toeplitz operator denoted in [6] by $T(\Theta^\sim)$); we have

$$(2.2) \qquad (\Theta_\times^* u)(e^{it}) = [\Theta(e^{it})^* u(e^{it})]_+ \quad (u \in H^2(\mathfrak{E})),$$

where $[\cdot]_+$ denotes the natural orthogonal projection of any (scalar or vector valued function space) L^2 onto its subspace H^2.

Observe that for any fixed $\mu \in D$ the function

$$(2.3) \qquad p_\mu(\lambda) = (1 - \bar{\mu}\lambda)^{-1}$$

belongs to H^2, and has norm

$$\|p_\mu\|_{H^2} = (1 - |\mu|^2)^{-1/2}.$$

It is easy to deduce from (2.2) that

$$(2.4) \qquad \Theta_\times^*(p_\mu a) = p_\mu \Theta(\mu)^* a \quad \text{for any} \quad a \in \mathfrak{E}_*.$$

The following functional η_Θ on H^2 will play an important part:

$$(2.5) \quad \eta_\Theta(\varphi) = \inf_{\mathfrak{A} \in \Phi} \sup_{a \in \mathfrak{A}} s(\varphi, a), \quad \text{where} \quad s(\varphi, a) = \frac{\|\Theta_\times^* \varphi a\|_{H^2(\mathfrak{E})}}{\|\varphi a\|_{H^2(\mathfrak{E}_*)}} \; (= 0 \text{ if } \varphi a = 0)$$

and Φ denotes the family of subspaces of $\mathfrak{E}_*$ with *finite codimension.*

Obviously, $\eta_\Theta(c\varphi) = \eta_\Theta(\varphi)$ for any complex number $c \neq 0$. By virtue of (2.4) we have, in particular,

$$(2.6) \qquad \eta_\Theta(p_\mu) = \inf_{\mathfrak{A} \in \Phi} \sup_{a \in \mathfrak{A}} \frac{\|\Theta(\mu)^* a\|_{\mathfrak{E}}}{\|a\|_{\mathfrak{E}_*}}$$

In what follows we shall assume that $\mathfrak{E}_*$ is infinite dimensional.

Lemma 1. *Given any sequence $\{\varphi_j\}_1^\infty$ of elements of H^2 there exists an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{E}_*$ such that,*

$$(2.7) \qquad s(\varphi_j, a_n) \leqq \eta_\Theta(\varphi_j) + \frac{1}{n} \quad \text{for} \quad j = 1, 2, \ldots, n; \; n = 1, 2, \ldots.$$

Proof. By virtue of the definition (2.5) there exist $\mathfrak{A}_{j,n} \in \Phi$ $(j = 1, 2, \ldots; n = 1, 2, \ldots)$ such that

$$\sup_{a \in \mathfrak{A}_{j,n}} s(\varphi_j, a) \leqq \eta_\Theta(\varphi_j) + \frac{1}{n}.$$

Set

$$\mathfrak{A}_n = \Big(\bigvee_{\substack{1\le j\le n\\ 1\le m\le n}} \mathfrak{A}_{j,m}^{\perp}\Big)^{\perp} \quad (n = 1, 2, \ldots).$$

Clearly, $\mathfrak{A}_n \in \Phi$, $\mathfrak{A}_n \subset \mathfrak{A}_{n-1}$ and $\mathfrak{A}_n \subset \mathfrak{A}_{j,m}$ $(1 \le j \le n,\ 1 \le m \le n)$. From the last inclusion we infer

$$\sup_{a\in\mathfrak{A}_n} s(\varphi_j, a) \le \sup_{a\in\mathfrak{A}_{j,n}} s(\varphi_j, a) \le \eta_\Theta(\varphi_j) + \frac{1}{n}.$$

Choose inductively a sequence $\{a_n\}_1^\infty$ of unit vectors in $\mathfrak{E}_*$ such that $a_n \in \mathfrak{A}_n$ and that a_n be orthogonal to $a_1, \ldots, a_{n-1}$ $(n=2, 3, \ldots)$. Then we shall have (2.7.)

Notice, for further use, that each infinite orthonormal sequence weakly converges to 0.

