Acta Sci. Math., 41 (1979), 411—418

A note on the Radon—Nikodym theorem of Pedersen and Takesaki

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0. Introduction. The Radon—Nikodym theorem of PEDERSEN and TAKESAKI [4] shows the existence and uniqueness of a density of certain semi-finite weights ψ with respect to a given normal, faithful and semi-finite weight φ, the density being a self-adjoint, positive operator. Here, it is shown that — with a suitable extension of the definition of a density — this theorem remains true without the assumption of semifiniteness of the weight ψ. Paragraph 2 sums up some facts about projections which are used in the sequel.

This paper is based on the first part of the author's thesis which was supervised by Professor D. Kölzow. The author would like to express his gratitude to Professor Kölzow for his constant assistance and encouragement which were formative in the preparation of this thesis.

1. Basic notations and definitions. Let $\mathfrak{A}$ be a von Neumann algebra. A weight φ on $\mathfrak{A}$ is a map defined on $\mathfrak{A}^+$ with values in $\overline{\mathbf{R}}^+ := \mathbf{R}^+ \cup \{\infty\}$ which is additive and positive homogeneous $(0 \cdot \infty := 0)$.

A weight φ on $\mathfrak{A}$ defines the left ideals

$$\mathfrak{n}_\varphi := \{A \in \mathfrak{A} \mid \varphi(A^*A) < \infty\} \quad \text{and} \quad N_\varphi := \{A \in \mathfrak{A} \mid \varphi(A^*A) = 0\}$$

and the convex cone

$$\mathfrak{m}_\varphi^+ := \{A \in \mathfrak{A}^+ \mid \varphi(A) < \infty\}.$$

The weight φ is called faithful if it is strictly positive, semi-finite if the identity of $\mathfrak{A}$ is the ultraweak limit of elements of $\mathfrak{m}_\varphi^+$, and normal if $\varphi(\sup A_i) = \sup \varphi(A_i)$ for every increasing bounded net in $\mathfrak{A}^+$.

If φ is semi-finite, normal and faithful, then on $\mathfrak{n}_\varphi$ an inner product is defined by $(A, B) := \dot{\varphi}(B^*A)$ ($\dot{\varphi}$ the canonical extension of φ to $\mathfrak{m}_\varphi := \mathfrak{m}_\varphi^+ - \mathfrak{m}_\varphi^+ = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$). The usual Gelfand—Naimark—Segal construction gives a faithful, normal represen-

Received April 20, 1978.

tation π_φ of $\mathfrak{A}$ on H_φ, the completion of $\mathfrak{n}_\varphi$ with respect to the inner product $(.,.)$. The involution * of $\mathfrak{A}$ extends from $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ to a closed conjugate linear operator S on H_φ. If $S = J\Delta^{1/2}$ (the polar decomposition of S) then J is a conjugate linear isometry and Δ is a self-adjoint, positive, non-degenerate operator. Every $t \in \mathbf{R}$ defines a unitary operator Δ^{it} on H_φ, $A \to \Delta^{-it} A \Delta^{it}$ leaves $\pi_\varphi(\mathfrak{A})$ invariant and so gives rise to a *-automorphism σ_t of $\mathfrak{A}$. The strongly continuous one parameter group $\Sigma_\varphi := \{\sigma_t | t \in \mathbf{R}\}$ is called the modular automorphism group of φ. The weight φ fulfils the Kubo—Martin—Schwinger (KMS) condition with respect to Σ_φ, i.e. for all $A, B \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ there is a continuous, bounded function f on $\{z \in \mathbf{C} | 0 \leqq \operatorname{Im} z \leqq 1\}$, holomorphic in the interior and such that for all $t \in \mathbf{R}$

$$f(t) = \varphi(\sigma_t(A)B),\ f(t+i) = \varphi(B\sigma_t(A)).$$

If Σ' is a strongly continuous one parameter group on $\mathfrak{A}$ and φ is KMS with respect to Σ', then $\Sigma' = \Sigma_\varphi$.

