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C_0-Fredholm operators. II

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SZ.-NAGY and FOIAŞ [16] proved that the operators T of class C_0 and of finite multiplicity have the following property:

(P) *any injection* $X \in \{T\}'$ *is a quasi-affiniti.*

In [3] we showed that property (P) also holds for weak contractions of class C_0. In sec. 4 of the present note we shall characterize the class $\mathscr{P}$ of C_0 operators having property (P).

UCHIYAMA [18] has shown that some quasi-affinities intertwining two contractions of class $C_0(N)$ induce isomorphisms between the corresponding lattices of hyper-invariant subspaces. This is not verified for arbitrary operators of class C_0 (cf. Example 2.10 below). For operators of the class $\mathscr{P}$ we show (cf. sec. 4) that any intertwining quasi-affinity induces isomorphisms between the corresponding lattices of invariant and hyper-invariant subspaces. However the other results proved in [18] for operators of the class $C_0(N)$ hold for arbitrary operators of class C_0; this is shown in sec. 2 of this note. In sec. 2 we also show which is the connection between the lattice of hyper-invariant subspaces of a C_0 operator and the corresponding lattice of the Jordan model.

In sec. 3 of this note we prove a continuity property of the Jordan model. This is useful when dealing with operators of class $\mathscr{P}$.

In [16] B. SZ.-NAGY and C. FOIAŞ made the conjecture that any operator T of class C_0 and of finite multiplicity has the property:

(Q) $T|\ker X$ *and* $T_{\ker X^*}$ *are quasisimilar for any* $X \in \{T\}'$.

This conjecture was infirmed in [3], Proposition 3.2, but was proved under the stronger assumption $X \in \{T\}''$ for any operator T of class C_0 (cf. also UCHIYAMA [19]).

Uchiyama began the study of the class of operators satisfying the property (Q) showing in particular that there exist operators of class $C_0(N)$ and multiplicity 2 wich have this property (cf. [19], Example 2). In sec. 5 of this note we characterise in terms of the Jordan model the class $\mathscr{Q}$ of C_0 operators having property (Q).

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In [3] the determinant function of a weak contraction was used for proving various index results. In sec. 6 of this note we extend the notion of inner function in order to find a substitute of the determinant function for the case of operators of class $\mathcal{P}$. In sec. 7 it is shown that the class of generalised inner functions (defined in sec. 6) naturally appears in the study of index problems. In sec. 8 we generalise the notion of C_0-fredholmness defined in [3]. All results of [3] are extended to this more general setting.

1. Notation and preliminaries

Let us recall that Lat (T) and Lat$_{\frac{1}{2}}$ (T) stand for the lattice of all invariant, respectively semi-invariant subspaces of the operator T. We shall denote by Hyp Lat (T) the lattice of hyper-invariant subspaces of T. If $\mathfrak{M}\in \mathrm{Lat}_{\frac{1}{2}}(T)$, $T_{\mathfrak{M}}$ stands for the compression of T to the subspace $\mathfrak{M}$ and $\mu_T(\mathfrak{M})$ stands for the multiplicity of $T_{\mathfrak{M}}$. The notations $T\prec T'$, $T\overset{i}{\prec} T'$ mean that T is a quasi-affine transform of T', respectively that T can be injected into T' (cf. e.g. [15]).

The following result will be frequently used in the sequel.

Lemma 1.1. *If T and T' are operators of class C_0 and $T\prec T'$ then T and T' are quasisimilar.*

Proof. Cf. [16], Theorem 1 or [4], Corollary 2.10.

Lemma 1.2. *Let $\{m_i\}_{i=0}^{\infty}$ be a sequence of pairwise relatively prime inner functions. If the operator $T=\bigoplus_{i=0}^{\infty} S(m_i)$ is of class C_0, the Jordan model of T is $S(m)$, $m=m_T$.*

Proof. If T is of class C_0 it follows that T is a weak contraction (cf. the proof of [6], Lemma 8.4) and from the assumption we easily infer $d_T=m_T$. The conclusion follows by [6], Theorem 8.7.

For two operators T and T' we denote by $\mathcal{I}(T', T)$ the set of intertwining operators

$$\mathcal{I}(T', T) = \{X\colon T'X = XT\}. \tag{1.1}$$

Let us recall (cf. [3], Definition 2.1) that $X\in\mathcal{I}(T', T)$ is a lattice-isomorphism if the mapping $\mathfrak{M}\mapsto(X\mathfrak{M})^-$ is an isomorphism of Lat (T) onto Lat (T').

Definition 1.3. An operator T has *property* (P) if any injection $A\in\{T\}'$ is a quasi-affinity.

We introduce the property (Q) as in [19]:

Definition 1.4. An operator T has *property* (Q) if for any $A\in\{T\}'$, $T|\ker A$ and $T_{\ker A^*}$ are quasisimilar.

Obviously (P) is implied by (Q).

Lemma 1.5. *The operator T of class C_0 acting on the Hilbert space $\mathfrak{H}$ has the property* (P) *if and only if there does not exist $\mathfrak{H}'\in\mathrm{Lat}(T)$, $\mathfrak{H}'\neq\mathfrak{H}$, such that T and $T|\mathfrak{H}'$ are quasisimilar.*

Proof. Let T be quasisimilar to $T|\mathfrak{H}'$, $\mathfrak{H}'\in\mathrm{Lat}(T)$ and let $X\colon \mathfrak{H}\to\mathfrak{H}'$ be a quasi-affinity such that $(T|\mathfrak{H}')X=XT$. Then $A=JX$ (where J denotes the inclusion of $\mathfrak{H}'$ into $\mathfrak{H}$) commutes with T and $\ker A=\{0\}$. If T has the property (P) we infer $\mathfrak{H}'=(A\mathfrak{H})^-=\mathfrak{H}$. Conversely, if $A\in\{T\}'$ is an injection, T and $T|(A\mathfrak{H})^-$ are quasi-similar by Lemma 1.1.

We shall denote by H_i^∞ the set of inner functions in H^∞. The set H_i^∞ is (pre)-ordered by the relation

$$(1.2)\qquad m\leqq m' \quad \text{if and only if} \quad |m(z)|\geqq|m'(z)|,\quad |z|<1.$$

Obviously $m\leqq m'$ if and only if m divides m'. The relations $m\leqq m'$ and $m'\leqq m$ imply that m and m' differ by a complex multiplicative constant of modulus one; we shall not distinguish between the functions m and m' in this case.

Let us recall (cf. [4]) that a Jordan operator is an operator of the form

$$(1.3)\qquad S(M)=\bigoplus_\alpha S(m_\alpha),\quad m_\alpha=M(\alpha)$$

where M is a model function, that is M is an inner function valued mapping defined on the class of ordinal numbers and

$$(1.4)\qquad \begin{cases} m_\alpha\leqq m_\beta & \text{whenever}\quad \alpha\geqq\beta;\\ m_\alpha=m_\beta & \text{whenever}\quad \bar\alpha=\bar\beta;\end{cases}$$

$$(1.5)\qquad m_\alpha=1\quad \text{for some } \alpha,$$

where $\bar\alpha$ denotes the cardinal number associated with the ordinal number α.

The Jordan model $S(M)$ is acting on a separable space if and only if $m_\omega=1$, where ω denotes the first transfinite ordinal number. In this case the Jordan operator is determined by the sequence $\{m_j\}_{j=0}^\infty$. If $m_n=1$ for some $n<\omega$, we shall also use the notation $S(m_0, m_1, \ldots, m_{n-1})$ for $S(M)$ (cf. [13]). If $S(M)$ is the Jordan model of the operator T of class C_0, we shall use the notation $m_\alpha[T]=M(\alpha)$ (cf. [4]).

2. Hyper-invariant subspaces of operators of class C_0

In this section we continue the study of hyper-invariant subspaces for the class C_0 begun by UCHIYAMA [18] (for the case of operators of class $C_0(N)$). The following Proposition extends [18], Theorem 3 and Corollaries 4 and 5 to the class of general Jordan operators.

Proposition 2.1. *Let $T=S(M)$ be a Jordan operator acting on the Hilbert space*

$$\mathfrak{H}(M) = \bigoplus_{\alpha} \mathfrak{H}(m_\alpha), \quad m_\alpha = M(\alpha). \tag{2.1}$$

(i) *A subspace $\mathfrak{M}\subset\mathfrak{H}(M)$ is hyper-invariant for T if and only if it is of the form*

$$\mathfrak{M} = \bigoplus_{\alpha} (m''_\alpha H^2 \ominus m_\alpha H^2), \quad m''_\alpha \leqq m_\alpha, \tag{2.2}$$

and the functions M' and M'' given by $M''(\alpha)=m''_\alpha$ and $M'(\alpha)=m_\alpha/m''_\alpha$ are model functions.

(ii) *If $\mathfrak{M}$ is a subspace of the form* (2.2) *then $T'=T|\mathfrak{M}$ is unitarily equivalent to $S(M')$ and $T''=T_{\mathfrak{M}^\perp}$ is unitarily equivalent to $S(M'')$. In particular,*

$$m_T = m_{T'}\, m_{T''} \tag{2.3}$$

if $\mathfrak{M}$ is hyper-invariant.

(iii) *If $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Hyp\,Lat}\,(T)$ are such that $T|\mathfrak{M}_1$ and $T|\mathfrak{M}_2$ are quasisimilar, we have $\mathfrak{M}_1=\mathfrak{M}_2$.*

Proof. We shall denote by $P_{\mathfrak{H}(m_\alpha)}$ the projection of H^2 onto $\mathfrak{H}(m_\alpha)$, by $\tilde{P}_{\mathfrak{H}(m_\alpha)}$ the projection of $\mathfrak{H}(M)$ onto $\mathfrak{H}(m_\alpha)$ and by J_α the inclusion of $\mathfrak{H}(m_\alpha)$ into $\mathfrak{H}(M)$. By the lifting Theorem (cf. [12], Theorem II.2.3) $\{T\}'$ is strongly generated by the operators $\psi(T)$, where $\psi\in H^\infty$, and the operators $A_{\beta\alpha}$ given by

$$\begin{cases} A_{\beta\alpha} = J_\beta P_{\mathfrak{H}(m_\beta)} \tilde{P}_{\mathfrak{H}(m_\alpha)} & \text{if} \quad \alpha \leqq \beta; \\ A_{\beta\alpha} = J_\beta (m_\beta/m_\alpha) \tilde{P}_{\mathfrak{H}(m_\alpha)} & \text{if} \quad \alpha > \beta, \end{cases} \tag{2.4}$$

and therefore the subspace $\mathfrak{M}\subset\mathfrak{H}(M)$ is a hyper-invariant subspace if and only it is invariant and $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ for each α and β. Let us assume that $\mathfrak{M}$ is hyper-invariant. Because $A_{\alpha\alpha}\mathfrak{M}=\tilde{P}_{\mathfrak{H}(m_\alpha)}\mathfrak{M}\subset\mathfrak{M}$ we have

$$\mathfrak{M} = \bigoplus_{\alpha} \mathfrak{M}_\alpha \tag{2.5}$$

where $\mathfrak{M}_\alpha\in\mathrm{Lat}\,(S(m_\alpha))$, say $\mathfrak{M}_\alpha=m''_\alpha H^2\ominus m_\alpha H^2$; therefore $\mathfrak{M}$ is of the form (2.2). Now let α and β be ordinal numbers such that $\alpha<\beta$; the conditions $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ and $A_{\beta\alpha}\mathfrak{M}\subset\mathfrak{M}$ are equivalent to $P_{\mathfrak{H}(m_\beta)}\mathfrak{M}_\alpha\subset\mathfrak{M}_\beta$ and $(m_\alpha/m_\beta)\mathfrak{M}_\beta\subset\mathfrak{M}_\alpha$. We infer $m''_\alpha\in m''_\beta H^2$ and $(m_\alpha/m_\beta)m''_\beta\in m''_\alpha H^2$ so that $m''_\alpha\geqq m''_\beta$ and $m_\alpha/m''_\alpha\geqq m_\beta/m''_\beta$, respectively; therefore M' and M'' are model functions.

Conversely, let $\mathfrak{M}$ be given by (2.2) and assume M' and M'' are model functions. It easily follows that $P_{\mathfrak{H}(m_\beta)}\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$ and $(m_\alpha/m_\beta)\mathfrak{M}_\beta \subset \mathfrak{M}_\alpha$ whenever $\alpha<\beta$. Thus $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ for each α and β so that $\mathfrak{M}\in \mathrm{Hyp\ Lat}\,(T)$ and (i) follows.

To prove (ii) let us remark that, if $\mathfrak{M}$ is given by (2.2), we have $T|\mathfrak{M}=\bigoplus_\alpha S(m_\alpha)|\mathfrak{M}_\alpha$ and $T_{\mathfrak{M}^\perp}=\bigoplus_\alpha S(m_\alpha)_{\mathfrak{M}_\alpha^\perp}$, where $\mathfrak{M}_\alpha=m''_\alpha H^2\ominus m_\alpha H^2$ and $S(m_\alpha)|\mathfrak{M}_\alpha$ is unitarily equivalent to $S(m'_\alpha)$ while $S(m_\alpha)_{\mathfrak{M}_\alpha^\perp}$ is unitarily equivalent to $S(m''_\alpha)$. If $\mathfrak{M}$ is hyper-invariant then $S(M')$ and $S(M'')$ are Jordan operators and therefore they are the Jordan models of T' and T'', respectively. In particular $m_{T'}=m'_0=m_0/m''_0=m_T/m_{T''}$ and (2.3) follows.

Finally, if $\mathfrak{M}_1, \mathfrak{M}_2\in \mathrm{Hyp\ Lat}\,(T)$ and $T|\mathfrak{M}_1$, $T|\mathfrak{M}_2$ are quasisimilar it follows that $T|\mathfrak{M}_1$ and $T|\mathfrak{M}_2$ have the same Jordan model. By (ii) $\mathfrak{M}_1$ is determined by the Jordan model of $T|\mathfrak{M}_1$. Therefore $\mathfrak{M}_1=\mathfrak{M}_2$ and (iii) follows.

Remark 2.2. The proof of Proposition 2.1 can be applied with minor changes to the description of Hyp Lat (T) when $T=\bigoplus_{j\in J} S(m_j)$ and $\{m_j\}_{j\in J}$ is a totally ordered subset of H_i^∞.

For further use let us note that the general form of a subspace $\mathfrak{M}\in \mathrm{Hyp\ Lat}\,(T)$ is

$$\mathfrak{M} = \bigoplus_{j\in J} (m''_j H^2\ominus m_j H^2),\quad m''_j \leq m_j \quad \text{for} \quad j\in J \tag{2.5}$$

where $m''_j\leq m''_k$ and $m_j/m''_j\leq m_k/m''_k$ whenever $m_j\leq m_k$.

