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On the very strong and mixed approximations

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1. Let f be a continuous and 2π-periodic function. Denote by $E_n(f)$, $\omega(f;\delta)$ and $s_k(x)=s_k(f;x)$ its best uniform approximation by trigonometric polynomials of degree at most n, its modulus of continuity, and the k-th partial sum of its Fourier series, respectively.

If ω is a modulus of continuity and $r\geqq 0$ is an integer we define W^rH^ω to be the class of those functions f for which $\omega(f^{(r)};\delta)\leqq K_f\omega(\delta)$ $(\delta\in[0,2\pi])$ holds with some constant K_f.

In [3], following works of ALEXITS, KRÁLIK and LEINDLER, we proved

Theorem A. *If* $p,\beta,\gamma>0$ *and* $f\in W^rH^\omega$ *then we have*

$$h_n(f,p,\beta;x)=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,\beta,n}\ ^{*)}$$

and

$$\sigma_n^\gamma|f,p;x|=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,1,n}\quad\left(A_n^\gamma=\binom{n+\gamma}{n}\right),$$

where

$$H_{r,\omega}^{p,\beta,n}=\left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\right\}^{\frac{1}{p}}.$$

Moreover, there are functions $f\in W^rH^\omega$ *for which*

$$h_n(f,p,\beta;0)\geqq cH_{r,\omega}^{p,\beta,n}\quad\text{and}\quad\sigma_n^\gamma|f,p;0|\geqq cH_{r,\omega}^{p,1,n}\quad(n=1,2,\dots)$$

for some $c>0$.

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*) K, c with or without subscripts denote constants not necessarily the same at each occurrence.

LEINDLER [2] raised the question: What can we say about the order of the strong approximation if we replace the sequence of the partial sums by a subsequence (very strong approximation) or by a permutation of such a subsequence (mixed approximation). In this paper we shall deal with these questions.

Our main result is

Theorem 1. *Let* $E_n(f) \leqq K\varrho_n$ $(n=1, 2, \ldots)$, *where the sequence* $\{\varrho_n\}$ *satisfies the condition*

$$i\varrho_{2^i n} \leqq K\varrho_n \quad (i, n = 1, 2, \ldots). \tag{1.1}$$

There exists a constant K_p, *independent of n and of the sequence* $\nu=\{\nu_k\}_{k=0}^{\infty}$ *for which*

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n} |s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq K_p\varrho_n \quad (p>0).$$

We shall use Theorem 1 to prove

Theorem 2. *Let us suppose that* $f \in W^r H^\omega$ *where either* $r \geqq 1$ *or* $r=0$, *and* ω *satisfies the condition*

$$i\omega\left(\frac{1}{2^i n}\right) \leqq K\omega\left(\frac{1}{n}\right) \quad (i, n_i = 1, 2, \ldots). \tag{1.2}$$

We have for any $\gamma, \beta, p > 0$ *and for an arbitrary sequence* $\nu=\{\nu_k\}$

$$\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n} (k+1)^{\beta-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,\beta,n} \tag{1.3}$$

and

$$\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n} A_{n-k}^{\gamma-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,1,n} \tag{1.4}$$

where K is independent of n and ν.

If, moreover, for every function $f \in W^r H^\omega$ *and for every sequence* $\{\nu_k\}$ *we have*

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n} |s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} = O\left(\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\right) \tag{1.5}$$

then either $r \geqq 1$ *or* $r=0$, *and* (1.2) *is true.*

If $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$ then (1.2) is satisfied and Theorem 2 shows that there is no difference with respect to the approximation order between the strong and the very strong approximation of functions in the classes $W^r \operatorname{Lip}\alpha$ $(r=0, 1, \ldots; 0<\alpha\leqq 1)$. This is an answer to one of Leindler's problems (see the last two question of [2]).

We mention that the assumption "either $r\geqq 1$ or $r=0$ and (1.2)" is also necessary that (1.3) and (1.4) should be satisfied, namely for $\beta>(r+1)p$ we obtain by Corollary of [3, Theorem 1]

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} = O\left(\left\{\frac{1}{(2n+1)^\beta}\sum_{k=0}^{2n}(k+1)^{\beta-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}}\right) =$$

$$= O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

so the second part of Theorem 2 is applicable.