A semi-finite, faithful, normal weight φ is a trace iff Σ_φ is trivial.

2. Semi-finite projections. Let φ be a fixed normal weight on $\mathfrak{A}$. If $A \in N_\varphi \cap \mathfrak{A}^+$ and if E is the spectral measure of A, then $\operatorname{supp} A = \sup E(]1/n, \infty[)$. Now, $0 \leqq 1/n E(]1/n, \infty[) \leqq A$, so $E(]1/n, \infty[)$ is in N_φ (since φ is additive), and we have that $\operatorname{supp} A$ is in N_φ (since φ is normal).

It follows that given two projections $P, Q \in N_\varphi$, their supremum (in the set of all projections of $\mathfrak{A}$) $P \vee Q = \operatorname{supp}(P+Q)$ is again in N_φ. So the set of all projections in N_φ is an increasing family with supremum P_φ. Since φ is normal, P_φ is again in N_φ, hence (Combes [1], p. 75): The set of all projections of N_φ has a largest element P_φ.

Remarks.

a) If $A \in N_\varphi$, then $\operatorname{supp} A^*A = \operatorname{supp} A$ is in N_φ, so $\operatorname{supp} A \leqq P_\varphi$. Thus, $A = A \operatorname{supp} A = AP_\varphi$ and $N_\varphi \subset \mathfrak{A}P_\varphi$. Since N_φ is a left ideal and $P_\varphi \in N_\varphi$, it follows that $N_\varphi = \mathfrak{A}P_\varphi$.

b) If g is a *-automorphism of $\mathfrak{A}$ and if φ is g-invariant, then $\varphi(g(P_\varphi)) = \varphi(P_\varphi) = \varphi(g^{-1}(P_\varphi)) = 0$, so $g(P_\varphi) \leqq P_\varphi \leqq g^{-1}(P_\varphi)$ and P_φ is g-invariant.

The following example shows that the set of projections of $\mathfrak{m}_\varphi^+$ is not an increasing family.

Example. Let H be an infinite-dimensional Hilbert space with an orthonormal basis $(e_n)_{n \in \mathbf{N}}$. Define $\mathfrak{A} := L(H)$, for $n \in \mathbf{N}$ define $f_n \in H$ by

$$f_n := (1 - 1/2^n)^{1/2} e_{2n-1} + (1/2^n)^{1/2} e_{2n},$$

and define projections $P_1, P_2 \in \mathfrak{A}$ by

$$P_1 := \sum_{n=1}^{\infty} e_{2n-1} \otimes e_{2n-1}, \quad P_2 := \sum_{n=1}^{\infty} f_n \otimes f_n.$$

Define the weight φ on $\mathfrak{A}$ by

$$\varphi := \sum_{n=1}^{\infty} \omega_{e_{2n}}.$$

Then φ is normal, $\varphi(P_1)=0$, $\varphi(\mathrm{Id})=\infty$ and $\varphi(e_{2n}\otimes e_{2n})=1$ for all $n\in\mathbf{N}$. Since $\mathrm{Id}=P_1+\sum_{n=1}^{\infty} e_{2n}\otimes e_{2n}$, this shows that φ is semi-finite.

$$\varphi(P_2) = \sum_{n=1}^{\infty} (P_2 e_{2n}, e_{2n}) = \sum_{n=1}^{\infty} ((e_{2n}, f_n) f_n, e_{2n}) = \sum_{n=1}^{\infty} |(e_{2n}, f_n)|^2 = \sum_{n=1}^{\infty} 1/2^n = 1.$$

Now, it is easy to see that $P_1 \vee P_2 = \mathrm{Id}$. Thus we have: P_1 and P_2 are in $\mathfrak{m}_\varphi^+$ and $P_1 \vee P_2$ is not, i.e. the set of all projections in $\mathfrak{m}_\varphi^+$ is not an increasing family.