3. A subset $\mathcal{S}$ of the (open) unit ball $\mathcal{D}$ of H^2 will be called *dominant* if

$$\sup_{\varphi\in\mathcal{S}} \|[\bar{u}\varphi]_+\|_{H^2} = \|u\|_{H^\infty} \quad \text{for every} \quad u \in H^\infty. \tag{3.1}$$

This is an obvious analogue of that a subset S of the unit disc D be dominant in the sense of [8], namely that

$$\sup_{\mu\in S} |u(\mu)| = \|u\|_{H^\infty} \quad \text{holds for every} \quad u \in H^\infty. \tag{3.2}$$

Moreover, if S is dominant in D in the sense (3.2) then

$$\mathcal{S}_S = \{(1-|\mu|^2)^{1/2} p_\mu :\ \mu \in S\} \tag{3.3}$$

is dominant in $\mathcal{D}$ in the sense (3.1). Indeed, $\mathcal{S}_S \subset \mathcal{D}$ is obvious and in analogy with (2.4) we easily obtain

$$[\bar{u}p_\mu]_+ = p_\mu \overline{u(\mu)} \quad \text{for} \quad u \in H^\infty \quad \text{and} \quad \mu \in D.$$

Hence,

$$\|[\bar{u}(1-|\mu|^2)^{1/2} p_\mu]_+\|_{H^2} = |u(\mu)| \tag{3.4}$$

so validity of (3.2) for S implies that of (3.1) for $\mathcal{S}_S$.

Lemma 2. *If $\mathcal{S}$ is dominant in the unit ball $\mathcal{D}$ of H^2 then the convex hull of the set*

$$\{(\psi\bar{\varphi})^{\cdot} :\ \varphi \in \mathcal{S},\ \psi \in \mathcal{D}\} \tag{3.5}$$

is dense in the unit ball of L^1/H_0^1.

Proof. If not, there exist in the Banach dual H^∞ of L^1/H_0^1 an element u and a unit vector f in L^1/H_0^1 such that

$$\text{(3.6)} \qquad \operatorname{Re}\langle f^\cdot, u\rangle > \sup_{\substack{\varphi\in\mathscr{S}\\ \psi\in\mathscr{D}}} \operatorname{Re}\langle(\psi\bar{\varphi})^\cdot, u\rangle = \sup_{\varphi\in\mathscr{S}}\sup_{\psi\in\mathscr{D}}\left|\int \overline{\bar{u}\varphi}\,\psi\,dm\right| =$$

$$= \sup_{\varphi\in\mathscr{S}}\sup_{\psi\in\mathscr{D}}\left|\int \overline{[\bar{u}\varphi]_+}\,\psi\,dm\right| = \sup_{\varphi\in\mathscr{S}}\|[\bar{u}\varphi]_+\|_{H^2}.$$

$\mathscr{S}$ being dominant in $\mathscr{D}$ the last member equals $\|u\|_{H^\infty}$, and hence is $\geqq \operatorname{Re}\langle f^\cdot, u\rangle$, in contradiction with the strict inequality in (3.6).

4. For fixed $\varphi\in H^2$ and $a\in\mathfrak{E}_*$ we denote

$$\text{(4.1)} \qquad \varphi\circ a = P_{\mathfrak{H}}(\varphi a\oplus 0).$$

It is easy to show that

$$\text{(4.2)} \qquad \varphi\circ a = (\varphi a - \Theta[\Theta^*\varphi a]_+)\oplus(-\Delta[\Theta^*\varphi a]_+).$$

For any $h=h_0\oplus h_1\in\mathfrak{H}$ $(h_0\in H^2(\mathfrak{E}_*),\ h_1\in\overline{\Delta L^2(\mathfrak{E})})$ we have therefore

$$\text{(4.3)} \qquad (\varphi\circ a)h^* = \varphi(a,h_0)_{\mathfrak{E}_*} - (\Theta[\Theta^*\varphi a]_+, h_0)_{\mathfrak{E}_*} - (\Delta[\Theta^*\varphi a]_+, h_1)_{\mathfrak{E}} =$$

$$= \varphi(a,h)_{\mathfrak{E}_*} - ([\Theta^*\varphi a]_+, \Theta^*h_0 + \Delta h_1)_{\mathfrak{E}},$$

where the last term belongs to H_0^1 since

$$\text{(4.4)} \qquad h_2 \overset{\text{def}}{=} \Theta^*h_0 + \Delta h_1 \in L^2(\mathfrak{E})\ominus H^2(\mathfrak{E})$$

because of the definition (1.1) of $\mathfrak{H}$.