Finally we turn to the mixed approximation. Let N be the collection of the natural numbers.

Theorem 3. *Let $\pi\colon N\to N$ be an injection, $p>0$ and $f\in W^rH^\omega$, where either $r\geqq 1$ or $r=0$, and* (1.2) *is true for ω.*

(i) *If $0<\beta\leqq 1$ then*

$$h_n(f,p,\beta,\pi;x)=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,\beta,n}.$$

(ii) *If $\beta>1$ and* $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p=\infty$ *then* $h_n(f,p,\beta,\pi;x)\leqq KH_{r,\omega}^{p,1,n}$.

(iii) *If $\beta>1$ and* $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p<\infty$ *then* $h_n(f,p,\beta,\pi;x)=o(H_{r,\omega}^{p,1,n})$, *uniformly in x.*

(iv) *If $0<p<1$ and* $\sum_{k=1}^{\infty}(k+1)^{\gamma-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p=\infty$ *then*

$$\sigma_n^\gamma|f,p,\pi;x|=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,\gamma,n}.$$

(v) *If $0<p<1$ and* $\sum_{k=1}^{\infty}(k+1)^{\gamma-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p<\infty$ *then*

$$\sigma_n^\gamma|f,p,\pi;x|=o(H_{r,\omega}^{p,\gamma,n}),$$

uniformly in x.

(vi) *If $\gamma\geqq 1$ then* $\sigma_n^\gamma|f,p,\pi;x|\leqq KH_{r,\omega}^{p,1,n}$.

The above constant K is independent of π, n and x.

These estimations are best possible, namely if $\varrho_n\to 0$ arbitrarily, then there exist $f\in W^rH^\omega$ and $c>0$ such that, according to the cases (i)—(vi) *separately, there* *are permutations π of N for which*

(1.8) $$h_n(f, p, \beta, \pi;\ x) = \begin{cases} cH^{p,\beta,n}_{r,\omega} & \text{(i)} \\ cH^{p,1,n}_{r,\omega} & \text{(ii)} \\ c\varrho_n H^{p,1,n}_{r,\omega} & \text{(iii)} \end{cases}$$

and

(1.9) $$\sigma^{\gamma}_n|f, p, \pi;\ 0| \geqq \begin{cases} cH^{p,\gamma,n}_{r,\omega} & \text{(iv)} \\ c\varrho_n H^{p,\gamma,n}_{r,\omega} & \text{(v)} \\ cH^{p,1,n}_{r,\omega} & \text{(vi)} \end{cases}$$

are satisfied for infinitely many n.

Corollary. *Under the assumptions of Theorem* 3, *for*

$$h_n(p, \beta) = \sup_{f:\ \omega(f;\ \delta) \leqq \omega(\delta)} \sup_{\pi;\ x} h_n(f, p, \beta, \pi;\ x)$$

and

$$\sigma_n(p, \beta) = \sup_{f:\ \omega(f;\ \delta) \leqq \omega(\delta)} \sup_{\pi;\ x} \sigma^{\beta}_n|f, p, \pi;\ x|$$

we have

$$c_1 H^{p,\beta^*,n}_{r,\omega} \leqq h_n(p, \beta),\ \sigma_n(p, \beta) \leqq c_2 H^{p,\beta^*,n}_{r,\omega} \quad (c_1 > 0,\ n = 1, 2, \dots)$$

where $\beta^* = \min(1, \beta)$.

2. To prove our theorems we require the following two lemmas.

Lemma 1. [3, Theorem 4] *There exists a K_p depending only on $p(>0)$ for which*

$$\left\{\frac{1}{r}\sum_{i=1}^{r} |s_{k_i} - f|^p\right\}^{\frac{1}{p}} \leqq K_p E_{k_1}(f) \log\frac{2n}{r},$$

whenever $1 \leqq k_1 < k_2 < \dots < k_r \leqq n$.

Lemma 2. [3, Lemma 5] *Let ω be an arbitrary modulus of continuity. Then there are functions $f \in W^0 H^{\omega}$ such that*

(2.1) $$|s_{n\pm\lambda}(f;\ 0) - f(0)| > 10^{-2}\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda} \quad (\lambda \leqq e^{-100} n)$$

is true for infinitely many n.

