Acta Sci. Math., 41 (1979), 429—433

Quasi-similarity of restricted C_0 contractions

MITSURU UCHIYAMA

1. A bounded linear operator X from a separable Hilbert space $\mathfrak{H}$ to a separable Hilbert space $\mathfrak{H}'$ is called a *quasi-affinity* if $K(X)=0$ and $K(X^*)=0$, where $K(X)$ denotes the kernel of X. The bounded operators T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$ are called *quasi-similar* and denoted by $T\sim T'$ if there are quasi-affinities X and Y such that $XT=T'X$ and $TY=YT'$.

In this note we say thet T has *property* (Q) if $T|K(A)$ and $((T^*|K(A^*)^*)$ are quasi-similar for every A in $(T)'$. Not every bounded operator has property (Q); it is easy to contstruct even a self adjoint operator which has not property (Q).

2. Lemma 1. *If T on $\mathfrak{H}$ and S on $\mathfrak{H}'$ are similar, then T has property* (Q) *if and only if so is S.*

Proof. Let T have property (Q) and suppose $XT=SX$ for some invertible X. Set $B=X^{-1}AX$ for A commuting with S. Then it is clear that B commutes with T and that $T|K(B)$ and $T^*|K(B^*)$ are similar to $S|K(A)$ and $S^*|K(A^*)$, respectively. Therefore $S|K(A)\sim(S^*|K(A^*))^*$.

Lemma 2. *If both T on $\mathfrak{H}$ and S on $\mathfrak{H}'$ have property* (Q) *and $\sigma(T)\cap\sigma(S)=\emptyset$, then the direct sum $T\oplus S$ on $\mathfrak{H}\oplus\mathfrak{H}'$ has property* (Q) *also.*

Proof. From Rosenblum's corollary, $(T\oplus S)'=(T)'\oplus(S)'$ [2]. The rest is omitted.

Proposition 1. *If $\mathfrak{H}$ is finite dimensional, then every normal operator on $\mathfrak{H}$ has property* (Q).

Proof. From Lemma 1 and Lemma 2, we may assume that $T=\alpha I$ for some scalar α. The rest is obvious.

We will use the above results in the last example.

Received September 29, 1978, in revised form February 1, 1979.

3. SZ.-NAGY and C. FOIAŞ [7] conjectured that all C_0 contractions with finite multiplicity have property (Q). In this section we present a counter example. About the terminology and the notations see [4] and [1].

Example 1. Let ψ_1 and ψ_2 be relatively prime scalar inner functions defined on the unit circle. And define the 2×2 diagonal matrix valued inner function M by

$$M = \psi_1^2\psi_2 \oplus \psi_1^3\psi_2^2.$$

Then the class $C_0(2)$ contraction $S(M)$ on $\mathfrak{H}(M)$ defined by

$$\mathfrak{H}(M) = H_2^2 \ominus MH_2^2, \quad S(M)h = P(zh),$$

where H_2^2 denotes the 2-dimensional vector valued Hardy class and P is the projection from H_2^2 onto $\mathfrak{H}(M)$, does not have property (Q).

Proof. Setting

$$\Delta = \begin{bmatrix} \psi_1^2 & \psi_1^3 \\ \psi_1^2\psi_2^2 & 0 \end{bmatrix},$$

$A = P\Delta|\mathfrak{H}(M)$ commutes with $S(M)$, because $\Delta MH_2^2 \subset MH_2^2$. First we show that

$$K(A) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \psi_1 \\ \psi_2 & -1 \end{bmatrix}\left\{H_2^2 \ominus \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_1 & \psi_1^3\psi_2 \\ \psi_1\psi_2 & 0 \end{bmatrix} H_2^2\right\}$$

and hence

$$S(M)|K(A) \sim S\left(\frac{1}{\sqrt{2}}\begin{bmatrix} \psi_1 & \psi_1^3\psi_2 \\ \psi_1\psi_2 & 0 \end{bmatrix}\right).$$

For this, it is sufficient to show that

$$\{h_1 \oplus h_2 \colon h_i \in H_2^2,\ \Delta(h_1 \oplus h_2) \in MH_2^2\} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \psi_1 \\ \psi_2 & -1 \end{bmatrix} H_2^2.$$