Definition. Let P be a projection in $\mathfrak{A}$.

a) P is called *semi-finite* (with respect to φ) if the restriction of φ to $P\mathfrak{A}^+P$ is semi-finite;

b) P is called *σ-finite* (with respect to φ) if there is a sequence $(P_n)_{n\in\mathbf{N}}$ of mutually orthogonal projections of $\mathfrak{m}_\varphi^+$ with $P=\sum P_n$.

Clearly, every σ-finite projection is semi-finite.

2.1 Lemma. *A projection P of $\mathfrak{A}$ is σ-finite iff there is an $A\in\mathfrak{m}_\varphi^+$ with $P=\mathrm{supp}\, A$.*

Proof. Let P be σ-finite, $P=\sum_{n=1}^{\infty} P_n$ and $P_n\in\mathfrak{m}_\varphi^+$. One can assume that $\varphi(P_n)\neq 0$ for all n. Define $A:=\sum_{n=1}^{\infty} \big(1/(2^n \max(\varphi(P_n), 1))\big)P_n$. Then $\varphi(A)\leqq 1$, so $A\in\mathfrak{m}_\varphi^+$. On the other hand, $\mathrm{supp}\, A=P$. To prove the other direction, let A be in $\mathfrak{m}_\varphi^+$ and $P:=\mathrm{supp}\, A$. Let E be the spectral measure of A. Define $E_1:=E(]1,\infty[)$, $E_n:=E(]1/n, 1/(n-1)[)$ for $n\geqq 2$. Then $1/nE_n\leqq A$, so $E_n\in\mathfrak{m}_\varphi^+$ and $P=\sum_{n=1}^{\infty} E_n$.

2.2 Corollary. *With P_1, P_2 σ-finite projections, $P_1\vee P_2$ is σ-finite.*

Proof. Let A_1, A_2 be in $\mathfrak{m}_\varphi^+$ with $\mathrm{supp}\, A_i=P_i$. Then $\mathrm{supp}(A_1+A_2)=P_1\vee P_2$.

2.3 Proposition (Characterization of semi-finite projections). *Let P be a projection in $\mathfrak{A}$. Then the following are equivalent:*

a) P is semi-finite;

b) $P=\vee P_i$, where $(P_i)_{i\in I}$ is a family of σ-finite projections;

c) $P=\vee P_i$, where $(P_i)_{i\in I}$ is a family of projections in $\mathfrak{m}_\varphi^+$;

d) $P=\sup P_i$, where $(P_i)_{i\in I}$ is an increasing family of σ-finite projections.

Proof. a)$\Rightarrow$b): Assume b) false and a) true.
Define $\mathfrak{S}:=\{S\in\mathfrak{A}|\ S\ \sigma\text{-finite projection},\ S\leqq P\}$, $Q:=P-\bigvee\mathfrak{S}$.

By assumption $Q\neq 0$, so there is an ultraweakly continuous state f on $\mathfrak{A}$ with $E:=\operatorname{supp} f\leqq Q$. Now, if $(A_i)_{i\in I}$ is a net in $P\mathfrak{A}P\cap\mathfrak{m}_\varphi^+$ converging ultraweakly to P, then for all $i\in I$, $\operatorname{supp} A_i$ is σ-finite and $\operatorname{supp} A_i\leqq P$; thus, $\operatorname{supp} A_i$ is in $\mathfrak{S}$, i.e. $A_iQ=0$. Now, $|f(P-A_i)|=|f(E(P-A_i)E)|=|f(EPE)|=1$, which is a contradiction to (A_i) converging ultraweakly to P. The proofs of the other implications are easy consequences of the definition of σ-finite.