Remark 2.3. Let the subspaces $\mathfrak{M}_j$ be given by

$$\mathfrak{M}_j = \bigoplus_\alpha (m_j(\alpha)H^2\ominus m_\alpha H^2),\quad j=1,2. \tag{2.6}$$

Then

$$\begin{cases} \mathfrak{M}_1\cap\mathfrak{M}_2 = \bigoplus_\alpha (m_1(\alpha)\vee m_2(\alpha)H^2\ominus m_\alpha H^2), \\ \mathfrak{M}_1\vee\mathfrak{M}_2 = \bigoplus_\alpha (m_1(\alpha)\wedge m_2(\alpha)H^2\ominus m_\alpha H^2); \end{cases} \tag{2.7}$$

in particular $\mathfrak{M}_1\subset\mathfrak{M}_2$ if and only if $m_1(\alpha)\geq m_2(\alpha)$ for each α.

We shall now characterize the Jordan operators having a totally ordered lattice of hyper-invariant subspaces thus extending [18], Theorem 6.

Proposition 2.4. *The lattice* Hyp Lat (T), $T=S(M)$, *is totally ordered if and only if one of the following situations* (i), (ii) *occurs:*

(i) $m_0=\left(\frac{z-a}{1-\bar a z}\right)^n$ *and* $m_\alpha\in\left\{1, \left(\frac{z-a}{1-\bar a z}\right)^{n-1}, \left(\frac{z-a}{1-\bar a z}\right)^n\right\}$ *for each* α, *with* $|a|<1$ *and a natural number* n.

(ii) $m_0=\exp\left(t\frac{z+a}{z-a}\right)$ *with* $|a|=1$, $t>0$, *and* $m_\alpha=m_0$ *whenever* $m_\alpha\neq 1$.

Proof. For two inner divisors m, m' of m_T we have $(\operatorname{ran} m(T))^- \subset (\operatorname{ran} m'(T))^-$ if and only if $m \geqq m'$ (cf. [4], Lemma 1.7). If Hyp Lat (T) is totally ordered it follows that the lattice of divisors of $m_T = m_0$ is also totally ordered. Therefore we have *either* $m_0 = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ ($|a|<1$, n a natural number) *or* $m_0 = \exp\left(t\frac{z+a}{z-a}\right)$ ($|a|=1$, $t>0$).

Let us consider the first situation. Then $m_\alpha = \left(\frac{z-a}{1-\bar{a}z}\right)^{n(\alpha)}$ where $n(\alpha)$ is a decreasing function of α. By Proposition 2.1 and Remark 2.3, Hyp Lat (T) is isomorphic to the lattice of natural number valued decreasing functions $k(\alpha)$ such that $k(\alpha) \leqq n(\alpha)$ and $n(\alpha)-k(\alpha)$ is also decreasing. Assume there exists α_0 such that $m = n(\alpha_0) \notin \{n, n-1, 0\}$ and define $k_1(\alpha) = \max\{n(\alpha)-1, 0\}$ and $k_2(\alpha) = \min\{m, n(\alpha)\}$. Then we have $k_1(0) = n-1 > k_2(0) = m$ and $k_1(\alpha_0) = m-1) < k_2(\alpha_0) = m$ so that k_1 and k_2 are incomparable. Thus we necessarily have $n(\alpha) \in \{n, n-1, 0\}$. Conversely, if $n(\alpha) \in \{n, n-1, 0\}$ for every α, let us take two functions k_1, k_2 of the type considered before. If k_1 and k_2 would not be comparable there would exist $\alpha < \beta$ such that $n(\beta) \neq 0$ and, by example, $k_1(\alpha) < k_2(\alpha)$, $k_1(\beta) > k_2(\beta)$. From the assumption it follows that $n(\alpha) \leqq n(\beta)+1$ so that $n(\beta) - k_2(\beta) \leqq n(\alpha) - k_2(\alpha) \leqq n(\beta) + 1 - k_2(\alpha)$ and therefore $k_2(\alpha) - 1 \leqq k_2(\beta)$. Now $k_1(\beta) \leqq k_1(\alpha) \leqq k_2(\alpha) - 1 \leqq k_2(\beta)$, a contradiction. This shows that Hyp Lat (T) is totally ordered in this case.

Now let us consider the case $m_0(z) = \exp\left(t\frac{z+a}{z-a}\right)$. Then $m_\alpha(z) = \exp\left(t(\alpha)\frac{z+a}{z-a}\right)$, where $t(\alpha)$ is a positive number valued decreasing function. Again by Proposition 2.1 and Remark 2.3, Hyp Lat (T) is isomorphic to the lattice of positive number valued decreasing functions $s(\alpha)$ such that $s(\alpha) \leqq t(\alpha)$ and $t(\alpha) - s(\alpha)$ is also decreasing. Assume there exists α_0 such that $t(\alpha_0) \notin \{t, 0\}$ and let us take $0 < \varepsilon < \min\{t(\alpha_0), t - t(\alpha_0)\}$. Then the functions $s_1(\alpha) = \max\{t(\alpha) - \varepsilon, 0\}$ and $s_2(\alpha) = \min\{t(\alpha), t(\alpha_0)\}$ are such that $s_1(0) = t(0) - \varepsilon > s_2(0) = t(\alpha_0)$ and

$$s_1(\alpha_0) = t(\alpha_0) - \varepsilon < s_2(\alpha_0) = t(\alpha_0);$$

therefore s_1 and s_2 are incomparable. Thus we necessarily have $t(\alpha) \in \{t, 0\}$ if Hyp Lat (T) is totally ordered.

Conversely, let us assume $t(\alpha) \in \{t, 0\}$ for each α. If s is a function of the type considered above and $t(\alpha) \neq 0$, we have $s(0) \geqq s(\alpha)$ and $t - s(0) \geqq t(\alpha) - s(\alpha) = t - s(\alpha)$ so that $s(\alpha) = s(0)$. Thus $s(\alpha) = s(0)$ if $t(\alpha) \neq 0$ and $s(\alpha) = 0$ if $t(\alpha) = 0$. It is obvious that Hyp Lat (T) is totally ordered in this case also. The Proposition is proved.

Uchiyama [18] has shown that two quasisimilar operators of class $C_0(N)$ have isomorphic lattices of hyper-invariant subspaces. This result is also verified, as we shall see in sec. 4, for operators of class C_0 having property (P). The same thing is not true for arbitrary operators of class C_0 (cf. Example 2.10). However we can find a connection between Hyp Lat (T) and Hyp Lat (S) if S is the Jordan model of the C_0 operator T. This allows us to extend [18], Corollaries 2 and 5 to arbitrary operators of class C_0.

Theorem 2.5. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $S=S(M)$ be the Jordan model of T. Let φ:* Hyp Lat $(S)\to$ Hyp Lat (T) *be defined by*

$$\varphi(\mathfrak{M}) = \bigvee_{X\in\mathscr{I}(T,S)} X\mathfrak{M} \tag{2.8}$$

and let ψ: Hyp Lat $(T)\to$ Hyp Lat (S),

ψ_*: Hyp Lat $(T^*)\to$ Hyp Lat (S^*)

be defined by analogous formulas.

(i) *There exist $Y\in\mathscr{I}(S,T)$ and $X\in\mathscr{I}(T,S)$ such that $\psi(\mathfrak{M})=(Y\mathfrak{M})^-=X^{-1}(\mathfrak{M})$, $\mathfrak{M}\in$* Hyp Lat (T). *In particular $S|\psi(\mathfrak{M})$ is unitarily equivalent to the Jordan model of $T|\mathfrak{M}$.*

(ii) $\psi\circ\varphi=\mathrm{id}_{\mathrm{Hyp\,Lat}(S)}$.

(iii) $\psi_*(\mathfrak{M}^\perp)=(\psi(\mathfrak{M}))^\perp$, $\mathfrak{M}\in$ Hyp Lat (T).

Proof. By [4], Theorem 3.4, there exists an almost-direct decomposition

$$\mathfrak{H}=\bigvee_\alpha \mathfrak{H}_\alpha, \quad \mathfrak{H}_\alpha\in \mathrm{Lat}(T), \tag{2.9}$$

such that $T|\mathfrak{H}_\alpha$ is quasisimilar to $S(m_\alpha)$ and $\mathfrak{H}_{\alpha+n}\perp\mathfrak{H}_{\beta+m}$ if α and β are different limit ordinals and m, $n<\omega$. If we put

$$\mathfrak{H}_\alpha^* = (\bigvee_{\beta\neq\alpha}\mathfrak{H}_\beta)^\perp\in\mathrm{Lat}(T^*) \tag{2.10}$$

we also have $\mathfrak{H}=\bigvee_\alpha \mathfrak{H}_\alpha^*$ by [4], Lemma 1.11; because

$$T_{\mathfrak{H}_\alpha^*}(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)=(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)(T|\mathfrak{H}_\alpha) \tag{2.11}$$

and obviously $P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha$ is a quasi-affinity, $T_{\mathfrak{H}_\alpha^*}$ is also quasisimilar to $S(m_\alpha)$. We choose quasi-affinities X_α: $\mathfrak{H}(m_\alpha)\to\mathfrak{H}_\alpha$, Y_α: $\mathfrak{H}_\alpha^*\to\mathfrak{H}(m_\alpha)$ such that $(T|\mathfrak{H}_\alpha)X_\alpha=X_\alpha S(m_\alpha)$ and $S(m_\alpha)Y_\alpha=Y_\alpha T_{\mathfrak{H}_\alpha^*}$ and moreover

$$\sum_{n<\omega}\|Y_{\alpha+n}\|\leqq 1, \quad \sum_{n<\omega}\|X_{\alpha+n}\|\leqq 1 \tag{2.12}$$

for each limit ordinal α. Then we can define quasi-affinities $X\in\mathcal{I}(T, S)$, $Y\in\mathcal{I}(S, T)$ by the formulas

$$Xh = \sum_{\alpha} X_\alpha h_\alpha,\ h = \bigoplus_{\alpha} h_\alpha \in \mathfrak{H}(M), \tag{2.13}$$

$$Yh = \bigoplus_{\alpha} J_\alpha Y_\alpha P_{\mathfrak{H}_\alpha^*} h, \quad h\in\mathfrak{H}.$$

Indeed, from (2.12) it follows that X and Y are bounded (of norm $\leqq 1$).

Let us remark that $Y_\alpha(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)X_\alpha\in\{S(m_\alpha)\}'$ is a quasi-affinity such that by Sarason's Theorem [10] we have

$$Y_\alpha(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)X_\alpha = u_\alpha(S(m_\alpha)), \quad u_\alpha\in H^\infty,\ u_\alpha\wedge m_\alpha = 1. \tag{2.14}$$

If $\mathfrak{M}\in\text{Hyp Lat}(S)$ we obviously have $\psi(\varphi(\mathfrak{M}))\subset\mathfrak{M}$. Now, let $\mathfrak{M}$ be given by (2.2) and denote $\mathfrak{M}_\alpha = m''_\alpha H^2\ominus m_\alpha H^2$. Then, by (2.14),

$$(YX\mathfrak{M})^- \supset (YX\mathfrak{M}_\alpha)^- = (YX_\alpha\mathfrak{M}_\alpha)^- = (Y_\alpha P_{\mathfrak{H}_\alpha^*}X_\alpha\mathfrak{M}_\alpha)^- =$$

$$= (u_\alpha(S(m_\alpha))\mathfrak{M}_\alpha)^- = \mathfrak{M}_\alpha \quad \text{and therefore} \quad \mathfrak{M} = (YX\mathfrak{M})^- \subset \psi(\varphi(\mathfrak{M}));$$

this proves (ii).

Let us consider the operators $R_{\beta\alpha}\in\{T\}'$ defined by

$$\begin{cases} R_{\beta\alpha} = X_\beta P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} & \text{if} \quad \alpha\leqq\beta, \\ R_{\beta\alpha} = X_\beta(m_\beta/m_\alpha)Y_\alpha P_{\mathfrak{H}_\alpha^*} & \text{if} \quad \alpha>\beta, \end{cases} \tag{2.15}$$

and let $A_{\beta\alpha}\in\{S\}'$ be defined by (2.4). Then, for $\alpha\leqq\beta$,

$$YR_{\beta\alpha} = J_\beta Y_\beta P_{\mathfrak{H}_\beta^*} X_\beta P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} =$$

$$= J_\beta u_\beta(S(m_\beta)) P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} =$$

$$= u_\beta(S) J_\beta P_{\mathfrak{H}(m_\beta)} \tilde{P}_{\mathfrak{H}(m_\alpha)} Y P_{\mathfrak{H}_\alpha^*} = u_\beta(S) A_{\beta\alpha} Y P_{\mathfrak{H}_\alpha^*}$$

and because $A_{\beta\alpha}YP_{(\mathfrak{H}_\alpha^*)^\perp}=0$ we obtain

$$YR_{\beta\alpha} = u_\beta(S)A_{\beta\alpha}Y \tag{2.16}$$

in this case. The relation (2.16) is proved analogously when $\alpha>\beta$. If $\mathfrak{N}\in\text{Hyp Lat}(T)$ and $\mathfrak{M}=(Y\mathfrak{N})^-$ we infer from (2.16) $u_\beta(S)A_{\beta\alpha}\mathfrak{M}\subset\mathfrak{M}$. Because $u_\alpha\wedge m_\alpha=1$ we infer by [3], Corollary 2.9, that $u_\alpha(S(m_\alpha))|(A_{\alpha\alpha}\mathfrak{M})^-$ is a quasi-affinity; therefore $\mathfrak{M}\supset(u_\alpha(S(m_\alpha))(A_{\alpha\alpha}\mathfrak{M})^-)^- = (A_{\alpha\alpha}\mathfrak{M})^- = (\tilde{P}_{\mathfrak{H}(m_\alpha)}\mathfrak{M})^-$. As in the proof of Proposition 2.1 it follows that $\mathfrak{M}=\bigoplus_\alpha\mathfrak{M}_\alpha$, $\mathfrak{M}_\alpha=m''_\alpha H^2\ominus m_\alpha H^2\in\text{Lat}(S(m_\alpha))$ and for $\alpha<\beta$, $u_\beta m''_\alpha\in m''_\beta H^2$ and $u_\alpha(m_\alpha/m_\beta)m''_\beta\in m''_\alpha H^2$. Because $u_\alpha\wedge m_\alpha=1$, $u_\beta\wedge m_\beta=1$ we also have $u_\alpha\wedge m''_\alpha=1$, $u_\beta\wedge m''_\beta=1$ so that from the preceding relations we infer $m''_\alpha\in m''_\beta H^2$, respectively $(m_\alpha/m_\beta)m''_\beta\in m''_\alpha H^2$. By Proposition 2.1 we proved

$$(Y\mathfrak{N})^-\in\text{Hyp Lat}(S) \quad \text{whenever} \quad \mathfrak{N}\in\text{Hyp Lat}(T). \tag{2.17}$$

Analogously we infer

(2.17)* $\quad (X^*\mathfrak{N})^- \in \operatorname{Hyp Lat}(S^*)$ whenever $\mathfrak{N} \in \operatorname{Hyp Lat}(T^*)$.