It is clear that the right hand side set is included to the left hand side set. Suppose that an element $h_1 \oplus h_2$ in the left hand side set is orthogonal to the right hand set. Then there are f_1 and f_2 in H_2^2 such that

$$h_1 + \psi_1 h_2 = \psi_2 f_1, \quad h_1 = \psi_1 f_2, \quad \text{and, therefore,} \quad \psi_1(f_2 + h_2) = \psi_2 f_1.$$

Since ψ_1 and ψ_2 are relatively prime, there exists f in H_2^2 such that $f_1 = \psi_1 f$ so $f_2 + h_2 = \psi_2 f$. On the other hand, for every g_1 and g_2 in H_2^2 it follows that

$$(h_1, \psi_1 g_2) + (h_2, \psi_2 g_1 - g_2) = 0.$$

Thus we have $f_2 = h_2$ and $(h_2, \psi_2 g_1) = 0$, which imply $f = 0$ and hence $h_1 = h_2 = 0$.

Next we show that

$$\text{closure of range } A = (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2 \ominus MH_2^2$$

and hence $(S(M)^*|K(\Delta^*))^* \sim S(\psi_1^2 \oplus \psi_1^2\psi_2^2)$. For this it suffices to show that

$$\Delta H_2^2 \vee MH_2^2 = (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2.$$

Since

$$\Delta = \begin{bmatrix} \psi_1^2 & 0 \\ 0 & \psi_1^2\psi_2^2 \end{bmatrix} \begin{bmatrix} 1 & \psi_1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M = (\psi_1^2 \oplus \psi_1^2\psi_2^2)(\psi_2 \oplus \psi_1),$$

$\Delta H_2^2 \vee MH_2^2 \subset (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2$. Suppose that $\psi_1^2 h_1 \oplus \psi_1^2\psi_2^2 h_2$ is orthogonal to $\Delta H_2^2 \vee MH_2^2$. Then $h_1 \oplus h_2$ is orthogonal to

$$\begin{bmatrix} 1 & \psi_1 \\ 1 & 0 \end{bmatrix} H_2^2 \vee (\psi_2 \oplus \psi_1) H_2^2.$$

From this it follows that $h_1 + h_2 = 0$, and that h_1 and h_2 are orthogonal to $\psi_2 H^2$ and $\psi_1 H^2$, respectively. Since ψ_1 and ψ_2 are relatively prime, we have $h_1 = h_2 = 0$.

Last we must show that $S(M)|K(\Delta)$ and $(S(M)^*|K(\Delta^*))^*$ are not quasi-similar. But this is clear, because the minimal functions of these operators are $\psi_1^3\psi_2^2$ and $\psi_1^2\psi_2^2$, respectively.

4. We denote the lattice of invariant subspaces for T and the lattice of hyper-invariant subspaces for T by Lat T and Hyplat T, respectively.

Let θ and θ' be $n \times n$ matrix valued inner functions. Suppose $S(\theta)$ on $\mathfrak{H}(\theta)$ and $S(\theta')$ on $\mathfrak{H}(\theta')$ defined as Example 1 are quasi-similar. Then there are $n \times n$ matrices Γ and Λ over H^∞ such that

$$\Gamma\theta = \theta'\Lambda \quad \text{and} \quad (\det \Gamma)(\det \Lambda) \wedge (\det \theta)(\det \theta') = 1 \quad [1].$$

Moreover, it follows that

$$(\det \Lambda)\Gamma^a \theta' = \theta (\det \Gamma) \Lambda^a,$$

where Γ^a denotes the classical adjoint of Γ [6]. In this case, setting $X = P'\Gamma|\mathfrak{H}(\theta)$ and $Y = P(\det \Lambda)\Gamma^a|\mathfrak{H}(\theta')$, where P' and P are the projections from H_n onto $\mathfrak{H}(\theta')$ and $\mathfrak{H}(\theta)$, respectively, X and Y are quasi-affinities satisfying $XS(\theta) = S(\theta')X$ and $YS(\theta') = S(\theta)Y$ [1]; moreover, $XY = \varphi(S(\theta'))$ and $YX = \varphi(S(\theta))$, where $\varphi = (\det \Gamma)(\det \Lambda)$.