We can also require that (2.1) *be true for infinitely many n belonging to a given sequence.*

3. Proof of Theorem 1. Let k_i be the number of those ν_t for which

$$2^i n < \nu_t \leqq 2^{i+1} n \quad (n < t \leqq 2n,\ i = 0, 1, \dots).$$

By Lemma 1 we have

$$\sum_{2^i n < \nu_t \leqq 2^{i+1} n} |s_{\nu_t}(x) - f(x)|^p \leqq K k_i (E_{2^i n}(f))^p \left(\log\frac{2^{i+1} n}{k_i}\right)^p \leqq K k_i \varrho^p_{2^i n} \left(\log\frac{2^{i+1} n}{k_i}\right)^p,$$

and thus it is enough to show that

$$S=\frac{1}{n}\sum_{k_i>0} k_i\left(\frac{\varrho_{2^i n}}{\varrho_n}\right)^p\left(\log\frac{2^{i+1}n}{k_i}\right)^p \leqq K, \tag{3.1}$$

where K is independent from v and n.

Now,

$$S \leqq K_p\left(\frac{1}{n}\sum_{k_i>0} k_i\left(\frac{\varrho_{2^i n}(i+1)}{\varrho_n}\right)^p+\frac{1}{n}\sum_{k_i>0}\left(\frac{\varrho_{2^i n}}{\varrho_n}\right)^p k_i\left(\log\frac{n}{k_i}\right)^p\right) = S_1+S_2.$$

(1.1) gives

$$S_1 \leqq K\frac{1}{n}\sum_{k_i>0} k_i \leqq K,$$

$$S_2 \leqq K\frac{1}{n}\sum_{k_i>0,\, i>0}\left(\frac{1}{i}\right)^p k_i\left(\log\frac{n}{k_i}\right)^p+\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p = S_{21}+S_{22}+\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p$$

(if $k_0=0$ then the last member is missing), where the summation in S_{21} is extended to the i's satisfying the condition $\frac{1}{i}\log\frac{n}{k_i}\leqq p$. We obtain

$$S_{21} \leqq K\sum_{k_i>0}\frac{k_i}{n} \leqq K.$$

In S_{22} we have $\log\frac{n}{k_i}>pi$ i.e. $\frac{n}{k_i}>e^{p_i}$, and so $\left(\log\frac{n}{k_i}\right)^p\Big/\frac{n}{k_i}\leqq (pi)^p/e^{p_i}$; hence,

$$S_{22} \leqq K\sum_{i=1}^{\infty}\frac{1}{i^p}\frac{(pi)^p}{e^{p_i}} \leqq K.$$

Finally, $(\log x)^p/x \leqq K_p$ $(x\geqq 1)$ and so $\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p \leqq K_p$.

Collecting the above estimations we obtain (3.1), and the proof is completed.

Proof of Theorem 2. $f\in W^r H^\omega$ implies by the well-known result of Jackson that $E_n(f)\leqq K\frac{1}{n^r}\omega\left(\frac{1}{n}\right)$; thus we can apply Theorem 1 with $\varrho_n=\frac{1}{n^r}\omega\left(\frac{1}{n}\right)$ and obtain

$$\left\{\frac{1}{n}\sum_{k=n=1}^{2n}|s_{v_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq K\frac{1}{n^r}\omega\left(\frac{1}{n}\right).$$

Using this, we get for $2^{m_0-1}<n\leqq 2^{m_0}$

$$\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\nu_k}x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{1}|s_{\nu_k}(x)-f(x)|^p+\right.$$

$$\left.+\frac{1}{(n+1)^\beta}\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=2^m+1}^{2^m}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq$$

$$\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}2^m\left(\frac{1}{2^{mr}}\,\omega\left(\frac{1}{2^m}\right)\right)^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,\beta,n}$$

which is (1.3).

(1.4) results by a similar argument using also the Hölder inequality (see e.g. the proof of [1, Theorem 3]).

Next we prove the last statement of Theorem 2. We have to show that if (1.2) is not satisfied then (1.5) does not hold for some f and ν.