If $\mathfrak{N} \in \operatorname{Hyp Lat}(T)$ we have $X^*(\mathfrak{N}^\perp) \subset (Y\mathfrak{N})^\perp$. Indeed, if $h \in \mathfrak{N}$, $g \in \mathfrak{N}^\perp$, we have $(Yh, X^*g) = (XYh, g) = 0$ because $XYh \in \mathfrak{N}$. An analogous argument shows that

(2.18) $\quad \psi_*(\mathfrak{N}^\perp) \subset (\psi(\mathfrak{N}))^\perp, \quad \mathfrak{N} \in \operatorname{Hyp Lat}(T).$

In particular we have

$$T^*|\mathfrak{N}^\perp \prec S^*|(X^*\mathfrak{N}^\perp)^- \overset{i}{\prec} S^*|\psi_*(\mathfrak{N}^\perp) \overset{i}{\prec} S^*|(\psi(\mathfrak{N}))^\perp \overset{i}{\prec} S^*|(Y\mathfrak{N})^\perp.$$

Because $P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp$ has dense range and $S_{(Y\mathfrak{N})^\perp}(P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp) = (P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp) T_{\mathfrak{N}^\perp}$ it follows that $S^*|(Y\mathfrak{N})^\perp \overset{i}{\prec} T^*|\mathfrak{N}^\perp$. By [16], Theorem 1 (cf. also [4], Corollary 2.10) the operators $T^*|\mathfrak{N}^\perp$, $S^*|(X^*\mathfrak{N}^\perp)^-$, $S^*|\psi_*(\mathfrak{N}^\perp)$, $S^*|(\psi(\mathfrak{N}))^\perp$ and $S^*|(Y\mathfrak{N})^\perp$ are pairwise quasisimilar. Because S^* is also (unitarily equivalent to) a Jordan operator it follows by Proposition 2.1 (iii) that $(X^*\mathfrak{N}^\perp)^- = \psi_*(\mathfrak{N}^\perp) = (\psi(\mathfrak{N}))^\perp = (Y\mathfrak{N})^\perp$. This proves the assertions (i) and (iii) of the Theorem.

The following Corollary extends [18], Corollary 5, to arbitrary operators of class C_0.

Corollary 2.6. *If T is an operator of class C_0 on $\mathfrak{H}$ and* $T = \begin{bmatrix} T' & X \\ 0 & T'' \end{bmatrix}$ *is the triangularization of T with respect to the decomposition* $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, $\mathfrak{M} \in \operatorname{Hyp Lat}(T)$, *we have*

(2.19) $$m_T = m_{T'} m_{T''}.$$

Proof. If ψ is as in Theorem 2.5, T' is quasisimilar to $S|\psi(\mathfrak{M})$ and T'' is quasisimilar to $S_{(\psi(\mathfrak{M}))^\perp}$. The Corollary follows by Proposition 2.1 (ii).

Corollary 2.7. *Let T and T' be two quasisimilar operators of class C_0, let S be their Jordan model and let* η: Hyp Lat $(T) \to$ Hyp Lat (T'), ψ: Hyp Lat $(T) \to$ Hyp Lat (S), ψ': Hyp Lat $(T') \to$ Hyp Lat (S) *be defined by formulas analogous to* (2.8).

(i) $\psi' \circ \eta = \psi$; *in particular* $T|\mathfrak{M}$ *and* $T'|\eta(\mathfrak{M})$ *are quasisimilar for* $\mathfrak{M} \in \operatorname{Hyp Lat}(T)$.

(ii) *If* $\mathfrak{M} \in \operatorname{Hyp Lat}(T)$, $\mathfrak{M}' \in \operatorname{Hyp Lat}(T')$ *are such that* $T|\mathfrak{M}$ *and* $T'|\mathfrak{M}'$ *are quasisimilar, then* $T_{\mathfrak{M}^\perp}$ *and* $T'_{\mathfrak{M}'^\perp}$ *are also quasisimilar.*

Proof. The inclusion $(\psi' \circ \eta)(\mathfrak{M}) \subset \psi(\mathfrak{M})$ is obvious for $\mathfrak{M} \in \operatorname{Hyp Lat}(T)$. Then by Theorem 2.5 (i) we infer $T|\mathfrak{M} \overset{i}{\prec} S|(\psi' \circ \eta)(\mathfrak{M}) \overset{i}{\prec} S|\psi(\mathfrak{M}) \prec T|\mathfrak{M}$. By [16], Theorem 1, $T|\mathfrak{M}$, $S|(\psi' \circ \eta)(\mathfrak{M})$, $S|\psi(\mathfrak{M})$ are pairwise quasisimilar and the equality $\psi' \circ \eta = \psi$ follows by Proposition 2.1 (iii). Now it is obvious by Theorem 2.5 (i) that $T|\mathfrak{M}$ and $T'|\eta(\mathfrak{M})$ are both quasisimilar to $S|\psi(\mathfrak{M})$; (i) follows.

To prove (ii) we remark that, by Theorem 2.5 (i), $S|\psi(\mathfrak{M})$ and $S|\psi'(\mathfrak{M}')$ are quasisimilar and therefore $\psi(\mathfrak{M})=\psi'(\mathfrak{M}')$ by Proposition 2.1 (iii). Again by Theorem 2.5 it follows that $T_{\mathfrak{N}^\perp}$ and $T'_{\mathfrak{M}'^\perp}$ are both quasisimilar to $S_{\mathfrak{N}^\perp}$ where $\mathfrak{N}=\psi(\mathfrak{M})=\psi'(\mathfrak{M}')$. Corollary follows.

Corollary 2.8. *Let T, S, φ, ψ be as in Theorem 2.5 and let* φ_*: Hyp Lat $(S^*)\to$ Hyp Lat (T^*) *be defined by a formula analogous to* (2.8). *Among the spaces* $\mathfrak{N}\in$ Hyp Lat (T) *such that* $T|\mathfrak{N}$ *is quasisimilar to* $S|\mathfrak{M}$ *for a given* $\mathfrak{M}\in$ Hyp Lat (S), $\varphi(\mathfrak{M})$ *is the least one and* $(\varphi^*(\mathfrak{M}^\perp))^\perp$ *is the greatest one.*

Proof. If $T|\mathfrak{N}$ is quasisimilar to $S|\mathfrak{M}$ we have $\psi(\mathfrak{N})=\mathfrak{M}$ by Theorem 2.5 (i) and Proposition 2.1 (iii) and therefore $\varphi(\mathfrak{M})=\varphi(\psi(\mathfrak{N}))\subset\mathfrak{N}$. Now, by Corollary 2.7, $T|\mathfrak{N}$ and $S|\mathfrak{M}$ are quasisimilar if and only if $T_{\mathfrak{N}^\perp}$ and $S_{\mathfrak{M}^\perp}$ are quasisimilar. Because $\varphi_*(\mathfrak{M}^\perp)$ is the least hyper-invariant subspace of T^* such that $T_{\varphi_*(\mathfrak{M}^\perp)}$ and $S_{\mathfrak{M}^\perp}$ are quasisimilar, the last assertion of the Corollary follows.

Corollary 2.9. *Let T, S, ψ, φ, φ_* be as before. The following assertions are equivalent:*

(i) *φ is a bijection;*

(ii) *φ_* is a bijection;*

(iii) $\varphi(\mathfrak{M})^\perp=\varphi_*(\mathfrak{M}^\perp)$ *for* $\mathfrak{M}\in$ Hyp Lat (S);

(iv) *if* $\mathfrak{N}_1, \mathfrak{N}_2\in$ Hyp Lat (T) *and* $T|\mathfrak{N}_1$, $T|\mathfrak{N}_2$ *are quasisimilar, we have* $\mathfrak{N}_1=\mathfrak{N}_2$.

Proof. By Theorem 2.5 (ii) φ is a bijection if and only if ψ is one-to-one. By Theorem 2.5 (i) and Proposition 2.1 (iii) ψ is one-to-one if and only (iv) holds. Thus the equivalence (i)$\Leftrightarrow$(iv) is established.

By Theorem 2.5 (iii) we have $\psi_*(\mathfrak{M}^\perp)=\psi(\mathfrak{M})^\perp$ so that ψ is one-to-one if and only if ψ_* is one-to-one. This establishes the equivalence (i)$\Leftrightarrow$(ii).

$T|\varphi(\mathfrak{M})$ and $T|(\varphi_*(\mathfrak{M}^\perp))^\perp$ are both quasisimilar to $S|\mathfrak{M}$ so that $\varphi(\mathfrak{M})=(\varphi_*(\mathfrak{M}^\perp))^\perp$ if (iv) holds. Conversely, if (iii) holds and $T|\mathfrak{N}_1$, $T|\mathfrak{N}_2$ are quasisimilar, by the preceding Corollary we have $\varphi(\mathfrak{M})\subset\mathfrak{N}_j\subset(\varphi_*(\mathfrak{M}^\perp))^\perp=\varphi(\mathfrak{M})$, $j=1, 2$, where $\mathfrak{M}=\psi(\mathfrak{N}_1)=\psi(\mathfrak{N}_2)$. Thus $\mathfrak{N}_1=\mathfrak{N}_2=\varphi(\mathfrak{M})$ and the Corollary is proved.

Example 2.10. Let $S=S(m^2)^{(\aleph_0)}$ and $T=S\oplus S(m)$, where $m\in H_i^\infty$ and $S(m^2)^{(\aleph_0)}$ denotes the direct sum of $\aleph_0$ copies of $S(m^2)$. By [2], Corollary 1, S is the Jordan model of T. The subspaces $\ker m(T)$, $\operatorname{ran} m(T)$ are hyper-invariant for T and $T|\ker m(T)$, $T|\operatorname{ran} m(T)$ are both quasisimilar to $S(m)^{(\aleph_0)}$. By Corollary 2.9 it follows that in this case φ is not onto, ψ is not one-to-one.

If we take in particular $m(z)=z^2$ $(|z|<1)$ it is easily seen that card (Hyp Lat (T))$=9$ and card (Hyp Lat (S))$=5$. Thus Hyp Lat (T) and Hyp Lat (S) are not isomorphic. Moreover, one can verify, by the proof of Proposition 2.4, that Hyp Lat (T) is not totally ordered while Hyp Lat (S) is totally ordered.

3. A theorem on monotonic sequences of invariant subspaces

If T is an operator of class C_0 acting on $\mathfrak{H}$ and $\mathfrak{H}_j \in \operatorname{Lat}(T)$ are such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$, $j=0, 1, \dots$, and $\mathfrak{H} = \bigvee_{j \geqq 0} \mathfrak{H}_j$, it is clear that m_T is the least common inner multiple of the functions $m_{T|\mathfrak{H}_j}$, $j=0, 1, \dots$. The following Theorem shows that the same thing is verified for all the functions appearing in the Jordan model of T.

Theorem 3.1. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $\{\mathfrak{H}_j\}_{j=0}^{\infty} \subset \operatorname{Lat}(T)$ be such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$, $0 \leqq j < \infty$, and $\mathfrak{H} = \bigvee_{j \geqq 0} \mathfrak{H}_j$. Then*

$$m_\alpha[T] = \bigvee_{j \geqq 1} m_\alpha[T|\mathfrak{H}_j] \tag{3.1}$$

for each ordinal number α.

Proof. Because $T|\mathfrak{H}_j \overset{i}{\prec} T|\mathfrak{H}_{j+1} \overset{i}{\prec} T$ it follows that $m_\alpha[T|\mathfrak{H}_j] \leqq m_\alpha[T|\mathfrak{H}_{j+1}] \leqq \leqq m_\alpha[T]$ for each α (cf. [4], Corollary 2.9). Let us consider firstly the case $\alpha \geqq \omega$ and denote $m = \bigvee_{j \geqq 0} m_\alpha[T|\mathfrak{H}_j]$; then m divides $m_\alpha[T]$. Because $m_\alpha[T|\mathfrak{H}_j]$ divides m we have $\mu_{T|(m(T)\mathfrak{H}_j)^-} = \mu_{T|\mathfrak{H}_j}(m) \leqq \bar{\alpha}$ (cf. [4], Remark 2.12). Because obviously $(m(T)\mathfrak{H})^- = \bigvee_{j \geqq 0} m(T)\mathfrak{H}_j$ we infer $\mu_T(m) = \mu_{T|(m(T)\mathfrak{H})^-} \leqq \aleph_0 \cdot \bar{\alpha} = \bar{\alpha}$ and therefore $m_\alpha[T]$ divides m by [4], Definition 2.4. Thus $m_\alpha[T] = m$ and (3.1) is proved for $\alpha \geqq \omega$.

Now let us recall that by [4], Theorem 3.3, there exists an orthogonal decomposition

$$\mathfrak{H} = \bigoplus_{\alpha} \mathfrak{M}_\alpha, \quad \mathfrak{M}_\alpha \in \operatorname{Lat}(T), \tag{3.2}$$

such that $T|\mathfrak{M}_\alpha$ is quasisimilar to $\bigoplus_{j<\omega} S(m_{\alpha+j}[T])$ for each limit ordinal α. If we define $\mathfrak{K}_j = (P_{\mathfrak{M}_0} \mathfrak{H}_j)^-$ we obviously have $\mathfrak{M}_0 = \bigvee_{j \geqq 0} \mathfrak{K}_j$ and $T^*_{\mathfrak{K}_j} \overset{i}{\prec} T^*_{\mathfrak{H}_j}$ so that $T|\mathfrak{K}_j \overset{i}{\prec} T|\mathfrak{H}_j$ by [4], Corollary 2.9. Again by [4], Corollary 2.9 we infer $m_\alpha[T|\mathfrak{K}_j] \leqq \leqq m_\alpha[T|\mathfrak{H}_j]$, $\alpha < \omega$, and therefore it will be enough to prove the relation (3.1) for $\mathfrak{H} = \mathfrak{M}_0$ and $\mathfrak{H}_j = \mathfrak{K}_j$, that is for T acting on a separable space.