2.4 Corollary. *For every family (P_i) of semi-finite projections, $\bigvee P_i$ is semi-finite.*

2.5 Corollary (cf. PEDERSEN—TAKESAKI [4]). *The set of all semi-finite projections (with respect to φ) contains a largest element denoted by Q_φ.*

2.6 Corollary. *If g is a *-automorphism of $\mathfrak{A}$ and φ is g-invariant, then Q_φ is g-invariant.*

Proof. The Proposition shows that

$$Q_\varphi=\bigvee\{P|\ P\ \text{projection},\ P\in\mathfrak{m}_\varphi^+\}.$$

This set is g-invariant by assumption and g is a *-automorphism, thus

$$g(Q_\varphi)=\bigvee\{g(P)|\ P\ \text{projection},\ P\in\mathfrak{m}_\varphi^+\}=Q_\varphi.$$

If H is an infinite-dimensional Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbf{N}}$, then $\varphi:=\sum n\cdot\omega_{e_n}$ defines a normal, semi-finite weight on $\mathfrak{A}:=L(H)$.

If $x:=\sum_{n\in\mathbf{N}} e_n/n$ and P_x is the projection on $\langle x\rangle$, then $\varphi(P_x)=\infty$. Every $A\in P_x\mathfrak{A}P_x$ is a multiple of P_x, so P_x is not semi-finite. This shows that if P is a semi-finite projection and Q is a projection with $Q\leqq P$, then Q is not necessarily semi-finite. However, if φ is semi-finite, normal and faithful and if P is Σ_φ-invariant, then P is semi-finite (cf. COMBES—DELAROCHE [2]). For then by [4], thm 3.6, $P\mathfrak{m}_\varphi\subset\mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi P\subset\mathfrak{m}_\varphi$; so, if (A_i) is a net in $\mathfrak{m}_\varphi^+$ which converges ultraweakly to the identity, (PA_iP) is a net in $P\mathfrak{A}P\cap\mathfrak{m}_\varphi^+$ which converges ultraweakly to P.

3. The Radon—Nikodym theorem. For the rest of this paragraph let $\mathfrak{A}$ be a von Neumann algebra and φ a semi-finite, normal, faithful weight on $\mathfrak{A}$. The von Neumann algebra of all invariant elements of $\mathfrak{A}$ with respect to the modular automorphism group Σ_φ will be denoted by $\mathfrak{A}^\varphi$.

For the convenience of the reader some of the notations and results of [4] will be given.

Let H be a self-adjoint, positive operator in $\mathfrak{A}^\varphi$. Then, the map $A \to \varphi(H^{1/2}AH^{1/2})$ is a normal, semi-finite, Σ_φ-invariant weight on $\mathfrak{A}$, denoted by φ_H. If H is a self-adjoint, positive operator affiliated to $\mathfrak{A}^\varphi$, then for every $\varepsilon > 0$ the operator H_ε is defined by $H_\varepsilon := H(1+\varepsilon H)^{-1}$. Then, $H_\varepsilon \in \mathfrak{A}^{\varphi+}$ and the map $A \to \sup_{\varepsilon>0} \varphi_{H_\varepsilon}(A)$ is a normal, semi-finite, Σ_φ-invariant weight on $\mathfrak{A}$, again denoted by φ_H.

Then, the main result of [4] is the following:

Theorem (Radon—Nikodym theorem of Pedersen and Takesaki). *Let ψ be a semi-finite, normal Σ_φ-invariant weight on $\mathfrak{A}$. Then, there is a unique self-adjoint operator H affiliated with $\mathfrak{A}^\varphi$ such that $\psi = \varphi_H$.*

There is a commutative analogue of this theorem, cf. [3], p. 245, lemme 1: If $\mathfrak{A}$ is commutative, i.e. isomorphic to an $L^\infty(Z, \mu)$ with locally compact Z and positive Radon measure μ, denote by $\widehat{\mathfrak{Z}}^+$ the set of all positive, measurable functions on Z with values in $\overline{\mathbf{R}}^+$ modulo locally null-functions. The weight φ is then a semi-finite, normal, faithful trace on $\mathfrak{A}$. Then, for every normal weight ψ (=normal trace) on $\mathfrak{A}$ there is a unique $H \in \widehat{\mathfrak{Z}}^+$ such that

$$\hat{\psi}(A) = \hat{\varphi}(HA) \quad \text{for all} \quad A \in \widehat{\mathfrak{Z}}^+,$$

where $\hat{\psi}$, $\hat{\varphi}$ denote the canonical extensions of ψ and φ to $\widehat{\mathfrak{Z}}^+$.