Thus let us suppose that (1.2) is not true. Then for every n there are m_n and i_n such that

$$i_n\omega\left(\frac{1}{2^{i_n}m_n}\right)>n\omega\left(\frac{1}{m_n}\right).$$

Since $\omega\left(\frac{1}{2m}\right)\geqq\frac{1}{2}\,\omega\left(\frac{1}{m}\right)$, we may suppose that the sequence $\{i_n\}_{n=1}^{\infty}$ is increasing and that $m_{n+1}>2m_n$ $(n=1,2,\ldots)$.

Taking into account that surely $i_n\to\infty$ if $n\to\infty$, we have $2^{i_n}>e^{100}$ for all sufficiently large n. Now Lemma 2 gives a function $f\in W^0H^\omega$ such that

$$|s_{2^{i_n}m_n+\lambda}(f;\,0)-f(0)|>10^{-2}(\log 2^{i_n})\,\omega\left(\frac{1}{2^{i_n}m_n}\right)\quad(0<\lambda\leqq m_n)\tag{3.2}$$

holds for infinitely many n. Hence, if we construct a sequence $\{\nu_k\}$ for which

$$\nu_{m_n+1}=2^{i_n}m_n+1,\ \nu_{m_n+2}=2^{i_n}m_n+2,\ \ldots,\ \nu_{2m_n}=2^{i_n}m_n+m_n$$

for all n (this is clearly possible) then we get for infinitely many n

$$\left\{\frac{1}{m_n}\sum_{k=m_n+1}^{2m_n}|s_{\nu_k}(f;\,0)-f(0)|^p\right\}^{\frac{1}{p}}>10^{-2}(\log 2^{i_n})\,\omega\left(\frac{1}{2^{i_n}m_n}\right)\geqq$$

$$\geqq\frac{1}{2}\,10^{-2}\,i_n\omega\left(\frac{1}{2^{i_n}m_n}\right)>\frac{1}{2}\,10^{-2}n\omega\left(\frac{1}{m_n}\right)$$

i.e. f and $\{\nu_k\}$ do not satisfy (1.5).

We have completed our proof.

Proof of Theorem 3. First we prove (i) for $\beta=1$:

$$h_n(f,p,1,\pi;x)\leqq K\left(\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}+\left\{\frac{1}{n+1}\sum_{\substack{0\leqq k\leqq m\\ \pi(k)>n}}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\right)\leqq$$

$$\leqq KH_{r,\omega}^{p,1,n}+K\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\leqq KH_{r,\omega}^{p,1,n},$$

where we use Theorem A and Theorem 1.

This gives

$$\sum_{k=1}^{n}|s_{\pi(k)}(x)-f(x)|^p\leqq K\sum_{k=1}^{n}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p, \tag{3.3}$$

by which we have for $2^{m_0-1}<n\leqq 2^{m_0}$ and for $\beta<1$

$$\sum_{k=1}^{n}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\leqq K\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=2^m}^{2^{m+1}}|s_{\pi(k)}(x)-f(x)|^p\leqq$$

$$\leqq K\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=1}^{2^{m+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\leqq K\sum_{k=1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\sum_{m=\overset{2}{\log}k-1}^{m_0-1}(2^m)^{\beta-1}\leqq$$

$$\leqq K\sum_{k=1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p(2^{\overset{2}{\log}k})^{\beta-1}\leqq K\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p$$

and this is exactly (i).

(ii) follows from (3.3) since

$$h_n(f,p,\beta,\pi;x)\leqq\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(n+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,1,n}.$$

Now let us suppose that $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p<\infty$. Lemma 1 gives that

$$\sum_{k=n}^{2n}|s_k(x)-f(x)|^p\leqq K\sum_{k=n}^{2n}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p$$

by which

$$\sum_{k=M}^{\infty}{}'|s_k(x)-f(x)|^p\leqq K\sum_{k=M}^{\infty}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p=o(1)\quad(M\to\infty).$$