We may assume that T is a functional model, that is

$$\mathfrak{H} = \mathfrak{H}(\Theta) = H^2(\mathfrak{U}) \ominus \Theta H^2(\mathfrak{U}) \tag{3.3}$$

where $\mathfrak{U}$ is a separable Hilbert space, Θ is a two-sided inner function, $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{U}))$, and

$$Th = S(\Theta)h = P_{\mathfrak{H}(\Theta)} \chi h, \; \chi(z) = z, \quad h \in \mathfrak{H}(\Theta). \tag{3.4}$$

With each subspace $\mathfrak{H}_j$ we can associate by [12], Theorem VII.1.1 a factorisation

$$\Theta = \Theta_j^{(2)} \Theta_j^{(1)} \tag{3.5}$$

such that the functions $\Theta_j^{(1)}$ and $\Theta_j^{(2)}$ are two-sided inner,

$$\mathfrak{H}_j = \Theta_j^{(2)} H^2(\mathfrak{U}) \ominus \Theta H^2(\mathfrak{U}), \tag{3.6}$$

and $T|\mathfrak{H}_j$ is unitarily equivalent to $S(\Theta_j^{(1)})$. The inclusion $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$ is equivalent to $\Theta_j^{(2)} H^2(\mathfrak{U}) \subset \Theta_{j+1}^{(2)} H^2(\mathfrak{U})$ and therefore

$$\Theta_j^{(2)} = \Theta_{j+1}^{(2)} \Omega_j \quad \text{for some} \quad \Omega_j \in H_i^\infty(\mathscr{L}(\mathfrak{U})). \tag{3.7}$$

The condition $\mathfrak{H} = \bigvee_{j \geq 0} \mathfrak{H}_j$ is equivalent to $H^2(\mathfrak{U}) = \bigvee_{j \geq 0} \Theta_j^{(2)} H^2(\mathfrak{U})$. In particular, if $u \in \mathfrak{U}$, we have $\lim_{j \to \infty} \|u - P_{\Theta_j^{(2)} H^2(\mathfrak{U})} u\| = 0$. It is easily seen that $P_{\Theta_j^{(2)} H^2(\mathfrak{U})} u = \Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$. Indeed, it is enough to verify that the scalar product

$$\left(u - \Theta_j^{(2)}(z) \Theta_j^{(2)}(0)^* u,\ \Theta_j^{(2)}(z) z^n v\right)$$

vanishes for $v \in \mathfrak{U}$ and natural n; this is a simple computation. Thus we have $u = \lim_{j \to \infty} \Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$, $u \in \mathfrak{U}$. Because the functions $\Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$ are uniformly bounded we also have $u_1 \wedge u_2 \wedge \ldots \wedge u_n = \lim_{j \to \infty} (\Theta_j^{(2)})^{\wedge n} (\Theta_j^{(2)}(0)^*)^{\wedge n} (u_1 \wedge \ldots \wedge u_n)$, $u_1, u_2, \ldots, u_n \in \mathfrak{U}$, and therefore

$$\bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \supset \mathfrak{U}^{\wedge n}.$$

Because $\bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n})$ is invariant with respect to the unilateral shift on $H^2(\mathfrak{U}^{\wedge n})$ we necessarily have

$$H^2(\mathfrak{U}^{\wedge n}) = \bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}). \tag{3.8}$$

The subspaces

$$\mathfrak{H}_j^n = (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \ominus \Theta^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \tag{3.9}$$

are invariant with respect to $S(\Theta^{\wedge n})$ and because $\Theta^{\wedge n} = (\Theta_j^{(2)})^{\wedge n} (\Theta_j^{(1)})^{\wedge n}$ is a regular factorization, $S(\Theta^{\wedge n})|\mathfrak{H}_j^n$ is unitarily equivalent to $S((\Theta_j^{(1)})^{\wedge n})$. By (3.7) we have $(\Theta_j^{(2)})^{\wedge n} = (\Theta_{j+1}^{(2)})^{\wedge n} \Omega_j^{\wedge n}$ and therefore $\mathfrak{H}_j^n \subset \mathfrak{H}_{j+1}^n$ for $0 \leq j < \infty$. Finally, relation (3.8) shows that $\mathfrak{H}(\Theta^{\wedge n}) = \bigvee_{j \geq 0} \mathfrak{H}_j^n$ and therefore

$$m_0[S(\Theta^{\wedge n})] = \bigvee_{j \geq 0} m_0[S(\Theta^{\wedge n})|\mathfrak{H}_j^n]. \tag{3.10}$$

By [6], Corollary 3.3, and relation (2.5) we have $m_0[S(\Theta^{\wedge n})] = m_0[T] m_1[T] \ldots m_{n-1}[T]$ and $m_0[S(\Theta^{\wedge n})|\mathfrak{H}_j^n] = m_0[S((\Theta_j^{(1)})^{\wedge n})] = m_0[T|\mathfrak{H}_j] m_1[T|\mathfrak{H}_j] \ldots m_{n-1}[T|\mathfrak{H}_j]$. Let us put $m_k = \bigvee_{j \geq 0} m_k[T|\mathfrak{H}_j]$ for $k < \omega$; then m_k divides $m_k[T]$ and relation (3.10) shows that

$$m_0[T] m_1[T] \ldots m_{n-1}[T] = m_0 m_1 \ldots m_{n-1}, \quad 1 \leq n < \omega.$$

Therefore we have necessarily $m_k[T]=m_k$ and (3.1) is proved for $\alpha<\omega$. The Theorem follows.

Remark 3.2. The relation (3.1) is not verified if the sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty}$ is replaced by an arbitrary totally ordered family of invariant subspaces. Indeed, let us take a Jordan operator $T=S(M)$ such that $m_\Omega=1$, where Ω denotes the first uncountable ordinal number. The subspaces $\mathfrak{H}_\alpha=\bigoplus_{\beta<\alpha}\mathfrak{H}(m_\beta)$ for $\alpha<\Omega$ are separable and $\mathfrak{H}(M)=\bigvee_{\alpha<\Omega}\mathfrak{H}_\alpha$. The relation (3.1) is not verified in this case because $m_\omega[T|\mathfrak{H}_\alpha]=1$ while it is possible to have $m_\omega[T]\neq 1$. However the relation (3.1) is verified for $\alpha<\omega$ and for any totally ordered family $\{\mathfrak{H}_j\}_{j\in J}$ of invariant subspaces such that $\mathfrak{H}=\bigvee_{j\in J}\mathfrak{H}_j$. Indeed, if $\mathfrak{H}$ is separable we can select an increasing sequence $\{\mathfrak{H}_{j_n}\}_{n=0}^{\infty}$ such that $\mathfrak{H}=\bigvee_{n\geq 0}\mathfrak{H}_{j_n}$ and then apply Theorem 3.1. If $\mathfrak{H}$ is not separable, the proof of Theorem 3.1 shows how to reduce the problem of verifying (3.1) to the separable case.

Let us recall that for a contraction T of class C_0 and for a subspace $\mathfrak{M}\in\operatorname{Lat}_{\frac{1}{2}}(T)$ such that $T_{\mathfrak{M}}$ is a weak contraction, $d_T(\mathfrak{M})$ denotes the determinant function of $T_{\mathfrak{M}}$ (cf. [3], Definition 1.1).

Corollary 3.3. *Let T be a weak contraction of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\operatorname{Lat}(T)$, $0\leqq j<\infty$.*

(i) *If $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$ and $\bigvee_{j\geqq 0}\mathfrak{H}_j=\mathfrak{H}$, we have $d_T=\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$.*

(ii) *If $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}$ and $\bigcap_{j\geqq 0}\mathfrak{H}_j=\{0\}$, we have $\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)=1$.*

Proof. (i) Obviously $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ divides d_T. Now, $m_0[T|\mathfrak{H}_j]m_1[T|\mathfrak{H}_j]\dots m_n[T|\mathfrak{H}_j]$ divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ for every natural n; by Theorem 3.1 it follows that $m_0[T]m_1[T]\dots$ $\dots m_n[T]$ divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ and therefore d_T divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$.

(ii) Since T^* is also a weak contraction we infer by (i) $d_T=\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j^\perp)$. Because $d_T=d_T(\mathfrak{H}_j)d_T(\mathfrak{H}_j^\perp)$ (cf. [6], Proposition 8.2) we obtain

$$d_T=\Big(\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)\Big)\cdot\Big(\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j^\perp)\Big)=\Big(\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)\Big)\cdot d_T.$$

The Corollary follows.

Proposition 3.4. *Let T be an operator of class C_0 acting on the separable Hilbert space $\mathfrak{H}$. Then*

$$\bigwedge_{j<\omega}m_j[T]=1 \tag{3.11}$$

if and only if for any sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty} \subset \mathrm{Lat}\,(T)$ *such that* $\mathfrak{H}_j \supset \mathfrak{H}_{j+1}$ *and* $\bigcap_{j \geqq 0} \mathfrak{H}_j = \{0\}$, *we have*

(3.12) $$\bigwedge_{j \geqq 0} m_0[T|\mathfrak{H}_j] = 1.$$

Proof. As shown in the proof of [5], Theorem 1, there exists a decreasing sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty} \subset \mathrm{Lat}\,(T)$ such that $\bigcap_{j \geqq 0} \mathfrak{H}_j = \{0\}$ and $m_0[T|\mathfrak{H}_j] = m_j[T]$ so that (3.11) follows from (3.12).

Conversely, let us assume (3.11) holds. For any natural number k we have the decomposition

$$\mathfrak{H}_j = (m_k(T)\mathfrak{H}_j)^- \oplus \mathfrak{N}_j^k = \mathfrak{M}_j^k \oplus \mathfrak{N}_j^k, \quad m_k = m_k[T].$$

Because obviously $m_0[T_{\mathfrak{N}_j^k}]$ divides m_k, it follows by [12], Proposition III.6.1, that

(3.13) $$m_0[T|\mathfrak{H}_j] \text{ divides } m_0[T|\mathfrak{M}_j^k] \cdot m_k, \quad 0 \leqq j < \infty.$$

Now, $\mathfrak{M}_j^k \subset (m_k(T)\mathfrak{H})^-$ and $T|(m_k(T)\mathfrak{H})^-$ is an operator of finite multiplicity, in particular a weak contraction (cf. [6], Theorem 8.5). Because $\bigcap_{j \geqq 0} \mathfrak{M}_j^k \subset \bigcap_{j \geqq 0} \mathfrak{H}_j = \{0\}$ we infer by the preceding Corollary $\bigwedge_{j \geqq 0} d_T(\mathfrak{M}_j^k) = 1$, in particular $\bigwedge_{j \geqq 0} m_0[T|\mathfrak{M}_j^k] = 1$. By (3.13) $\bigwedge_{j \geqq 0} m_0[T|\mathfrak{H}_j]$ necessarily divides m_k and the relation (3.12) follows from the assumption. The Proposition is proved.

4. Operators of class C_0 having property (P)

In [16], Theorem 2, the operators of class C_0 and of finite multiplicity were shown to have property (P). In [3], Corollary 2.8 we extended this result to the class of weak contractions of class C_0. We are now going to characterise the class of C_0 operators having property (P).

Theorem 4.1. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$. Then T has property* (P) *if and only if*

(4.1) $$\bigwedge_{j < \omega} m_j[T] = 1.$$

In particular, if T has property (P), *$\mathfrak{H}$ is separable and T^* also has property* (P).

Proof. Let us assume (4.1) holds and denote $m_j = m_j[T]$. For each $j < \omega$ the subspace

(4.2) $$\mathfrak{H}_j = (m_j(T)\mathfrak{H})^-$$

is hyper-invariant for T and $\mu_T(\mathfrak{H}_j)<\infty$ (cf. [4], Remark 2.12). If $A\in\{T\}'$ is an injection then $A|\mathfrak{H}_j\in\{T|\mathfrak{H}_j\}'$ is also an injection and by [16], Theorem 2,

$$(4.3)\qquad (A\mathfrak{H})^-\supset(A\mathfrak{H}_j)^-=\mathfrak{H}_j.$$

We have $(\bigvee_{j<\omega}\mathfrak{H}_j)^\perp=\bigcap_{j<\omega}\ker m_j^\sim(T^*)=\mathfrak{H}^0$ and the minimal function m^0 of $T^*|\mathfrak{H}^0$ divides $m_j^\sim$, $j<\omega$. By the assumption we infer $m^0=1$ so that $\mathfrak{H}^0=\{0\}$ and therefore $\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$. From (4.3) we infer

$$(4.4)\qquad (A\mathfrak{H})^-\supset\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$$

that is, A is a quasi-affinity. The injection A being arbitrary it follows that T has property (P).

Conversely, let us assume that (4.1) does not hold. We claim that there exist an inner function m such that T and $T\oplus S(m)$ are quasisimilar. If $\mathfrak{H}$ is separable we may take $m=\bigwedge_{j<\omega}m_j[T]$ and apply [1], Lemma 3. If $\mathfrak{H}$ is nonseparable we may take $m=m_\omega[T]$. Then $T\oplus S(m)$ and T have the same Jordan model so that they are quasisimilar. Let us take a quasi-affinity X such that

$$(4.5)\qquad (T\oplus S(m))X=XT.$$

Let us put

$$(4.6)\qquad \mathfrak{M}=(X^*(\{0\}\oplus\mathfrak{H}(m)))^-,\quad \mathfrak{N}=\mathfrak{H}\ominus\mathfrak{M}.$$

Then $\mathfrak{M}\in\operatorname{Lat}(T^*)$ and $T^*|\mathfrak{M}$ is quasisimilar to $S(m)^*$. If P_1 and P_2 denote the orthogonal projections of $\mathfrak{H}\oplus\mathfrak{H}(m)$ onto $\mathfrak{H}$, $\mathfrak{H}(m)$, respectively, the operator

$$(4.7)\qquad Y=P_1X|\mathfrak{N}$$

satisfies the relation

$$(4.8)\qquad TY=Y(T|\mathfrak{N}).$$

We claim that Y is a quasi-affinity. We show firstly that ran Y^* is dense in $\mathfrak{N}$. Indeed, because $P_\mathfrak{N}X^*|\{0\}\oplus\mathfrak{H}(m)=0$ (by the definition (4.6) of $\mathfrak{M}$ and $\mathfrak{N}$), we have

$$(4.9)\qquad \operatorname{ran}Y^*=P_\mathfrak{N}X^*(\mathfrak{H}\oplus\{0\})=P_\mathfrak{N}X^*(\mathfrak{H}\oplus\mathfrak{H}(m))$$

which shows that

$$(4.10)\qquad (\operatorname{ran}Y^*)^-=(P_\mathfrak{N}(\operatorname{ran}X^*)^-)^-=P_\mathfrak{N}\mathfrak{H}=\mathfrak{N}.$$

Now let us show that $\ker Y^*=\{0\}$. To do this let us remark that the subspace

$$(4.11)\qquad \mathfrak{K}=\ker Y^*\oplus\mathfrak{H}(m)=\{u\in\mathfrak{H}\oplus\mathfrak{H}(m);\ X^*u\in\mathfrak{M}\}$$

is invariant with respect to $(T\oplus S(m))^*$, $(X^*\mathfrak{K})^-=\mathfrak{M}$ and $(T^*|\mathfrak{M})X^*=$ $=X^*(T\oplus S(m))^*|\mathfrak{K}$ so that $T^*|\mathfrak{M}$ and $(T\oplus S(m))^*|\mathfrak{K}$ are quasisimilar. By the remark following relation (4.6), $(T\oplus S(m))^*|\mathfrak{K}$ is quasisimilar to $S(m)^*$. But

$(T\oplus S(m))^*|\{0\}\oplus\mathfrak{H}(m)$ is unitarily equivalent to $S(m)^*$ so that $\mathfrak{K}=\{0\}\oplus\mathfrak{H}(m)$ by [14], Theorem 2, and the injectivity of Y^* is proved. Relation (4.8) and Lemma 1.1 show that T and $T|\mathfrak{N}$ are quasisimilar. Because $\mathfrak{M}\neq\{0\}$, we have $\mathfrak{N}\neq\mathfrak{H}$ so that T does not have property (P) by Lemma 1.5.

Theorem is proved.