Proposition 2. *The mapping τ from* Lat $S(\theta)$ *to* Lat $S(\theta')$ *defined by $\tau\mathfrak{L} = \overline{X\mathfrak{L}}$ is a lattice isomorphism, and its inverse is given by $\tau^{-1}\mathfrak{L} = \overline{Y\mathfrak{L}}$.* Hyplat $S(\theta)$ *and* Hyplat $S(\theta')$ *are isomorphic. Similarly, the mapping τ' from* Lat $S(\theta)^*$ *to* Lat $S(\theta')^*$ *defined by $\tau'\mathfrak{L} = \overline{Y^*\mathfrak{L}}$ is a lattice isomorphism, and its inverse is given by $\tau'^{-1}\mathfrak{L} = \overline{X^*\mathfrak{L}}$.* Hyplat $S(\theta)^*$ *and* Hyplat $S(\theta')^*$ *are isomorphic.*

Proof. Let $\mathfrak{L} \neq 0$ belong to Lat $S(\theta)$. Then $\overline{X\mathfrak{L}} \neq 0$ belongs to Lat $S(\theta')$. Since $(X|\mathfrak{L})(S(\theta)|\mathfrak{L}) = (S(\theta')|\overline{X\mathfrak{L}})(X|\mathfrak{L})$, we have $S(\theta)|\mathfrak{L} \sim S(\theta')|\overline{X\mathfrak{L}}$ [1]. Similarly,

$S(\theta')|\overline{X\mathfrak{L}} \sim S(\theta)|\overline{YX\mathfrak{L}}$. Since $\overline{YX\mathfrak{L}}=\overline{\varphi(S(\theta))\mathfrak{L}}\subset\mathfrak{L}$, we have $\overline{YX\mathfrak{L}}=\mathfrak{L}$ (see [5] or [7]). Therefore, τ is one to one. Surjectivity is similarly shown. That τ preserve the lattice structure is obvious. That Hyplat $S(\theta)$ and Hyplat $S(\theta')$ are isomorphic was shown in [8]. Since

$$X^*Y^* = \tilde{\varphi}(S(\theta)^*) \quad \text{and} \quad Y^*X^* = \tilde{\varphi}(S(\theta')^*)$$

we can show the rest similarly.

Proposition 3. *If $S(\theta)$ and $S(\theta')$ are quasi-similar, then $S(\theta)$ has property* (Q) *if and only if so is $S(\theta')$.*

Proof. Assume that $S(\theta')$ has property (Q). For each A commuting with $S(\theta)$ set $B=XAY$. Then B commutes with $S(\theta')$ and $Y\,K(B)\subset K(A)$. Since

$$BX = XAYX = XA\varphi(S(\theta)) = X\varphi(S(\theta))\,A$$

we have $XK(A)\subset K(B)$. Thus, by Proposition 2, it follows that

$$K(A) \supset \overline{YK(B)} \supset \overline{YXK(A)} = K(A).$$

Therefore, we have $K(A)=\overline{YK(B)}$ and $\overline{XK(A)}=\overline{XYK(B)}=K(B)$. Thus

$$S(\theta)|K(A) = S(\theta)|\overline{YK(B)} \sim S(\theta')|K(B).$$

Similarly, we have

$$S(\theta)^*|K(A^*) = S(\theta)^*|\overline{X^*K(B^*)} \sim S(\theta')^*|K(B^*).$$

Since $S(\theta')|K(B)\sim(S(\theta')^*|K(B^*))^*$, it follows that

$$S(\theta)|K(A) \sim (S(\theta)|K(A^*))^*,$$

concluding the proof.