Let $\varepsilon>0$ be arbitrary and let us choose M so that

$$\sum_{k=M}^{\infty}|s_k(x)-f(x)|^p<\varepsilon$$

be satisfied for all x. If $N \geqq \max_{0 \leqq i \leqq M} \pi^{-1}(i)$ then

$$\frac{1}{(n+1)^{\beta}} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_{\pi(k)}(x)-f(x)|^p \leqq$$

$$\leqq \frac{(N+1)^{\beta-1}}{(n+1)^{\beta}} \sum_{k=0}^{M} |s_k(x)-f(x)|^p + \frac{1}{(n+1)^{\beta}} \sum_{\substack{0 \leqq k \leqq n \\ \pi(k) \geqq M}} (n+1)^{\beta-1} |s_{\pi(k)}(x)-f(x)|^p =$$

$$= o(n^{-1}) + \frac{1}{n+1} \sum_{k=M}^{\infty} |s_k(x)-f(x)|^p \leqq \frac{2\varepsilon}{n+1}$$

for all n large enough. Thus we have proved (iii), too.

(iv) follows from (i):

$$\sigma_n^{\gamma}|f, p, \pi;\, x| \leqq K \left\{ \frac{1}{(n+1)^{\gamma}} \sum_{k=0}^{n} (n+1-k)^{\gamma-1} |s_{\pi(k)}(x)-f(x)|^p \right\}^{\frac{1}{p}} =$$

$$= K \left\{ \frac{1}{(n+1)^{\gamma}} \sum_{k=0}^{n} (k+1)^{\gamma-1} |s_{\pi(n-k)}(x)-f(x)|^p \right\}^{\frac{1}{p}} \leqq K H_{r,\omega}^{p,\gamma n},$$

where we used the inequalities

$$c_1(\alpha) k^{\alpha} \leqq A_k^{\alpha} \leqq c_2(\alpha) k^{\alpha} \quad (\alpha > -1,\ c_1(\alpha) > 0,\ k = 1, 2, \ldots).$$

(vi) could be proved similarly with the aid of (ii) and (iii).

Finally let us suppose that $\gamma < 1$ and $\sum_{k=1}^{\infty} (k+1)^{\gamma-1} \left(\frac{1}{k^r} \omega \left(\frac{1}{k} \right) \right)^p < \infty$. It is known that the last condition implies

$$\sum_{k=1}^{n} \left(\frac{1}{k^r} \omega \left(\frac{1}{k} \right) \right)^p = o((n+1)^{1-\gamma}).$$

Thus to every $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ for which

$$\sum_{k=2^M+1}^{\infty} (k+1)^{\gamma-1} \left(\frac{1}{k^r} \omega \left(\frac{1}{k} \right) \right)^p < \varepsilon^p \quad \text{and} \quad \sum_{k=1}^{2^M+1} \left(\frac{1}{k^r} \omega \left(\frac{1}{k} \right) \right)^p < \varepsilon^p (2^M)^{1-\gamma}$$

are satisfied. It is easy to see that (1.2) implies $\omega(\delta) \log \delta = o(1)$ $(\delta \to 0)$. Now the Dini—Lipschitz test gives that $s_k(x) - f(x) = o(1)$ uniformly in x, and so

$$|s_{\pi(n-k)}(x) - f(x)| < \frac{\varepsilon^p}{2^M+1} \quad (k = 0, 1, \ldots, 2^M)$$

for $n \geqq n_{\varepsilon}$.

Using the previous estimations and (3.3) we have for $n \geqq n_\varepsilon$ and $2^{m_0-1} < < n \leqq 2^{m_0}$

$$\sigma_n^\gamma |f, p, \pi; x| \leqq K\left(\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{2^M} A_k^{\gamma-1}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}\sum_{k=2^m+1}^{2^{m+1}} A_k^{\gamma-1}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq K\left(\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{2^M}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} + \right.$$

$$+\left.\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\sum_{k=2^m+1}^{2^{m+1}}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\right) \leqq K\left(\left\{\frac{1}{A_n^\gamma}(2^M+1)\frac{\varepsilon^p}{2^M+1}\right\}^{\frac{1}{p}} + \right.$$

$$+\left.\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\right\}^{\frac{1}{p}}\right) \leqq K\left(\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\right\}^{\frac{1}{p}} + \left\{\frac{1}{A_n^\gamma}\sum_{k=2^{M+1}+1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p \cdot\right.$$