Corollary 4.2. *An operator T of class C_0 has property* (P) *if and only if there does not exist T' of class C_0 on a nontrivial Hilbert space such that T and $T\oplus T'$ are quasisimilar.*

Proof. Let T and $T\oplus T'$ be quasisimilar. Since T' acts on a nontrivial space, there exists a nonconstant inner function m such that $T\oplus S(m)\overset{i}{\prec} T$. Because obviously $T\overset{i}{\prec} T\oplus S(m)$, $T\oplus S(m)$ and T are quasisimilar by [16], Theorem 1. By the proof of Theorem 4.1 it follows that T does not have the property (P). The converse assertion of the Corollary follows from the proof of Theorem 4.1.

Corollary 4.3. *If T and T' are two quasisimilar operators of class C_0, then T has property* (P) *if and only if T' has property* (P).

Proof. Theorem 4.1 exprimes the property (P) in terms of the Jordan model so that the Corollary is obvious.

Proposition 4.4. *Let $T=\begin{bmatrix}T' & X\\ 0 & T''\end{bmatrix}$ be the triangularization of the operator T of class C_0 with respect to the decomposition $\mathfrak{H}=\mathfrak{H}'\oplus\mathfrak{H}''$, $\mathfrak{H}'\in\operatorname{Lat}(T)$. Then T has property* (P) *if and only if T' and T'' have property* (P).

Proof. Let $S(M)$, $S(M')$, $S(M'')$ be the Jordan models of T, T', T'', respectively. Let us assume that T has property (P). Because $S(M')\overset{i}{\prec} S(M)$ it follows that m'_α divides m_α for each α (cf. [4], Corollary 2.9), therefore by Theorem 4.1 we have $\bigwedge_{j<\omega} m'_j=1$ and T' has property (P). Analogously T''^* has property (P) because T^* has property (P) and it follows by Theorem 4.1 that T'' also has property (P).

Conversely, let us assume that T' and T'' have property (P) so that

$$\bigwedge_{j<\omega} m'_j=\bigwedge_{j<\omega} m''_j=1. \tag{4.12}$$

We consider firstly the case $\mu_{T'}<\infty$. In this case the space

$$\mathfrak{H}_j=(m''_j(T)\mathfrak{H})^-\in\operatorname{Hyp\,Lat}(T),\quad j<\omega, \tag{4.13}$$

is contained in $\mathfrak{H}'\oplus(m''_j(T'')\mathfrak{H}'')^-$ so that $\mu_T(\mathfrak{H}_j)<\infty$ and by [16], Theorem 2, $T|\mathfrak{H}_j$ has property (P). Because $\bigwedge_{j<\omega} m''_j=1$ we have $\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$ (cf. the proof of

Theorem 4.1) and the first part of the proof of Theorem 4.1 shows that T has property (P).

Considering the operator T^* instead of T, it follows that T has property (P) in the case $\mu_{T''}<\infty$ also.

We are now considering the general case $\mu_{T'}=\mu_{T''}=\aleph_0$. Let us define the hyperinvariant subspaces $\mathfrak{H}_j$ by (4.13). The operator $T|\mathfrak{H}'\oplus(m_j''(T'')\mathfrak{H}'')^-$ has property (P) because $\mu_{T''|(m_j''(T'')\mathfrak{H}'')^-}<\infty$ and from the first part of the proof of our Proposition it follows that $T|\mathfrak{H}_j$ also has the property (P). Because $\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$ we infer as in the first part of the proof of Theorem 4.1 that T has property (P). The proposition is proved.

Corollary 4.5. *If T is an operator of class C_0 having property* (P) *and* $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$, *then $T_{\mathfrak{M}}$ also has property* (P).

Proof. We have $\mathfrak{M}=\mathfrak{U}\ominus\mathfrak{V}$, $\mathfrak{U}, \mathfrak{V}\in\mathrm{Lat}(T)$ and $T|\mathfrak{U}$ has property (P) by Proposition 4.4. Again by Proposition 4.4 and Theorem 4.1 it follows that $T_{\mathfrak{M}}$ has property (P) because $T_{\mathfrak{M}}^*=(T|\mathfrak{U})^*|\mathfrak{M}$.

Proposition 4.6. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let* $\mathfrak{H}_j\in\mathrm{Lat}(T)$ *be such that* $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $j<\omega$, $\mathfrak{H}_0=\{0\}$ *and* $\mathfrak{H}=\bigvee_{j<\omega}\mathfrak{H}_j$. *Then T has property* (P) *if and only if* $T_{\mathfrak{K}_j}$, $\mathfrak{K}_j=\mathfrak{H}_{j+1}\ominus\mathfrak{H}_j$ $(j<\omega)$ *have property* (P) *and*

$$\bigwedge_{j<\omega} m_0[T_{\mathfrak{H}_j^\perp}]=1. \tag{4.14}$$

Proof. If T has property (P) then $T_{\mathfrak{K}_j}$ have property (P) by Corollary 4.5. By Theorem 4.1 and Proposition 3.4 we infer the necessity of (4.14).

Conversely let us assume that $T_{\mathfrak{K}_j}$ have property (P) and (4.14) holds; let us put $m_j=m_0[T_{\mathfrak{H}_j^\perp}]$. If we define

$$\mathfrak{L}_j=(m_j(T)\mathfrak{H})^-\in\mathrm{Hyp\,Lat}(T) \tag{4.15}$$

then, as in the proof of Theorem 4.1, from (4.14) we infer $\bigvee_{j<\omega}\mathfrak{L}_j=\mathfrak{H}$ and the first part of the proof of Theorem 4.1 shows us that it is enough to prove that $T|\mathfrak{L}_j$ have property (P). Now, obviously $\mathfrak{L}_j\subset\mathfrak{H}_j$ so that by Corollary 4.5 we have only to show that $T|\mathfrak{H}_j$ have property (P). This easily proved inductively since the triangularization of $T|\mathfrak{H}_{j+1}$ with respect to the decomposition $\mathfrak{H}_{j+1}=\mathfrak{H}_j\oplus\mathfrak{K}_j$ is of the form $T|\mathfrak{H}_{j+1}=\begin{bmatrix}T|\mathfrak{H}_j & X_j\\ 0 & T_{\mathfrak{K}_j}\end{bmatrix}$. The Proposition follows.

Corollary 4.7. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let* $\mathfrak{H}_j\in\mathrm{Lat}(T)$ *be such that* $\mathfrak{H}_{j+1}\subset\mathfrak{H}_j$, $j<\omega$, $\mathfrak{H}_0=\mathfrak{H}$ *and* $\bigcap_{j<\omega}\mathfrak{H}_j=\{0\}$. *Then T has property* (P) *if and only if* $T_{\mathfrak{K}_j}$, $\mathfrak{K}_j=\mathfrak{H}_j\ominus\mathfrak{H}_{j+1}$ $(j<\omega)$, *have property* (P) *and*

$$\bigwedge_{j<\omega} m_0[T|\mathfrak{H}_j]=1. \tag{4.16}$$

Proof. By Theorem 4.1, T has property (P) if and only if T^* has property (P). Therefore we have only to replace T by T^*, $\mathfrak{H}_j$ by $\mathfrak{H}_j^\perp$ and apply the preceding Proposition.

We are now going to extend [18], Theorem 1, and [3], Corollaries 2.4, 2.8 and 2.9 to the case of C_0 contractions having property (P).

Proposition 4.8. *Let T and T' be two quasisimilar operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and having property* (P). *Let us define*

$$\xi\colon \mathrm{Hyp\,Lat}(T) \to \mathrm{Hyp\,Lat}(T') \quad and \quad \eta\colon \mathrm{Hyp\,Lat}(T') \to \mathrm{Hyp\,Lat}(T)$$

by

$$\xi(\mathfrak{M}) = \bigvee_{X\in\mathscr{I}(T',T)} X\mathfrak{M}, \quad \eta(\mathfrak{N}) = \bigvee_{Y\in\mathscr{I}(T,T')} Y\mathfrak{N}. \tag{4.17}$$

(i) *Each injection $A\in\mathscr{I}(T',T)$ is a lattice-isomorphism.*

(ii) $\xi(\mathfrak{M})=(A\mathfrak{M})^-=B^{-1}\mathfrak{M}$, $\mathfrak{M}\in\mathrm{Hyp\,Lat}(T)$, *for any quasi-affinities $A\in\mathscr{I}(T',T)$, $B\in\mathscr{I}(T,T')$.*

(iii) *ξ is bijective and $\eta=\xi^{-1}$.*

Proof. (i) If $A\in\mathscr{I}(T',T)$ is an injection, T is quasisimilar to $T'|(A\mathfrak{H})^-$ so that T' and $T'|(A\mathfrak{H})^-$ are quasisimilar. Now T' has property (P) so that $(A\mathfrak{H})^-=\mathfrak{H}'$ by Lemma 1.5 and A is a quasi-affinity.

Let $\mathfrak{K}', \mathfrak{K}''\in\mathrm{Lat}(T)$ be such that $(A\mathfrak{K}')^-=(A\mathfrak{K}'')^-=\mathfrak{K}^*$; then we also have $(A\mathfrak{K})^-=\mathfrak{K}^*$ with $\mathfrak{K}=\mathfrak{K}'\vee\mathfrak{K}''$. The operators $T|\mathfrak{K}'$, $T|\mathfrak{K}''$ and $T|\mathfrak{K}$ are quasisimilar to $T'|\mathfrak{K}^*$. By Proposition 4.4 $T|\mathfrak{K}$ has the property (P) and therefore $\mathfrak{K}'=\mathfrak{K}''=\mathfrak{K}$ by Lemma 1.5. Thus we have shown that the mapping $\mathfrak{K}\to(A\mathfrak{K})^-$ is one-to-one on $\mathrm{Lat}(T)$. Because we have shown that A is a quasi-affinity, the same argument can be applied to T'^*, T^* and A^* thus proving, via [3], Lemma 1.4, that A is a lattice-isomorphism.

(ii) Let us take any quasi-affinities $A\in\mathscr{I}(T',T)$ and $B\in\mathscr{I}(T,T')$; by (i) A and B are lattice isomorphisms. For each $\mathfrak{M}\in\mathrm{Hyp\,Lat}(T)$, $BA\in\{T\}'$ so that $BA\mathfrak{M}\subset\mathfrak{M}$ and since $T|\mathfrak{M}$ also has property (P) by Proposition 4.4 and $BA|\mathfrak{M}\in\{T|\mathfrak{M}\}'$ is one-to-one, we infer by (i) $(BA\mathfrak{M})^-=\mathfrak{M}$. Now, B is a lattice-isomorphism so that we infer

$$B^{-1}(\mathfrak{M}) = (A\mathfrak{M})^-. \tag{4.18}$$

If $X\in\mathscr{I}(T',T)$, we have $BX\in\{T\}'$ so that $BX\mathfrak{M}\subset\mathfrak{M}$ and by (4.18) $X\mathfrak{M}\subset\subset B^{-1}(\mathfrak{M})=(A\mathfrak{M})^-$; it follows that $\xi(\mathfrak{M})\subset(A\mathfrak{M})^-$. Because the inclusion $(A\mathfrak{M})^-\subset\subset\xi(\mathfrak{M})$ is obvious, (ii) is proved.

(iii) If $A\in\mathscr{I}(T',T)$, $B\in\mathscr{I}(T,T')$ are quasi-affinities we have by (ii) $(BA\mathfrak{M})^-=\mathfrak{M}$ and $(AB\mathfrak{N})^-=\mathfrak{N}$ for any $\mathfrak{M}\in\mathrm{Hyp\,Lat}(T)$, $\mathfrak{N}\in\mathrm{Hyp\,Lat}(T')$. Because, again by (ii), $\xi(\mathfrak{M})=(A\mathfrak{M})^-$ and $\eta(\mathfrak{N})=(B\mathfrak{N})^-$, (iii) follows.

The Proposition is proved.

Corollary 4.9. *Let T, S, φ, ψ be as in Theorem 2.5. If T has property* (P), *φ is a bijection and $\psi=\varphi^{-1}$.*

Proof. Obviously follows from the preceding Proposition.

The following result extends [3], Proposition 2.3, to the class of C_0 operators having property (P).

Proposition 4.10. *Let T, T', T'' be operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, $\mathfrak{H}''$, respectively, and let $A\in\mathcal{I}(T,T')$, $B\in\mathcal{I}(T,T'')$ be such that $A\mathfrak{H}'\subset(B\mathfrak{H}'')^-$. If T has property* (P) *then*

(i) $\left(A^{-1}(B\mathfrak{H}'')\right)^-=\mathfrak{H}'$ *and* (ii) $(A\mathfrak{H}'\cap B\mathfrak{H}'')^-\supset A\mathfrak{H}'$.

Proof. Because (ii) easily follows from (i), we have only to prove (i). We may assume that A is one-to-one, B is a quasi-affinity and T has the property (P). Indeed, we have only to replace T, T', T'', A, B, by $T|(B\mathfrak{H}'')^-$, $T'_{(\ker A)^\perp}$, $T''_{(\ker B)^\perp}$, $A|(\ker A)^\perp$, $B|(\ker B)^\perp$, respectively. Now the operator T'' has property (P) being quasisimilar to T (cf. Corollary 4.3) and T' has property (P) being quasisimilar to $T|(A\mathfrak{H}')^-$ (cf. Proposition 4.4). Then the operators $T'\oplus T''$ and $T'\oplus T$ are quasisimilar and have property (P) by Proposition 4.4. The operator $X\colon \mathfrak{H}'\oplus\mathfrak{H}''\to\mathfrak{H}'\oplus\mathfrak{H}$ given by

$$X(h'\oplus h'')=h'\oplus(Ah'-Bh''),\quad h'\oplus h''\in\mathfrak{H}'\oplus\mathfrak{H}'', \tag{4.19}$$

is an injection. Indeed, $X(h'\oplus h'')=0$ implies $h'=0$ and $Bh''=Ah'=0$, thus $h''=0$ by the injectivity of B. Because $X\in\mathcal{I}(T'\oplus T, T'\oplus T'')$ it follows by Proposition 4.8(i) that X is a lattice-isomorphism. In particular $X(X^{-1}(\mathfrak{H}'\oplus\{0\}))$ is dense in $\mathfrak{H}'\oplus\{0\}$. But

$$X(X^{-1}(\mathfrak{H}'\oplus\{0\}))=\{h'\oplus 0;\ h'\in\mathfrak{H}' \text{ and } Ah'=Bh'' \text{ for some } h''\}$$

so that (i) follows and the Proposition is proved.