Therefore,

$$\text{(4.5)} \qquad (\varphi\circ a)h^* \equiv \varphi(a,h_0)_{\mathfrak{E}_*} \bmod H_0^1.$$

It also follows from (4.3) and (4.4) that

$$\text{(4.6)} \qquad h(\varphi\circ a)^* = \overline{(\varphi\circ a)h^*} = \bar{\varphi}(h_0, a)_{\mathfrak{E}_*} - (h_2, [\Theta^*\varphi a]_+)_{\mathfrak{E}}.$$

Suppose $\{a_n\}$ is a sequence of vectors in $\mathfrak{E}_*$, tending weakly to 0. Then by (4.5) and by the Lebesgue dominated convergence theorem,

$$\|(\varphi\circ a_n)h^*\|_{L^1/H_0^1} \leqq \|\varphi(a_n,h_0)_{\mathfrak{E}_*}\|_{L^1} \leqq \|\varphi\|_{H^2}\left[\int|(a_n, h_0(e^{it}))|^2\,dm\right]^{1/2} \to 0 \quad (n\to 0).$$

We shall also show that $\|h(\varphi\circ a_n^*)\|_{L^1/H_0^1}\to 0$. Since $\|\bar{\varphi}(h_0,a_n)_{\mathfrak{E}_*}\|_{L^1}\to 0$ by part of the preceding argument, by (4.6) it suffices to prove that

$$\text{(4.7)} \qquad \|(h_2, [\Theta^*\varphi a_n]_+)_{\mathfrak{E}}\|_{L^1} \to 0 \quad \text{as} \quad n\to 0.$$

It even suffices to prove (4.7) for $\varphi=e^{irt}$ $(r=0, 1, \ldots)$. Indeed, (4.7) then holds for all partial sums $\varphi_N(e^{it})$ of the L^2-expansion $\varphi(e^{it})=\sum_0^\infty c_r e^{irt}$, and since $\{a_n\}$ is bounded, say $\|a_n\|_{\mathfrak{E}_*}\leqq A$, we have, setting $\psi_N=\varphi-\varphi_N$,

$$\|(h_2, [\Theta^*\psi_N a_n]_+)_{\mathfrak{E}}\|_{L_1} = \int \|h_2\|_{\mathfrak{E}}\|[\Theta^*\psi_N a_n]_+\|_{\mathfrak{E}}\, dm = \tag{4.8}$$
$$\leqq \|h_2\|_{L^2(\mathfrak{E})}\|[\Theta^*\psi_N a_n]_+\|_{L^2(\mathfrak{E})} \leqq \|h_2\|_{L^2(\mathfrak{E})}\|\psi_N\|_{H^2}A,$$

and this bound is independent of n and as small as we wish upon choosing N large enough.

Now to prove (4.7) for $\varphi=e^{irt}$ $(r\geqq 0)$ observe that if $\Theta(\lambda)=\Theta_0+\lambda\Theta_1+\lambda^2\Theta_2+\dots$ then we have

$$[\Theta(e^{it})^* e^{irt}a_n]_+ = \sum_{j=0}^{r}\Theta_{r-j}e^{ijt}a_n,$$

and hence,

$$\|(h_2, [\Theta^* e^{irt}a_n]_+)_{\mathfrak{E}}\|_{L^1} = \int\left|\sum_{j=0}^{r}(e^{-ijt}\Theta_{r-j}^* h_2, a_n)_{\mathfrak{E}_*}\right| dm,$$

which tends to 0 as $n\to 0$, again by the weak convergence of $\{a_n\}$ to 0 and by the Lebesgue dominated convergence theorem.