Here, ψ need not be semi-finite. In the following it is shown that the same is true for the theorem of Pedersen and Takesaki with a suitable definition of the density H.

Definition. Let $\mathfrak{B}$ be a von Neumann algebra. A spectral measure on the Borel sets $B(\overline{\mathbf{R}}^+)$ of the extended positive real line with values in the set of self-adjoint projections of $\mathfrak{B}$ is called a ($\mathfrak{B}$-*valued*) *extended spectral measure*. The set of all $\mathfrak{B}$-valued extended spectral measures is denoted by $\widehat{\mathfrak{B}^+}$.

3.1 Lemma. *If $E \in \widehat{\mathfrak{A}^{\varphi+}}$ and $A \in \mathfrak{A}^+$, then the map $m_{\varphi,A}$ on $B(\overline{\mathbf{R}}^+)$ with values in $\overline{\mathbf{R}}^+$, defined by*

$$m_{\varphi,A}(\Delta) := \varphi\big(E(\Delta)AE(\Delta)\big) \quad (\Delta \in B(\overline{\mathbf{R}}^+))$$

is a measure on $\overline{\mathbf{R}}^+$.

Proof. Since $E(\Delta) \in \mathfrak{A}^\varphi$, prop. 4.1 of [4] shows that $m_{\varphi,A}$ is additive. The spectral measure E is σ-additive and, by prop. 4.2 of [4], the map $E(\Delta) \to \varphi\big(E(\Delta)AE(\Delta)\big)$ is normal.

Definition. Let E be in $\widehat{\mathfrak{A}^{\varphi+}}$. For $A \in \mathfrak{A}^+$ define $m_{\varphi,A}$ as in Lemma 3.1. Define φ_E: $\mathfrak{A}^+ \to \overline{\mathbf{R}}^+$ by

$$\varphi_E(A) := \int_{\overline{\mathbf{R}}^+} \lambda \, dm_{\varphi,A}(\lambda) \Big(= \int_{\overline{\mathbf{R}}^+} \lambda \, d\varphi(E_\lambda A E_\lambda)\Big) \quad (A \in \mathfrak{A}^+).$$

3.2 Lemma. *If* $E\in\widehat{\mathfrak{A}^{\varphi+}}$, *then* φ_E *is a normal and* Σ_φ*-invariant weight on* $\mathfrak{A}$.

Proof. By [4], prop. 4.1 the map $A\to\varphi(E(\Delta)AE(\Delta))$ is a weight on $\mathfrak{A}$ for every $\Delta\in B(\overline{\mathbf{R}}^+)$. Thus, the map φ_E is additive and positive homogeneous.

Next, take a $g\in\Sigma_\varphi$. Then, for all $A\in\mathfrak{A}^+$ and for all $\Delta\in B(\overline{\mathbf{R}}^+)$ we have

$$\varphi(E(\Delta)g(A)E(\Delta))=\varphi(E(\Delta)AE(\Delta)),$$

from which it follows that φ_E is Σ_φ-invariant.

Now, we show that φ_E is normal. Take an increasing family (A_i) in $\mathfrak{A}^+$ with $\sup A_i=A$. Then by the positivity of all occurring values the following holds:

$$\begin{aligned}\varphi_E(A)&=\int_{\overline{\mathbf{R}}^+}\lambda\,d\varphi(E_\lambda AE_\lambda)=\\&=\sup\{\textstyle\sum\lambda_j\varphi(E(\Delta_j)AE(\Delta_j))\,|\,\sum\Delta_j=\overline{\mathbf{R}}^+,\ \lambda_j=\inf\Delta_j\}=\\&=\sup\{\textstyle\sum\lambda_j\varphi(E(\Delta_j)(\sup_i A_i)E(\Delta_j))\}=\\&=\sup_i\sup\{\textstyle\sum\lambda_j\varphi(E(\Delta_j)A_iE(\Delta_j))\}=\\&=\sup_i\varphi_E(A_i).\end{aligned}$$

The following lemma shows that the definition of φ_E is indeed an extension of the definition of φ_H by Pedersen and Takesaki.