Corollary 4.11. *Let T, T', T'', A and B be as in the preceding Proposition. If T' is multiplicity-free then $A^{-1}(B\mathfrak{H}'')$ contains cyclic vectors of T'.*

Proof. Let us denote by P the orthogonal projection of $\mathfrak{H}'\oplus\mathfrak{H}$ onto $\mathfrak{H}'$. From Proposition 4.10 it follows that $A^{-1}(B\mathfrak{H}'')=PX(X^{-1}(\mathfrak{H}'\oplus\{0\}))$ is dense in $\mathfrak{H}'$ (where X is defined by relation (4.19)). Let us denote $\mathfrak{H}_0=(X^{-1}(\mathfrak{H}'\oplus\{0\}))\ominus$ $\ominus\ker(X|X^{-1}(\mathfrak{H}'\oplus\{0\}))\in\mathrm{Lat}_{\frac{1}{2}}(T'\oplus T'')$. Then we have

$$T'(PX|\mathfrak{H}_0)=(PX|\mathfrak{H}_0)\,(T'\oplus T'')_{\mathfrak{H}_0}$$

and by Lemma 1.1 T' and $(T'\oplus T'')_{\mathfrak{H}_0}$ are quasisimilar; in particular $(T'\oplus T'')_{\mathfrak{H}_0}$

is also multiplicity-free. If h_0 is any cyclic vector of $(T' \oplus T'')_{\mathfrak{H}_0}$ then $PXh_0 \in A^{-1}(B\mathfrak{H}'')$ is a cyclic vector of T'. Corollary follows.

Finally let us remark that the result of [4] concerning the quasi-direct decomposition of the space on which a weak contraction acts can be extended, via Proposition 4.8 (i), to the class of C_0 operators having property (P).

Corollary 4.12. *Let T be an operator of class C_0 having property* (P) *and acting on the (necessarily separable) Hilbert space $\mathfrak{H}$ and let $\bigoplus_{j<\omega} S(m_j)$ be the Jordan model of T. There exists a decomposition of $\mathfrak{H}$*

$$\mathfrak{H} = \bigvee_{j<\omega} \mathfrak{H}_j \tag{4.19}$$

into a quasi-direct sum of invariant subspaces of T such that $T|\mathfrak{H}_j$ is quasisimilar to $S(m_j)$.

Proof. Cf. the proof of [4], Proposition 3.5.

5. Operators of class C_0 having property (*Q*)

The following Lemma extends [19], Proposition 3, to the entire class of C_0 operators.

Lemma 5.1. *Let T and T' be two quasisimilar operators of class C_0. Then T has property* (Q) *if and only if T' has property* (Q).

Proof. Because (Q) implies (P), by Corollary 4.3 it is enough to prove the Lemma for T and T' having the property (P). Let $X \in \mathscr{I}(T, T')$, $Y \in \mathscr{I}(T', T)$ be two quasi-affinities. By Proposition 4.8 (i) X and Y are lattice-isomorphisms. Let us take $A \in \{T'\}'$; then $B = XAY \in \{T\}'$. Obviously $\ker B = Y^{-1}(\ker A)$, X being an injection. Because Y is a lattice-isomorphism we have $(Y(\ker B))^- = \ker A$ so that $Y|\ker B$ is a quasi-affinity from $\ker B$ into $\ker A$. Because

$$Y|\ker B \in \mathscr{I}(T'|\ker A, T|\ker B)$$

it follows by Lemma 1.1 that $T|\ker B$ and $T'|\ker A$ are quasisimilar. Analogously $T_{\ker B^*}$ and $T'_{\ker A^*}$ are quasisimilar. If T has the property (Q), the operators $T|\ker B$ and $T_{\ker B^*}$ are quasisimilar and it follows from the preceding considerations that $T'|\ker A$ and $T'_{\ker A^*}$ are quasisimilar. Since $A \in \{T'\}'$ is arbitrary it follows that T' has the property (Q). The Lemma is proved.

Lemma 5.2. *For any inner function m and natural number k the operator $T = S(\underbrace{m, m, \ldots, m}_{k \text{ times}})$ has the property* (Q).

Proof. By the lifting Theorem (cf. [12], Theorem II.2.3) any operator $X\in\{T\}'$ is given by

$$(5.1)\qquad Xh = P_{\mathfrak{H}}Ah,\ h\in\mathfrak{H} = \underbrace{\mathfrak{H}(m)\oplus\mathfrak{H}(m)\oplus\ldots\oplus\mathfrak{H}(m)}_{k\text{ times}}$$

where $A=[a_{ij}]_{1\le i,j\le k}$ is an arbitrary matrix over H^∞. As shown by NORDGREN [9] (cf. also SZŰCS [17] and SZ.-NAGY [11]) there exist matrices B, U, V which determine by formulas analogous to (5.1) operators Y, K, L in $\{T\}'$ such that

$$(5.2)\qquad (\det U)(\det V)\wedge m = 1;$$

$$(5.3)\qquad AU = VB,$$

$$(5.4)\qquad B=[b_{ij}]_{1\le i,j\le k},\quad b_{ij}=0 \quad \text{for} \quad i\ne j.$$

From (5.2) we infer as in [8] that K and L are quasi-affinities and therefore lattice-isomorphisms by Proposition 4.8 (i). From (5.3) we infer

$$(5.5)\qquad XK = LY$$

so that $K(\ker Y)\subset\ker X$ and $K^{-1}(\ker X)\subset\ker Y$; because K is a lattice-isomorphism it follows that $(K(\ker Y))^-=\ker X$ and therefore $T|\ker X$ and $T|\ker Y$ are quasisimilar. Analogously $T_{\ker X*}$ and $T_{\ker Y*}$ are quasisimilar. We have $Y=\bigoplus_{j=1}^{k} b_{jj}(S(m))$ and $\ker Y=\bigoplus_{j=1}^{k}(\ker b_{jj}(S(m)))$ so that $T|\ker Y$ is unitarily equivalent (cf. [15], p. 315) to $\bigoplus_{j=1}^{k} S(m_j)$, where $m_j=m\wedge b_{jj}$. Analogously we can show that $T_{\ker Y*}$ is unitarily equivalent to $\bigoplus_{j=1}^{k} S(m_j)$. We have shown $T|\ker Y$ and $T_{\ker Y*}$ are unitarily equivalent; we infer that $T|\ker X$ and $T_{\ker X*}$ are quasisimilar. Because X is arbitrary in $\{T\}'$, the Lemma follows.

Lemma 5.3. *If $T\oplus S$ has the property* (Q) *then T and S also have the property* (Q).

Proof. It is obvious since $\{T\oplus S\}'\supset\{T\}'\oplus I\cup I\oplus\{S\}'$.

The following Theorem characterizes the class of C_0 operators having the property (Q) in terms of the Jordan model.

Theorem 5.4. *An operator T of class C_0 has property* (Q) *if and only if*

(i) $\bigwedge_{j<\omega} m_j = 1,\quad m_j = m_j[T],$ and

(ii) *the functions $m_0/m_1, m_1/m_2, \ldots$ are pairwise relatively prime.*

In particular, if T has property (Q), *then T acts on a separable Hilbert space and T^* also has property* (Q).

Proof. Let T have property (Q). Then T also has property (P) so that the necessity of (i) follows by Theorem 4.1. By Lemma 5.1 the Jordan model $S(M)$ of T also has the property (Q) so that $S^j = S(m_j) \oplus S(m_{j+1}), j < \omega$, must have property (Q) by Lemma 5.3. The matrix

(5.6) $$A = \begin{bmatrix} 0 & m_j/m_{j+1} \\ 0 & 0 \end{bmatrix}$$

determines an operator $X \in \{S^j\}'$ by the formula

(5.7) $$Xh = P_{\mathfrak{H}_j} Ah, \quad h \in \mathfrak{H}_j = \mathfrak{H}(m_j) \oplus \mathfrak{H}(m_{j+1}).$$

Obviously

$$\ker X = \mathfrak{H}(m_j) \oplus \{0\}$$

so that $S^j | \ker X$ is unitarily equivalent to $S(m_j)$. Now

$$\operatorname{ran} X = ((m_j/m_{j+1}) H^2 \ominus m_j H^2) \oplus \{0\}$$

so that $\ker X^* = \mathfrak{H}(m_j/m_{j+1}) \oplus \mathfrak{H}(m_{j+1})$ and it follows that $S^j_{\ker X^*}$ is unitarily equivalent to $S(m_j/m_{j+1}) \oplus S(m_{j+1})$. The Jordan model of $S(m_j/m_{j+1}) \oplus S(m_{j+1})$ is

$$S((m_j/m_{j+1}) \vee m_{j+1}) \oplus S((m_j/m_{j+1}) \wedge m_{j+1})$$

by [2], Lemma 4. Because S^j has the property (Q) this Jordan model must coincide with $S(m_j)$ so that $(m_j/m_{j+1}) \wedge m_{j+1} = 1$. In particular m_j/m_{j+1} and m_k/m_{k+1} are relatively prime for $k > j$; (ii) is proved.

Conversely, let us assume that conditions (i) and (ii) are satisfied. Let us denote

(5.8) $$u_j = m_j/m_{j+1}, \quad j < \omega.$$

Then by Lemma 1.2, $S(m_0)$ is quasisimilar to $\bigoplus_{j<\omega} S(u_j)$, $S(m_1)$ is quasisimilar to $\bigoplus_{1 \leq j < \omega} S(u_j), \ldots, S(m_k)$ is quasisimilar to $\bigoplus_{k \leq j < \omega} S(u_j)$ so that T is quasisimilar to

(5.9) $$S = \bigoplus_{j<\omega} T^j, \quad T^j = S(\underbrace{u_j, u_j, \ldots, u_j}_{j+1 \text{ times}}).$$

Because the functions $u_0, u_1, \ldots$ are pairwise relatively prime we have $(m_0/u_j) \wedge u_j = 1$ so that $(m_0/u_j)(T^k) = 0$, $k \neq j$, and $(m_0/u_j)(T^j)$ is a quasi-affinity. This implies that

$$\mathfrak{H}^j = \underbrace{\mathfrak{H}(u_j) \oplus \mathfrak{H}(u_j) \oplus \ldots \oplus \mathfrak{H}(u_j)}_{j+1 \text{ times}} = (\operatorname{ran}(m_0/u_j)(S))^-$$

is a hyper-invariant subspace of S. We are now able to prove that S, and therefore T, has property (Q). Any operator $X \in \{S\}'$ has the property $X\mathfrak{H}^j \subset \mathfrak{H}^j, j < \omega$, so that $X = \bigoplus_{j<\omega} X^j$, $X^j \in \{T^j\}'$. By Lemma 5.2, $T^j | \ker X^j$ and $T^j_{\ker X^{j*}}$ are quasisimilar. But obviously $\ker X = \bigoplus_{j<\omega} \ker X^j$, $\ker X^* = \bigoplus_{j<\omega} \ker X^{j*}$ so that $S|\ker X = \bigoplus_{j<\omega} T^j|\ker X^j$ and $S_{\ker X^*} = \bigoplus_{j<\omega} T^j_{\ker X^{j*}}$; it follows that $S|\ker X$ and $S_{\ker X^*}$ are quasisimilar. The Theorem is proved.

We are now able to give a complete description of the lattice of hyper-invariant subspaces of an operator of class C_0 having property (Q).

Proposition 5.5. *An operator of class C_0 having property* (P) *has property* (Q) *if and only if*

$$\text{Hyp Lat}(T) = \{(\operatorname{ran} m(T))^- : m \in H_i^\infty,\ m \leqq m_0[T]\}. \tag{5.10}$$

Proof. As usual $S(M)$ denotes the Jordan model of T. Assume (5.10) holds; by Proposition 4.8 (iii), (5.10) also holds for $S(M)$. In particular,

$$\ker m_{j+1}(S(M)) = \bigoplus_{i \leqq j} ((m_i/m_{j+1})H^2 \ominus m_i H^2) \oplus \bigoplus_{j+1 \leqq i < \omega} \mathfrak{H}(m_i)$$

is of the form $(\operatorname{ran} u(S(M)))^-$ for some inner divisor u of m_0. Because $\operatorname{ran} u(S(m_0)) = (m_0/m_{j+1})H^2 \ominus m_0 H^2$ we must have $u = m_0/m_{j+1}$. We have also

$$(m_0/m_{j+1}) \wedge m_{j+1} = 1 \tag{5.11}$$

because $u(S(m_{j+1}))$ must have dense range. From (5.11) we infer $(m_j/m_{j+1}) \wedge m_{j+1} = 1$, $j<\omega$. By Theorem 5.4 it follows that T has property (Q).

Conversely, let us assume that T has property (Q). By the proof of Theorem 5.4, T is quasisimilar to

$$S = \bigoplus_{j<\omega} S^j \quad \text{on} \quad \mathfrak{H} = \bigoplus_{j<\omega} \mathfrak{H}^j, \tag{5.12}$$

where

$$S^j = S(\underbrace{u_j, u_j, \ldots, u_j}_{j+1 \text{ times}}), \quad \mathfrak{H}^j = \underbrace{\mathfrak{H}(u_j) \oplus \mathfrak{H}(u_j) \oplus \ldots \oplus \mathfrak{H}(u_j)}_{j+1 \text{ times}}, \tag{5.13}$$

$$u_j = m_j/m_{j+1}, \tag{5.14}$$

and

$$\mathfrak{H}^j = ((m_0/u_j)(S)\mathfrak{H})^- \in \text{Hyp Lat}(S). \tag{5.15}$$

Let us take $\mathfrak{M} \in \text{Lat}(S)$ and denote $\mathfrak{M}_j = ((m_0/u_j)(S)\mathfrak{M})^-$. We claim that

$$\mathfrak{M} = \bigoplus_{j<\omega} \mathfrak{M}_j \quad \text{and} \quad \mathfrak{M}_j = \mathfrak{M} \cap \mathfrak{H}^j. \tag{5.16}$$

The inclusion $\mathfrak{M} \supset \bigoplus_{j<\omega} \mathfrak{M}_j$ is obvious. Now, the minimal function m of $S_{\mathfrak{N}}$, $\mathfrak{N} = \mathfrak{M} \ominus (\bigoplus_{j<\omega} \mathfrak{M}_j) = \bigcap_{j<\omega} \ker (m_0/u_j)^\sim((S|\mathfrak{M})^*)$ divides m_0/u_j, $j<\omega$, so that $m \wedge u_j = 1$. It follows that $m=1$, $\mathfrak{N} = \{0\}$ and (5.16) is proved.

Moreover, by (5.16), $\mathfrak{M}_j$ is a hyper-invariant subspace of S^j if $\mathfrak{M}\in\text{Hyp Lat}(S)$. By Proposition 2.1 (i) we have $\mathfrak{M}_j=\underbrace{\mathfrak{M}_j^0\oplus\mathfrak{M}_j^0\oplus\ldots\oplus\mathfrak{M}_j^0}_{j+1\text{ times}}$ where $\mathfrak{M}_j^0=u_j'H^2\ominus u_jH^2$ so that $\mathfrak{M}_j=u_j'(S^j)\mathfrak{H}^j$. Let us denote by m the limit of an arbitrary converging subsequence of $\{u_0'u_1'\ldots u_k'\}_{k<\omega}$; we shall have $(m/u_j')\wedge u_j=1$ so that $\mathfrak{M}_j=(m(S^j)\mathfrak{H}^j)^-$. Using (5.16) we infer $\mathfrak{M}=(m(S)\mathfrak{H})^-$ and by Proposition 4.8 (iii) the proof is done.

Let us denote by $\mathscr{L}_m^k$ the lattice $\text{Lat}(S(\underbrace{m, m, \ldots, m}_{k\text{ times}}))$ $(m\in H_i^\infty,\ 1\leq k<\omega)$. The preceding proof also characterizes Lat (T) for T having property (Q).