So we have proved, in particular,

Lemma 3. *If $\{a_n\}$ converges to 0 weakly in $\mathfrak{E}_*$ then for any $\varphi\in H^2$ and $h\in\mathfrak{H}$ we have*

$$\|(\varphi\circ a_n)h^*\|_{L^1/H_0^1}\to 0,\quad \|h(\varphi\circ a_n)^*\|_{L^1/H_0^1}\to 0\quad \text{as}\quad n\to\infty.$$

We shall also need the following

Lemma 4. *For all φ, $\psi\in H^2$ and $a\in\mathfrak{E}_*$ we have*

$$\|(\psi\circ a)(\varphi\circ a)^*-\psi\bar\varphi\,\|a\|_{\mathfrak{E}_*}^2\|_{L^1/H_0^1}\leqq \|\psi\|_{H^2}\|\Theta_\times^*\varphi a\|_{H^2(\mathfrak{E})}\|a\|_{\mathfrak{E}_*}. \tag{4.9}$$

Proof. By virtue of (4.5) and (4.2) we have

$$(\psi\circ\varphi)(\varphi\circ a)^* = \varphi(a, \varphi a-\Theta[\Theta^*\varphi a]_+)_{\mathfrak{E}_*}\bmod H_0^1.$$

Because

$$\|\psi(a, \Theta[\Theta^*\varphi a]_+)_{\mathfrak{E}_*}\|_{L^1} = \|\psi\|_{H^2}\|a\|_{E_*}\|\Theta_\times^*\varphi a\|_{L^2(\mathfrak{E})}$$

(in analogy to (4.8)) we conclude to (4.9).

5. Next we prove the following

Lemma 5. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional and suppose $h, k\in\mathfrak{H}$; $\varphi_1, \dots, \varphi_r, \psi_1, \dots, \psi_r\in H^2$, and $\varepsilon>0$ are given. Then there exist $h', k'\in\mathfrak{H}$ such that*

$$\left\|(h+h')(k+k')^*-hk^*-\sum_1^r\psi_j\bar\varphi_j\right\|_{L^1/H_0^1}\leqq\sum_1^r\|\psi_j\|_{H^2}\|\varphi_j\|_{H^2}\eta_\Theta(\varphi_j)+\varepsilon, \tag{5.1}$$

$$\|h'\|^2\leqq\sum_1^r\|\psi_j\|_{H^2}^2,\quad \|k'\|^2\leqq\sum_1^r\|\varphi_j\|_{H^2}^2. \tag{5.2}$$

Proof. Let $\delta>0$ be fixed and choose by virtue of Lemma 1 an orthonormal sequence $\{a_n\}$ in $\mathfrak{E}_*$ such that

$$\|\Theta_\times^* \varphi_j a_n\|_{H^2(\mathfrak{E})} \leqq \left(\eta_\Theta(\varphi_j)+\frac{1}{n}\right)\|\varphi_j\|_{H^2} \quad \text{for} \quad j=1,\dots,r \quad \text{and} \quad n \geqq r.$$

Hence, and from Lemma 3 we deduce that for n large enough, say for $n\geqq n_0$, and for $j=1,\dots,r$ we have

$$\|\Theta_\times^* \varphi_j a_n\|_{H^2(\mathfrak{E})} \leqq \left(\eta_\Theta(\varphi_j)+\delta\right)\|\varphi_j\|_{H^2} \tag{5.3}$$

and

$$\|h(\varphi_j\circ a_n)^*\|_{L^1/H_0^1} \leqq \delta, \quad \|(\psi_j\circ a_n)k^*\|_{L^1/H_0^1} \leqq \delta. \tag{5.4}$$

Again by virtue of Lemma 3 we can choose, step by step, the integers $(n_0\leqq)n_1< <n_2<\dots<n_r$ such that

$$\|(\psi_i\circ a_{n_i})(\varphi_j\circ a_n)^*\|_{L^1/H_0^1} \leqq \delta, \quad \|(\psi_j\circ a_n)(\varphi_i\circ a_{n_i})^*\|_{L^1/H_0^1} \leqq \delta \tag{5.5}$$

$$(j=1,\dots,r;\ i=1,\dots,j-1;\ n\geqq n_j).$$

Rename a_{n_j} by b_j $(j=1,\dots,r)$ and set

$$h' = \sum_1^r (\psi_j\circ b_j)', \quad k' = \sum_1^r (\varphi_j\circ b_j). \tag{5.6}$$