$$\left.\cdot\sum_{m=\overset{2}{\log}k-1}^{m_0-1}(2^m)^{\gamma-1}\right\}^{\frac{1}{p}}\Bigg) \leqq K\left(\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} + \left\{\frac{1}{A_n^\gamma}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p(2^M)^{\gamma-1}\right\}^{\frac{1}{p}} + \right.$$

$$+\left.\left\{\frac{1}{A_n^\gamma}\sum_{k=2^M+1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p(k+1)^{\gamma-1}\right\}^{\frac{1}{p}}\right) \leqq K\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} \leqq K\varepsilon H_{r,\omega}^{p,\gamma,n}$$

which was to be proved.

So far we have proved (i)—(vi). It remains to show that these estimations are best possible.

Let f be the function given in Theorem A.

The first row of (1.8) immediately follows from Theorem A.

Let us suppose that $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p=\infty$ and that $\beta>1$. We shall define a π permutation of N as follows: If $\pi(0), \ldots, \pi(n_{m-1})$ are already known and $\pi(i) \leqq M_m$ $(i=0, 1, \ldots, n_{m-1})$, let

$$\pi(n_m+1) = 2M_m+1,\ \pi(n_m+2) = 2M_m+2, \ldots, \pi(n_m+n_m) = 2M_m+n_m,$$

where n_m will be chosen later.

However should π be defined between n_{m-1} and n_m we have in any case

$$\left\{\frac{1}{(2n_m+1)^\beta}\sum_{k=0}^{2n_m}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \geqq$$

$$\geqq \left\{\frac{1}{(2n_m+1)^\beta}(n_m)^{\beta-1}\sum_{k=n_m+1}^{2n_m}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \geqq \left\{\frac{C}{n_m+1}\sum_{k=2M_m+1}^{n_m}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}$$

and therefore

(3.4)

$$h_{2n_m}(f, p, \beta, \pi; 0) \geqq c\left\{\frac{1}{n_m+1}\sum_{k=0}^{n_m}|s_k(0)-f(0)|^p-\frac{\sum_{k=0}^{2M_m}|s_k(0)-f(0)|^p}{n_m+1}\right\}^{\frac{1}{p}} \geqq \frac{c}{2}H_{r,\omega}^{p,1,n_m}$$

if n_m is large enough in comparison with M_m, because $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p=\infty$ is equivalent with $(H_{r,\omega}^{p,1,n})^p\neq O\left(\frac{1}{n}\right)$.

Let us choose n_m so large that the above estimation should be satisfied, and then continue the procedure.

It is clear that the above, partly defined π could be extended to a permutation of N, and so (3.4) shows that (ii) cannot be improved.

Finally, if $\varrho_n\to 0$ arbitrarily, we follow the above construction and get

$$h_{2n_m}(f, p, \beta, \pi; 0) \geqq c\left\{\frac{1}{n_m+1}\sum_{k=2M_m+1}^{n_m}|s_k(0)-f(0)|^p\right\}^{\frac{1}{p}} \geqq$$

$$\geqq c\left\{\frac{1}{n_m}\sum_{k=2M_m+1}^{n_m}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\right\}^{\frac{1}{p}} \geqq c\varrho_{n_m}\left(\frac{1}{n_m}\right)^{\frac{1}{p}} \geqq c\varrho_{n_m}H_{r,\omega}^{p,1,n}$$

(at the second inequality we used that for f we have $|s_k(0)-f(0)|\geqq c\,\frac{1}{k^r}\omega\left(\frac{1}{k}\right)$ $(5\cdot 2^\nu-2^{\nu-1}\leqq k\leqq 5\cdot 2^\nu+2^{\nu-1})$ (see the proof of [3, Theorem 1]) if n_m is large enough.

Thus the proof of (1.8) is completed.

The proof of (1.9) is similar, we omit the details.

The proof of the Corollary on the basis of the above arguments is easy. The right-hand estimations follow from the proof of (i)—(vi), while the left-hand sides are easy consequences of Theorem A.

The proof of Theorem 3 is thus completed.

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