3.3 Lemma. *Let* H *be a self-adjoint, positive operator affiliated to* $\mathfrak{A}^\varphi$. *If* E *is the canonical spectral measure on* $B(\overline{\mathbf{R}}^+)$ *defined by* H, *then* $\varphi_H=\varphi_E$.

Proof. Take $A\in\mathfrak{A}^+$. First, if f is a simple real-valued function on $\overline{\mathbf{R}}^+$, $f=\sum_{i=1}^n\alpha_i1_{\Delta_i}$, we have

$$\varphi_{f(H)}(A)=\sum_{i=1}^n\alpha_i\varphi(1_{\Delta_i}(H)A1_{\Delta_i}(H))=\int_{\overline{\mathbf{R}}^+}f(\lambda)\,d\varphi(E_\lambda AE_\lambda).$$

Next, take an increasing sequence (f_n) of simple functions which converges to $\lambda(1+\varepsilon\lambda)^{-1}$. Then $H_\varepsilon=\sup_j f_n(H)$, so

$$\begin{aligned}\varphi_{H_\varepsilon}(A)&=\sup_n\varphi_{f_n(H)}(A)=\sup_n\int_{\overline{\mathbf{R}}^+}f_n(\lambda)\,d\varphi(E_\lambda AE_\lambda)=\\&=\int_{\overline{\mathbf{R}}^+}\lambda(1+\varepsilon\lambda)^{-1}\,d\varphi(E_\lambda AE_\lambda).\end{aligned}$$

Finally, by definition we have

$$\begin{aligned}\varphi_H(A)&=\sup_\varepsilon\varphi_{H_\varepsilon}(A)=\sup_\varepsilon\int_{\overline{\mathbf{R}}^+}\lambda(1+\varepsilon\lambda)^{-1}\,d\varphi(E_\lambda AE_\lambda)=\\&=\int_{\overline{\mathbf{R}}^+}\lambda\,d\varphi(E_\lambda AE_\lambda)=\varphi_E(A).\end{aligned}$$

3.4 Theorem (Radon—Nikodym theorem of Pedersen and Takesaki — generalized version). *Let ψ be a normal Σ_φ-invariant weight on $\mathfrak{A}$. Then, there is a unique* $E\in\widehat{\mathfrak{A}^{\varphi+}}$ *such that* $\psi=\varphi_E$.

Proof. *Existence:* Define $\tilde{\mathfrak{A}}:=Q_\psi\mathfrak{A}Q_\psi$. For a map f on $\mathfrak{A}^+$ define $\tilde{f}$ to be the restriction of f to $\tilde{\mathfrak{A}}^+$.

Since ψ is Σ_φ-invariant, so is Q_ψ (2.6), i.e. $Q_\psi\in\mathfrak{A}^\varphi$ and so Q_ψ is semi-finite with respect to φ (see the end of § 2). Thus, $\tilde{\varphi}$ is semi-finite, normal and faithful and $\tilde{\psi}$ is semi-finite and normal.

We show that $\tilde{\psi}$ is $\Sigma_{\tilde{\varphi}}$-invariant: Since Q_ψ is Σ_φ-invariant, Σ_φ leaves $\tilde{\mathfrak{A}}$ invariant; now, $\tilde{\varphi}$ fulfils the KMS condition with respect to the restriction of Σ_φ to $\tilde{\mathfrak{A}}$, so $\tilde{\Sigma}_\varphi$ and $\Sigma_{\tilde{\varphi}}$ coincide. Thus, for all $\hat{g}\in\Sigma_{\tilde{\varphi}}$ there is a $g\in\Sigma_\varphi$ such that $\hat{g}=\tilde{g}$; hence, $\tilde{\psi}(\hat{g}(A))= =\tilde{\psi}(\tilde{g}(A))=\psi(g(A))=\psi(A)=\tilde{\psi}(A)$ $(A\in\tilde{\mathfrak{A}}^+)$.