Then we have

$$(h+h')(k+k')^* - hk^* - \sum_1^r \psi_j\overline{\varphi_j} = hk'^* + h'k^* + h'k'^* - \sum_1^r \psi_j\overline{\varphi_j} =$$

$$= \sum_1^r h(\varphi_j\circ b_j)^* + \sum_1^r (\psi_j\circ b_j)k^* + \sum_1^r \left[(\psi_j\circ b_j)(\varphi_j\circ b_j) - \psi_j\overline{\varphi_j}\right] +$$

$$+ \sum_{j=1}^r \sum_{i=1}^{j-1} \left[(\psi_i\circ b_i)(\varphi_j\circ b_j)^* + (\psi_j\circ b_j)(\varphi_i\circ b_i)^*\right] = \Omega.$$

Taking account of (5.3), (5.4), (5.5), and Lemma 4 we deduce that

$$\|\Omega\|_{L^1/H_0^1} \leqq r\delta + r\delta + \sum_1^r \|\psi_j\|_{H^2}\left(\eta_\Theta(\varphi_j)+\delta\right)\|\varphi_j\|_{H^2} + \frac{r(r-1)}{2}\delta + \frac{r(r-1)}{2}\delta$$

so we arrive at the conclusion (5.3) by choosing δ small enough, namely such that

$$\left[(r(r+1) + \sum_1^r \|\psi_j\|_{H^2}\|\varphi_j\|_{H^2}\right]\delta \leqq \varepsilon.$$

Finally, (5.2) follows at once from (5.6) and (4.1); e.g.,

$$\|h'\|^2 = \left\|P_{\mathfrak{H}}\left(\sum_1^r \psi_j b_j \oplus 0\right)\right\|^2 \leqq \left\|\sum_1^r \psi_j b_j\right\|^2_{H^2(\mathfrak{E}_*)} = \sum_{j,i=1}^r (\psi_j,\psi_i)_{H^2}(b_j,b_i)_{\mathfrak{E}_*} = \sum_1^r \|\psi_j\|^2$$

because of orthonormality of $\{b_j\}_1^r$.

Remark. The pair h', k' can obviously be replaced by any of the pairs $h^{(n)}, k^{(n)}$ $(n=1, 2, \ldots)$ defined by

$$h^{(n)} = \sum_{j=1}^{r} (\psi_j \circ b_{j+n}), \quad k^{(n)} = \sum_{j=1}^{r} (\varphi_j \circ b_{j+n}). \tag{5.7}$$

Then, for every $l \in \mathfrak{H}$,

$$l \circ h^{(n)*},\ h^{(n)} l^*,\ l k^{(n)*},\ k^{(n)} l^*$$

tend to 0 in L^1/H_0^1 as $n \to \infty$.

6. Now we are going to establish the main result of this paper.

Theorem. *Suppose $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ is a contractive analytic function, with separable $\mathfrak{E}, \mathfrak{E}_*$, and $\dim \mathfrak{E}_* = \infty$, and suppose that for some ϑ, $0 < \vartheta < 1$, the set*

$$\mathscr{S} = \{\varphi \in H^2 \colon \|\varphi\|_{H^2} = 1, \eta_\Theta(\varphi) \leqq \vartheta\}$$

is dominant in the unit ball $\mathscr{D}$ of H^2. Then

$$\{(hk^*)^\cdot \ : h, k \in \mathfrak{H}\} = L^1/H_0^1.$$

i.e. every $f \in L^1$ has a representation

$$f \equiv hk^* \bmod H_0^1 \quad \text{with} \quad h, k \in \mathfrak{H}.$$

Proof. Consider an $f \in L^1$ with $\|f\|_{L^1/H_0^1} \leqq v_0$; it does not restrict generality to assume $v_0 = 1$. Choose a number ω such that $\vartheta < \omega < 1$ and set $v_s = \omega^s$, $\varepsilon_s = \dfrac{\omega - \vartheta}{2}\, \omega^s$; then