Corollary 5.6. *Let T be an operator of class C_0 having the property* (Q). *Then* Lat (T) *is isomorphic to* $\prod_{j<\omega}\mathscr{L}_{u_j}^{j+1}$, *where* $u_j=m_j[T]/m_{j+1}[T],\ j<\omega$.

Proof. The decomposition (5.16) was proved for any $\mathfrak{M}\in\text{Lat}(S)$. The Corollary follows by Proposition 4.8 (i).

Example 5.7. There are operators T of class C_0 for which (5.10) holds without property (P). In fact it can be shown that *a Jordan operator $S(M)$ satisfies the condition* (5.10) *if and only if* $(m_0/m_\alpha)\wedge m_\alpha=1$ *for each ordinal number* α.

Proof. The necessity of the condition $(m_0/m_\alpha)\wedge m_\alpha=1$ is proved analogously with the proof of (5.11). Conversely, let us assume $(m_0/m_\alpha)\wedge m_\alpha=1$ and let $\mathfrak{M}\in\text{Hyp Lat}(S(M))$ be given by (2.2). Then m_α/m_α'' divides m_0/m_0'' so that m_0''/m_α'' divides m_0/m_α and therefore $(m_0''/m_\alpha'')\wedge m_\alpha=1$. We infer $(m_0''(S(m_\alpha))\mathfrak{H}(m_\alpha))^-=$ $=(m_\alpha''(S(m_\alpha))(m_0''/m_\alpha'')(S(m_\alpha))\mathfrak{H}(m_\alpha))^-=m_\alpha''H^2\ominus m_\alpha H^2$ because $(m_0''/m_\alpha'')(S(m_\alpha))$ is a quasi-affinity (cf. [12], Proposition III.4.7). We infer

$$\mathfrak{M}=(\text{ran}\, m_0''(S(M)))^-.$$

Remark 5.8. As shown by Example 2.10, property (5.10) is not stable with respect to quasisimilarities.

6. Generalized inner functions

Let us recall (cf. [7]) that a function $m\in H_i^\infty$ has a factorization

$$m=cbs \tag{6.1}$$

where c is a complex constant of modulus one, b is a Blaschke product

$$b(z)=\prod_k\frac{\bar a_k}{|a_k|}\cdot\frac{a_k-z}{1-\bar a_k z},\quad |a_k|<1,\quad \sum_k(1-|a_k|)<\infty \tag{6.2}$$

and s is a singular inner function, that is

$$s(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right) \tag{6.3}$$

where μ is a finite Borel measure on $[0, 2\pi]$, singular with respect to Lebesgue measure. Let us denote by $\sigma(z)$ the multiplicity of the zero z in the Blaschke product (6.2), that is,

$$\sigma(z) = \operatorname{card}\{k:\ a_k = z\}. \tag{6.4}$$

The convergence condition in (6.2) is equivalent to

$$\sum_{|z|<1} \sigma(z)(1-|z|) < \infty. \tag{6.5}$$

We shall denote by Γ the set of pairs $\gamma=(\sigma, \mu)$, where μ is a finite Borel measure singular with respect Lebesgue's measure on $[0, 2\pi]$, $\sigma(z)$ is a natural number for $|z|<1$ and the condition (6.5) is satisfied. With respect to the adition $(\sigma, \mu)+(\sigma', \mu')=(\sigma+\sigma', \mu+\mu')$, Γ becomes a commutative monoid. The set Γ is ordered by the relation $(\sigma, \mu)\leqq(\sigma', \mu')$ if and only if $\sigma\leqq\sigma'$ and $\mu\leqq\mu'$. Moreover, in Γ are defined the lattice operations:

$$(\sigma, \mu)\vee(\sigma', \mu') = (\sigma\vee\sigma',\ \mu\vee\mu'),$$

$$(\sigma, \mu)\wedge(\sigma', \mu') = (\sigma\wedge\sigma', \mu\wedge\mu')$$

where $\mu\vee\mu'$, $\mu\wedge\mu'$ have the usual sense and $\sigma\vee\sigma'=\max\{\sigma, \sigma'\}$, $\sigma\wedge\sigma'=\min\{\sigma, \sigma'\}$. A mapping γ: $H_i^\infty\to\Gamma$ is defined by $\gamma(m)=(\sigma, \mu)$, where σ is given by (6.4) and μ by (6.3) if m has the decomposition (6.1). We have also a mapping δ: $\Gamma\to H_i^\infty$ defined by

$$(\delta(\gamma))(z) = \prod_{|z|<1}\left(\frac{\bar a}{|a|}\cdot\frac{a-z}{1-\bar a z}\right)^{\sigma(a)}\cdot\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right) \tag{6.6}$$

where $\gamma=(\sigma, \mu)$. Then $\gamma\circ\delta=\mathrm{id}$ and $\delta(\gamma(m))=cm$ with c a complex constant of modulus one.

Let us recall that, for a function $f\in H^\infty$, the function $f^\sim$ is defined by $f^\sim(z)=\overline{f(\bar z)}$. For $\gamma=(\sigma, \mu)\in\Gamma$ we shall define the element $\gamma^\sim=(\sigma^\sim, \mu^\sim)\in\Gamma$ by $\sigma^\sim(z)=\sigma(\bar z)$ and $\mu^\sim=\mu\circ j$ where j: $[0, 2\pi]\to[0, 2\pi]$ is given by $j(t)=2\pi-t$.

Let us list some properties of the mapping γ.

Lemma 6.1. (i) $\gamma(m_1m_2)=\gamma(m_1)+\gamma(m_2)$, $m_1, m_2\in H_i^\infty$.

(ii) $\gamma(m_1)\leqq\gamma(m_2)$ *if and only if* $m_1\leqq m_2$; $\gamma(m_1)=\gamma(m_2)$ *if and only if* m_1 *and* m_2 *differ by a complex multiplicative constant of modulus one.*

(iii) $\gamma(m^\sim)=\gamma(m)^\sim$, $m\in H_i^\infty$.

(iv) *If* $\{m_j\}_{j=0}^{\infty}\subset H_i^{\infty}$, *then the family* $\{m_0m_1\dots m_j\}_{j=0}^{\infty}$ *has a least inner multiple* m *if and only if* $\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ *and in this case* $\gamma(m)=\sum_{j=0}^{\infty}\gamma(m_j)$.

Proof. (i), (ii) and (iii) are obvious. To prove (iv) let us assume firstly that $\{m_0m_1\dots m_j\}_{j=0}^{\infty}$ has a least inner multiple m. Then obviously $\gamma\geqq\gamma(m)$ if and only if $\gamma\geqq\sum_{j\leqq n}\gamma(m_j)$ for each natural n. Consequently $\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ and $\gamma(m)=\sum_{j=0}^{\infty}\gamma(m_j)$. Conversely if $\gamma=\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ then $\delta(\gamma)\geqq m_0m_1m_2\dots m_j$ for each j so that the family $\{m_0m_1\dots m_j\}_{j=0}^{\infty}$ has a least inner multiple. The Lemma is proved.

We shall now introduce the class $\mathcal{M}$ of (not necessarily finite) Borel measures μ on $[0,2\pi]$ for which there exists a finite Borel measure ν singular with respect to Lebesgue measure such that $\mu\prec\nu$, where the absolute continuity $\mu\prec\nu$ is understood as

$$\mu=\bigvee_n(\mu\wedge n\nu). \tag{6.7}$$

We shall denote by $\mathcal{M}_0$ the class of σ-finite measures $\mu\in\mathcal{M}$ and by $\mathcal{M}_\infty$ the class of measures $\mu\in\mathcal{M}$ which take the values 0 and ∞ only.

Lemma 6.2. (i) *If* $\mu\in\mathcal{M}$ *and* ν *is a finite measure such that* $\mu\prec\nu$, *we have a decomposition*

$$d\mu=fd\nu \tag{6.8}$$

where $f\colon[0,2\pi]\to[0,+\infty]$ *is a Borel function.*

(ii) *Every* $\mu\in\mathcal{M}$ *admits a unique decomposition* $\mu=\mu_0+\mu_\infty$, *where* $\mu_0\in\mathcal{M}_0$, $\mu_\infty\in\mathcal{M}_\infty$ *and* μ_0 *and* μ_∞ *are mutually singular.*

(iii) *If* $\{\mu_j\}_{j=0}^{\infty}\subset\mathcal{M}$ *then* $\sum_{j=0}^{\infty}\mu_j\in\mathcal{M}$.

Proof. (i) The measure $\mu_n=\mu\wedge n\nu$ is finite, $\mu_n\prec\nu$, and by the Radon—Nikodym theorem we have $d\mu_n=f_nd\nu$, where $f_n\colon[0,2\pi]\to[0,n]$ is a Borel function. Because $\mu_n\leqq\mu_{n+1}$ we have $f_n\leqq f_{n+1}$ $d\nu$-a.e.; replacing f_n by $f_n'=f_1\vee f_2\vee\dots\vee f_n$ we may assume $f_n\leqq f_{n+1}$. Now it is clear that the function $f=\lim_{n\to\infty}f_n$ satisfies the relation (6.8).

(ii) Let ν and f be as before; let us denote $A=\{t;f(t)=+\infty\}$ and $f_\infty=f\chi_A$, $f_0=f(1-\chi_A)$. Then we may take $d\mu_0=f_0\cdot d\nu$, $d\mu_\infty=f_\infty d\nu$.

(iii) Let us take finite measures ν_j such that $\mu_j\prec\nu_j$; then $\sum_{j=0}^{\infty}\mu_j\prec\nu$, where ν is defined by

$$\nu=\sum_{j=0}^{\infty}2^{-j}\nu_j/\nu_j([0,2\pi]).$$

Remark 6.3. Obviously, every measure μ of the form (6.8) belongs to $\mathscr{M}$ if ν is a finite singular measure on $[0, 2\pi]$.

Lemma 6.4. *If $\mu_j, \nu_j \in \mathscr{M}, j=0, 1, \dots$, are such that $\sum_{j=0}^{\infty} \mu_j = \sum_{j=0}^{\infty} \nu_j$ then there exist $\mu_{ij} \in \mathscr{M}$, $i, j=0, 1, \dots$, such that $\sum_{j=0}^{\infty} \mu_{ij} = \mu_i$, $\sum_{i=0}^{\infty} \mu_{ij} = \nu_j$, $i, j=0, 1, \dots$.*

Proof. Let us take a finite singular measure α such that $\mu_j \prec \alpha$, $\nu_j \prec \alpha$, $j=0, 1, \dots$. By Lemma 6.2 we have

$$d\mu_j = f_j d\alpha, \quad d\nu_j = g_j d\alpha, \quad 0 \leqq j < \infty. \tag{6.9}$$

By the hypothesis we have

$$\sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} g_j \quad d\alpha\text{-a.e.} \tag{6.10}$$

It will be enough to find Borel functions h_{ij} such that

$$\sum_{j=0}^{\infty} h_{ij} = f_i, \quad \sum_{i=0}^{\infty} h_{ij} = g_j \quad d\alpha\text{-a.e.}, \quad 0 \leqq i, j < \infty, \tag{6.11}$$

and then to define $d\mu_{ij} = h_{ij} d\alpha$.

If the sum (6.10) is $d\alpha$-a.e. finite we may define h_{ij} inductively by

$$\begin{cases} h_{00} = f_0 \wedge g_0, \quad h_{0j} = \left(f_0 - \sum_{k=0}^{j-1} h_{0k}\right) \wedge g_j, \quad 1 \leqq j < \infty; \\ h_{i0} = f_i \wedge \left(g_0 - \sum_{k=0}^{i-1} h_{k0}\right), \quad 1 \leqq i < \infty; \\ h_{ij} = \left(f_i - \sum_{r=0}^{j-1} h_{ir}\right) \wedge \left(g_j - \sum_{k=0}^{i-1} h_{kj}\right), \quad 1 \leqq i, j < \infty. \end{cases} \tag{6.12}$$

If the sum (6.10) is not $d\alpha$-a.e. finite we can find increasing sequences $\{f_i^{(n)}\}_{n=0}^{\infty}$, $\{g_j^{(n)}\}_{n=0}^{\infty}$ such that $f_i = \lim_{n \to \infty} f_i^{(n)}$, $g_j = \lim_{n \to \infty} g_j^{(n)}$ $d\alpha$-a.e., $0 \leqq i, j < \infty$, and $\sum_{i=0}^{\infty} f_i^{(n)} = \sum_{j=0}^{\infty} g_j^{(n)} < \infty$ $d\alpha$-a.e., $0 \leqq n < \infty$.

Let $h_{ij}^{(n)}$ be defined by (6.12) with f_i, g_j replaced by $f_i^{(1)}$, $g_j^{(1)}$ in case $n=0$, and by $f_i^{(n+1)} - f_i^{(n)}$, $g_j^{(n+1)} - g_j^{(n)}$ in case $n \geqq 1$. We can take $h_{ij} = \sum_{n=0}^{\infty} h_{ij}^{(n)}$ and the Lemma follows.

We shall now introduce the class $\tilde{\Gamma}$ of "generalized inner functions". An element γ of $\tilde{\Gamma}$ is a pair $\gamma = (\sigma, \mu)$ where $\mu \in \mathscr{M}$ and σ is a natural number valued function defined on $\{z;\ |z| < 1\}$ such that

$$\sum_{\sigma(z) \neq 0} (1 - |z|) < \infty. \tag{6.13}$$

The subclass $\tilde{\Gamma}_0 \subset \tilde{\Gamma}$ consists of the pairs $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ such that $\mu\in\mathcal{M}_0$. Analogously with Γ, $\tilde{\Gamma}$ is a commutative monoid and an ordered set in which the lattice operations are defined. For $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ we define $\gamma^{\sim}=(\sigma^{\sim}, \mu^{\sim})\in\tilde{\Gamma}$ as in the case $\gamma\in\Gamma$. Any $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ has a decomposition

$$\gamma = \gamma_0+\gamma_\infty, \quad \gamma_0 = (\sigma, \mu_0)\in\tilde{\Gamma}_0, \quad \gamma_\infty = (0, \mu_\infty) \tag{6.14}$$

where $\mu=\mu_0+\mu_\infty$ is the decomposition of μ given by Lemma 6.2 (ii).

Lemma 6.5. (i) *$\tilde{\Gamma}_0$ is the set of simplifiable elements of $\tilde{\Gamma}$, that is $\gamma\in\tilde{\Gamma}_0$ if and only if $\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ for $\gamma', \gamma''\in\tilde{\Gamma}$.*

(ii) *$\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ whenever $\gamma_\infty\leqq\gamma'\wedge\gamma''$.*

Proof. (i) It is obvious that $\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ whenever $\gamma\in\tilde{\Gamma}_0$. Conversely, if $\gamma\notin\tilde{\Gamma}_0$, we have $0\neq\gamma_\infty$ and $0+\gamma=\gamma_\infty+\gamma$.