So, the Radon—Nikodym theorem of Pedersen and Takesaki gives a unique self-adjoint and positive operator H affiliated to $\tilde{\mathfrak{A}}^{\tilde{\varphi}}$ such that $\tilde{\psi}=\tilde{\varphi}_H$. If E_H is the spectral measure of H, define $E\in\widehat{\mathfrak{A}^{\varphi+}}$ by $\mathrm{Rest}_{B(\mathbf{R}^+)}E:=E_H$ and $E(\{\infty\}):=Q_\psi^\perp$.

Next we show that $\psi=\varphi_E$:

Let A be in $\mathfrak{A}^+$.

Case 1: $Q_\psi^\perp AQ_\psi^\perp\neq 0$. Since φ is faithful, $\varphi(Q_\psi^\perp AQ_\psi^\perp)\neq 0$ and it follows that $\varphi_E(A)\geqq\infty\cdot\varphi(Q_\psi^\perp AQ_\psi^\perp)=\infty$. Assume that $\psi(A)$ is finite. Then, by Lemma 2.1, supp A is σ-finite, and it follows that supp $A\leqq Q_\psi$ and $Q_\psi^\perp AQ_\psi^\perp=0$ which is a contradiction. Thus $\psi(A)=\infty$ and so $\psi(A)=\varphi_E(A)$.

Case 2: $Q_\psi^\perp AQ_\psi^\perp=0$. Now, since A is positive, we have that $Q_\psi AQ_\psi=A$ (i.e. $A\in\tilde{\mathfrak{A}}^+$) and $\psi(A)=\tilde{\psi}(A)=\tilde{\varphi}_H(A)=\tilde{\varphi}_{E_H}(A)=\varphi_E(A)$.

Uniqueness. Suppose $F\in\widehat{\mathfrak{A}^{\varphi+}}$ with $\psi=\varphi_F$. Then $Q:=F(\overline{\mathbf{R}}^+)\in\mathfrak{A}^\varphi$, so Q is semi-finite with respect to φ. If f is a map on $\mathfrak{A}^+$, denote the restriction of f to $(Q\mathfrak{A}Q)^+$ by $\hat{f}$. Then, $\hat{\varphi}$ is semi-finite, normal and faithful and $\widehat{\varphi_F}(.)=\int_{\overline{\mathbf{R}}^+}\lambda d\varphi(F.F)$.

If K is the (canonical) self-adjoint operator with spectral measure $\mathrm{Rest}_{B(\mathbf{R}^+)}F$, then by 3.3, $\widehat{\varphi_F}=\hat{\varphi}_{\mathrm{Rest}_{B(\mathbf{R}^+)}F}=\hat{\varphi}_K$, so $\widehat{\varphi_F}$ is semi-finite by [4], prop. 4.2, from which it follows that Q is semi-finite with respect to $\varphi_F=\psi$. Thus $Q\leqq Q_\psi=Q_{\varphi_F}$. On the other hand, if $P\in\mathfrak{A}$ is a projection with $\varphi_F(P)<\infty$, then by the faithfulness of φ, $F(\{\infty\})PF(\{\infty\})=0$, so $P\leqq Q$. These facts together give $Q=Q_\psi$ (see § 2).

In particular, the argument applies to the spectral measure E (where E is as in the proof of existence), so one has $E(\mathbf{R}^+)=F(\mathbf{R}^+)$ (i.e. $\tilde{\psi}=\hat{\psi}=\hat{\varphi}_K$), and by the uniqueness of K it follows that $E=F$.

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