$$v_{s+1} = \vartheta v_s + 2\varepsilon_s. \tag{6.1}$$

Setting $h_0, h_{-1}, k_0, k_{-1} = 0$ (in $\mathfrak{H}$) we are going to prove that there exist $h_s, k_s \in \mathfrak{H}$ $(s=1, 2, \ldots)$ such that

$$\begin{aligned} &\|f - h_s k_s^*\|_{L^1/H_0^1} \leqq v_s \\ &\|h_s - h_{s-1}\|^2 \leqq v_{s-1}, \quad \|k_s - k_{s-1}\|^2 \leqq v_{s-1} \end{aligned} \qquad (s = 1, 2, \ldots). \tag{6.2}$$

This being obvious for $s=0$ we shall proceed by induction. Suppose h_s, k_s have been already found for $s = 0, \ldots, q$, satisfying (6.2), and perform the step $q \to q+1$ as follows. Set

$$f' = f - h_q k_q^*; \tag{6.3}$$

then $\|f'\|_{L^1/H_0^1} \leqq v_q$ by (6.2) for $s=q$. It now follows from Lemma 2 that there exist $\varphi_j \in \mathscr{S}$, $\psi_j \in \mathscr{D}$, $c_j \geqq 0$ $(j=1, \ldots, r)$, with $\sum_1^r c_j = 1$ and

$$\Big\| f' - \sum_{j=1}^{r} c_j v_q \psi_j \overline{\varphi_j} \Big\|_{L^1/H_0^1} \leqq \varepsilon_q. \tag{6.4}$$

On the other hand, from Lemma 5 it follows that there exist

$h_{q+1}=h_q+h'$, $k_{q+1}=k_q+k'\in\mathfrak{H}$ such that

$$\left\|h_{q+1}k_{q+1}^*-h_qk_q^*-\sum_{j=1}^r c_j v_q \psi_j\overline{\varphi_j}\right\|_{L^1/H_0^1}\leqq \tag{6.5}$$

$$\leqq\sum_{j=1}^r\|\sqrt{c_jv_q}\psi_j\|_{H^2}\|\sqrt{c_jv_q}\varphi_j\|_{H^2}\,\eta_\Theta(\varphi_j)+\varepsilon_q\leqq\sum_{j=1}^r c_jv_q\vartheta+\varepsilon_q=v_q\vartheta+\varepsilon_q$$

and

$$\|h_{q+1}-h_q\|^2\leqq\sum_{j=1}^r\|\sqrt{c_jv_q}\psi_j\|_{H^2}^2\leqq v_q,\quad \|k_{q+1}-k_q\|^2\leqq\sum_{j=1}^r\|\sqrt{c_jv_q}\varphi_j\|_{H^2}^2\leqq v_q. \tag{6.6}$$

Because of the relation

$$f-h_{q+1}k_{q+1}^*=\left(f'-\sum_{j=1}^r c_jv_q\psi_j\overline{\varphi_j}\right)-\left(h_{q+1}k_{q+1}^*-h_qk_q^*-\sum_{j=1}^r c_jv_q\psi_j\overline{\varphi_j}\right),$$

from (6.4), (6.5) and (6.1) we deduce

$$\|f-h_{q+1}k_{q+1}^*\|_{L^1/H_0^1}\leqq\vartheta v_q+2\varepsilon_q=v_{q+1}; \tag{6.7}$$

and (6.6), (6.7) yield (6.2) for the h_{q+1}, k_{q+1} just defined. The construction by induction is thus established for all s.

From (6.2) now follows that h_s, k_s converge (strongly in $\mathfrak{H}$) to some limits h, k, and that $h_sk_s^*$ converges in L^1/H_0^1 to $f^\cdot$. Since $h_s\to h$, $k_s\to k$ obviously also imply $\|h_sk_s^*-hk^*\|_{L^1}\to 0$ we conclude that $\|f-hk^*\|_{L^1/H_0^1}=0$; thus completing the proof of the theorem.

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