Proposition 4. *If A belongs to $(S(\theta))''$, then*

$$S(\theta)|K(A) \sim (S(\theta)^*|K(A^*))^*.$$

Proof. Let $\theta'=\psi_1\oplus\dots\oplus\psi_n$ be the normal form of θ. Then $B=XAY$ belongs to $(S(\theta'))''$ so $B=\eta(S(\theta'))$ for some η in H^∞ [9]. Setting $\psi_i'=\psi_i/(\eta\wedge\psi_i)$ we have

$$K(B) = (\psi_1'\oplus\dots\oplus\psi_n')H_n^2\ominus(\psi_1\oplus\dots\oplus\psi_n)H_n^2.$$

Thus $S(\theta')|K(B)\sim S(\eta\wedge\psi_1\oplus\dots\oplus\eta\wedge\psi_n)$. On the other hand,

$$\eta H_n^2\vee\theta' H_n^2 = (\eta\wedge\psi_1\oplus\dots\oplus\eta\wedge\psi_n)H_n^2$$

implies that

$$(S(\theta')^*|K(B^*))^* \sim S(\eta\wedge\psi_1\oplus\dots\oplus\eta\wedge\psi_n).$$

Since, by the proof of Proposition 3,

$$S(\theta)|K(A) \sim S(\theta')|K(B) \quad \text{and} \quad S(\theta)^*|K(A^*) \sim S(\theta')^*|K(B^*),$$

we have $S(\theta)|K(A)\sim(S(\theta)^*|K(A^*))^*$.

Corollary. *If $S(\theta)$ has a cyclic vector, then $S(\theta)$ has Property* (Q).

Proof. Since $(S(\theta))' = (S(\theta))''$ (see [3] and [4]), it is obvious.

To conclude we present a counterexample to the converse assertion of Corollary.

Example 2. Set $\psi_1(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ for $|\alpha|<1$ and $\psi_2(z) = \exp\left(\frac{z+1}{z-1}\right)$. Then $\theta = (\psi_1 \oplus \psi_1\psi_2)$ is a 2×2 matrix valued inner function, and $S(\theta)$ has no cyclic vector [4]. But it follows that

$$S(\theta) = S(1 \oplus \psi_1 \oplus \psi_1\psi_2) \sim S(\psi_1 \oplus \psi_1 \oplus \psi_2) = S(\psi_1 \oplus \psi_1) \oplus S(\psi_2).$$

Since $S(\psi_1 \oplus \psi_1)$ is a 2×2 diagonal matrix, by Proposition 1, $S(\psi_1 \oplus \psi_1)$ has property (Q). Since $S(\psi_2)$ has a cyclic vector, by Proposition 4, $S(\psi_2)$ has property (Q). Lemma 2 and relation

$$\sigma(S(\psi_1 \oplus \psi_1)) \cap \sigma(S(\psi_2)) = \emptyset \quad \text{(cf. [4])},$$

imply that $S(\psi_1 \oplus \psi_1) \oplus S(\psi_2)$ has property (Q). Thus, by Proposition 3, $S(\theta)$ also has property (Q).

Note. After this paper was written, the author received a preprint*) from Hari Bercovici, which covers a great part of the results of this paper. The author thanks to H. Bercovici and B. Sz.-Nagy.

References

[1] B. Moore—E. A. Nordgren, On quasi-equivalence and quasi-similarity, *Acta Sci. Math.*, **34** (1973), 311—316.

[2] H. Radjavi—P. Rosenthal, *Invariant Subspaces*, Springer Verlag (Berlin, 1973).

[3] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[4] B. Sz.-Nagy—C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland/Akadémiai Kiadó (Amsterdam/Budapest, 1970).

[5] B. Sz.-Nagy—C. Foiaş, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **61** (1970), 91—115.

[6] B. Sz.-Nagy—C. Foiaş, Jordan model for contractions of class C_0, *ibidem*, **36** (1974), 305—322.

[7] B. Sz.-Nagy—C. Foiaş, On injections, intertwining operators of class C_0, *ibidem*, **40** (1978), 163—167.

[8] M. Uchiyama, Hyperinvariant subspaces of operators of class $C_0(N)$, *ibidem*, **39** (1977), 179—184.

[9] M. Uchiyama, Double commutants of $C_{.0}$ contractions. II, *Proc. Amer. Math. Soc.*, to appear.

DEPARTMENT OF MATHEMATICS
FUKUOKA UNIVERSITY OF EDUCATION
MUNAKATA, FUKUOKA, 811—41
JAPAN

*) It has appeared in the meantime in this journal: H. Bercovici, C_0-Fredholm operators. I, *Acta Sci. Math.*, **41** (1979), 15—27. (The Editor)