(ii) By (i) we can simplify γ_0 from the equality $\gamma'+\gamma=\gamma''+\gamma$ and we obtain $\gamma'+\gamma_\infty=\gamma''+\gamma_\infty$. Now the assumption implies $\gamma'+\gamma_\infty=\gamma'$ and $\gamma''+\gamma_\infty=\gamma''$; the Lemma follows.

We shall consider the cartesian product $\mathcal{K}=\tilde{\Gamma}\times\tilde{\Gamma}$ and on $\mathcal{K}$ we define the relation "$\sim$" by

$$(\gamma, \gamma_1) \sim (\gamma', \gamma_1') \quad \text{if and only if} \quad \gamma+\gamma_1' = \gamma'+\gamma_1. \tag{6.15}$$

The relation "$\sim$" is not an equivalence relation; however, as shown by Lemma 6.5 (i) the restriction of "$\sim$" on $\mathcal{K}_0=\tilde{\Gamma}_0\times\tilde{\Gamma}_0$ is an equivalence relation. The quotient $\mathcal{G}_0=\mathcal{K}_0/\sim$ is a group- the group of formal differences $\gamma-\gamma'$, $\gamma, \gamma'\in\tilde{\Gamma}_0$. We may assume $\tilde{\Gamma}_0\subset\mathcal{G}_0$ identifying the element $\gamma\in\tilde{\Gamma}_0$ with the class of $(\gamma, 0)$ in $\mathcal{K}_0/\sim$.

We shall now describe the connection of $\tilde{\Gamma}$ and $\tilde{\Gamma}_0$ with Γ.

Proposition 6.6. (i) *If $\{\gamma_j\}_{j=0}^{\infty}\subset\Gamma$ are such that*

$$\gamma_j \geqq \gamma_{j+1}, \quad 0\leqq j<\infty, \quad \bigwedge_{j\geqq 0}\gamma_j = 0, \tag{6.16}$$

then

$$\gamma = \sum_{j=0}^{\infty}\gamma_j\in\tilde{\Gamma}. \tag{6.17}$$

Conversely, each $\gamma\in\tilde{\Gamma}$ has a representation of the form (6.17) *such that* (6.16) *is satisfied.*

(ii) *If $\{\gamma_j\}_{j=0}^{\infty}\subset\Gamma$ satisfy* (6.16) *and, moreover,*

$$(\gamma_j-\gamma_{j+1})\wedge(\gamma_k-\gamma_{k+1}) = 0, \quad j\neq k, \tag{6.18}$$

then the element γ defined by (6.17) *belongs to $\tilde{\Gamma}_0$. Conversely, each $\gamma\in\tilde{\Gamma}_0$ has a representation of the form* (6.17) *such that* (6.16) *and* (6.18) *are verified.*

Proof. (i) If $\gamma_j=(\sigma_j, \mu_j)$, $0\leqq j<\infty$, we have $\mu=\sum_{j=0}^{\infty}\mu_j\in\mathcal{M}$ by Lemma 6.2 (iii); it remains to show that $\sigma=\sum_{j=0}^{\infty}\sigma_j$ is finite and the condition (6.13) is satisfied. But $\bigwedge_{j\geqq 0}\sigma_j=0$ imply that for each z, $\sigma_j(z)=0$ for some j and the finiteness of σ is obvious. The condition (6.13) is satisfied because $\sigma(z)\neq 0$ implies $\sigma_0(z)\neq 0$ and therefore

$$\sum_{\sigma(z)\neq 0}(1-|z|)\leqq\sum_{|z|<1}\sigma_0(z)(1-|z|)<\infty .$$

Conversely, if $\gamma=(\sigma, \mu)$ we define

$$\begin{cases}\sigma_j(z)=0 & \text{if} \quad \sigma(z)\leqq j\\ \qquad\;\; =1 & \text{if} \quad \sigma(z)>j, \; 0\leqq j<\infty.\end{cases} \tag{6.19}$$

To define μ_j let us write $d\mu=f\cdot dv$ for some finite measure v and put $d\mu_j=f_j\cdot dv$, where

$$f_0=f\wedge 1, \quad f_j=\Big(f-\sum_{k=0}^{j-1}f_k\Big)\wedge 1/(j+1), \quad 1\leqq j<\infty. \tag{6.20}$$

It is obvious that $\gamma_j=(\sigma_j, \mu_j)$ satisfy (6.16—17).

(ii) Let us put $\gamma_j=(\sigma_j, \mu_j)$; from (6.18) we infer the existence of a sequence of pairwise disjoint Borel subsets $A_j\subset[0, 2\pi]$ such that $[0, 2\pi]=\bigcup_{j=0}^{\infty}A_j$ and $\mu_j\big(\bigcup_{k<j}A_k\big)=0$. If $\mu=\sum_{j=0}^{\infty}\mu_j$, we have $\mu(A_j)=(\mu_0+\mu_1+\ldots+\mu_j)(A_j)<\infty$; thus μ is σ-finite. Conversely, let us take $\gamma=(\sigma, \mu)\in\tilde{\Gamma}_0$ and define σ_j by (6.19). If $d\mu=f\cdot dv$ and v is finite, f is dv-a.e. finite so that $[0, 2\pi]=\bigcup_{j=0}^{\infty}A_j$ where $A_j=\{x; f(x)\in[j, j+1)\}$. We define

$$f_j=\sum_{k=j}^{\infty}(k+1)^{-1}f\chi_{A_k}$$

and $d\mu_j=f_j\cdot dv$. It is clear that $\gamma_j=(\sigma_j, \mu_j)$ satisfy the conditions (6.16—18). Proposition 6.6 is proved.

Proposition 6.7. *If* $\{\gamma_j\}_{j=0}^{\infty}, \{\gamma_j'\}_{j=0}^{\infty}\subset\tilde{\Gamma}$ *are such that* $\sum_{j=0}^{\infty}\gamma_j=\sum_{j=0}^{\infty}\gamma_j'\in\tilde{\Gamma}$ *then there exist* $\{\gamma_{ij}\}_{0\leqq i,j<\infty}\subset\tilde{\Gamma}$ *such that* $\sum_{j=0}^{\infty}\gamma_{ij}=\gamma_i$, $\sum_{i=0}^{\infty}\gamma_{ij}=\gamma_j$, $0\leqq i, j<\infty$.

Proof. If $\gamma_j=(\sigma_j, \mu_j)$, $\gamma_j'=(\sigma_j', \mu_j')$, $0\leqq j<\infty$, we shall define $\gamma_{ij}=(\sigma_{ij}, \mu_{ij})$, where μ_{ij} are given by Lemma 6.4 and σ_{ij} are defined by formulas analogous to (6.12) with f_j and g_j replaced by σ_j and σ_j', respectively. The Proposition follows.

7. C_0-dimension of a subspace

We shall denote by $\mathscr{P}$ the class of C_0 operators having the property (P). If $T\in\mathscr{P}$ and $S(M)$ is the Jordan model of T we have $\bigwedge_{j<\omega}\gamma(m_j)=0$, $m_j=m_j[T]$, by Theorem 4.1 and Lemma 6.1. This fact and Proposition 6.6 suggest the following Definition.

Definition 7.1. The *dimension* γ_T of the operator $T\in\mathscr{P}$ is defined as

$$\gamma_T=\sum_{j=0}^{\infty}\gamma(m_j),\quad m_j=m_j[T]. \tag{7.1}$$

If T is an operator of class C_0 and $\mathfrak{M}\in\mathrm{Lat}_{\frac12}(T)$ is such that $T_{\mathfrak{M}}\in\mathscr{P}$, then the *T-dimension* $\gamma_T(\mathfrak{M})$ is defined as

$$\gamma_T(\mathfrak{M})=\gamma(\mathfrak{M})=\gamma_{T_{\mathfrak{M}}}. \tag{7.2}$$

Remark 7.2. (i) Because $m_j[T^*]=m_j[T]^\sim$ (cf. [4], Corollary 2.8) we have $\gamma_{T^*}=\gamma_T^\sim$, $T\in\mathscr{P}$. Moreover, if T is of class C_0 and $\mathfrak{M}\in\mathrm{Lat}_{\frac12}(T)$ is such that $T_{\mathfrak{M}}\in\mathscr{P}$, then

$$\gamma_{T^*}(\mathfrak{M})=\gamma_T(\mathfrak{M})^\sim. \tag{7.3}$$

(ii) It is clear that $\gamma_T=0$ *if and only if T acts on the trivial space* $\{0\}$.

(iii) *The dimension γ_T is a quasisimilarity invariant of T.* Indeed, γ_T is defined in terms of the Jordan model.

We shall say C_0-dimension instead of T-dimension if no confusion is possible. The usual dimension is a particular case of the C_0-dimension. Indeed, the operator $T=0\in\mathscr{L}(\mathfrak{H})$ is a C_0 operator and each subspace $\mathfrak{M}\subset\mathfrak{H}$ is invariant for T. By Theorem 4.1, $T|\mathfrak{M}$ has the property (P) if and only if $\dim\mathfrak{M}<\infty$ and in this case $\gamma_T(\mathfrak{M})=(\sigma,0)$ where $\sigma(0)=\dim\mathfrak{M}$ and $\sigma(z)=0$ otherwise.

Lemma 7.3. *An operator $T\in\mathscr{P}$ is a weak contraction if and only if $\gamma_T\in\Gamma$ and in this case*

$$\gamma_T=\gamma(d_T). \tag{7.4}$$

Proof. Obviously follows from Lemma 6.1 (iv), [6], Theorem 8.5 and [3], Definition 1.1.

By Proposition 6.6, Theorems 4.1 and 5.4, we have $\{\gamma_T;\ T\in\mathscr{P}\}=\tilde\Gamma$ and $\{\gamma_T;\ T \text{ has the property (Q)}\}=\tilde\Gamma_0$. It is natural to define $\mathscr{P}_0$ by

$$T\in\mathscr{P}_0 \quad\text{if and only if}\quad T\in\mathscr{P} \quad\text{and}\quad \gamma_T\in\tilde\Gamma_0. \tag{7.5}$$

Lemma 7.4. *If $T\in\mathscr{P}$ is acting on $\mathfrak{H}$ and $\mathfrak{H}_j\in\mathrm{Lat}(T)$ are such that $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $0\leqq j<\infty$, and $\bigvee_{j\geqq0}\mathfrak{H}_j=\mathfrak{H}$, we have*

$$\gamma_T=\bigvee_{j\geqq0}\gamma_T(\mathfrak{H}_j). \tag{7.6}$$

Proof. Because $T|\mathfrak{H}_j \overset{i}{\prec} T$, we have $m_k[T|\mathfrak{H}_j] \leqq m_k[T]$ for each natural number k; therefore $\gamma(m_k[T|\mathfrak{H}_j]) \leqq \gamma(m_k[T])$ and the inequality $\gamma_T \geqq \bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j)$ follows. Now, by Lemma 6.1 we shall have $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \sum_{k=0}^{n} \gamma\left(\bigvee_{j\geqq 0} m_k[T|\mathfrak{H}_j]\right)$ for each natural number n; by Theorem 3.1 we infer $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \sum_{k=0}^{n} \gamma(m_k[T])$. Since n is arbitrary the inequality $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \gamma_T$ follows. Lemma 7.4 is proved.

Remark 7.5. From (7.3) it follows that Lemma 7.4 also holds under the assumption $\mathfrak{H}_j \in \operatorname{Lat}(T^*)$ instead of $\mathfrak{H}_j \in \operatorname{Lat}(T)$, $0 \leqq j < \infty$.

Corollary 7.6. *If $T, T' \in \mathscr{P}$, we have* $\gamma_{T\oplus T'} = \gamma_T + \gamma_{T'}$.

Proof. By Remark 7.2 (iii) it is enough to prove the Corollary for $T = S(M)$, $T' = S(M')$. For each j the space $\mathfrak{K}_j = \mathfrak{H}_j \oplus \mathfrak{H}'_j \in \operatorname{Lat}(T \oplus T')$, where $\mathfrak{H}_j = \mathfrak{H}(m_0) \oplus \oplus \mathfrak{H}(m_1) \oplus \ldots \oplus \mathfrak{H}(m_j)$, $\mathfrak{H}'_j = \mathfrak{H}(m'_0) \oplus \mathfrak{H}(m'_1) \oplus \ldots \oplus \mathfrak{H}(m'_j)$ and $\mathfrak{H}(M) = \bigvee_{j\geqq 0} \mathfrak{H}_j$, $\mathfrak{H}(M') = \bigvee_{j\geqq 0} \mathfrak{H}'_j$. By Lemma 7.4 we have $\gamma_{T\oplus T'} = \bigvee_{j\geqq 0} \gamma_{T\oplus T'}(\mathfrak{K}_j)$, $\gamma_T = \bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j)$, $\gamma_{T'} = \bigvee_{j\geqq 0} \gamma_{T'}(\mathfrak{H}'_j)$. By Lemma 7.3 and [3], Theorem 1.3, the Corollary follows.

We shall now introduce a relation ϱ on the class $\mathscr{P}$, connected to index problems.

Definition 7.7. For $T_1, T_2 \in \mathscr{P}$ we write $T_1 \varrho T_2$ if there exist $T \in \mathscr{P}$ and $X \in \{T\}'$ such that T_1 and T_2 are quasisimilar to $T|\ker X$ and $T_{\ker X^*}$, respectively.

Lemma 7.8. *If $T \in \mathscr{P}$ and $\mathfrak{H} \in \operatorname{Lat}(T)$ then $T \varrho (T_{\mathfrak{H}} \oplus T_{\mathfrak{H}^\perp})$.*

Proof. The operator $S = T \oplus T_{\mathfrak{H}} \in \mathscr{P}$ by Proposition 4.4 and the operator X defined by $X(u \oplus v) = v \oplus 0$ commutes with S. It is easy to see that $S|\ker X$ is unitarily equivalent to T and $S_{\ker X^*}$ is unitarily equivalent to $T_{\mathfrak{H}} \oplus T_{\mathfrak{H}^\perp}$; Lemma 7.8 follows.

By Theorem 4.1 and Remark 7.2 (iii), $\gamma_{T_1} = 0$ if and only if $\gamma_{T_2} = 0$ if $T_1 \varrho T_2$. The connection between ϱ and γ is stronger than that, as it will be shown in the following propositions.

Theorem 7.9. *If $T_1, T_2 \in \mathscr{P}$ and $T_1 \varrho T_2$ then* $\gamma_{T_1} = \gamma_{T_2}$.

Proof. It is enough to show that for $T \in \mathscr{P}$ and $X \in \{T\}'$ we have $\gamma_T(\ker X) = \gamma_T(\ker X^*)$. Let T be acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. As shown in the proof of Theorem 4.1 we have

$$\mathfrak{H} = \bigvee_{j\geqq 0} \mathfrak{H}_j, \quad \mathfrak{H}_j = (m_j(T)\mathfrak{H})^- \in \operatorname{Hyp} \operatorname{Lat}(T). \tag{7.7}$$

For each natural j we have $X\mathfrak{H}_j \subset \mathfrak{H}_j$ and $X_j = X|\mathfrak{H}_j \in \{T|\mathfrak{H}_j\}'$. Because $T|\mathfrak{H}_j$ is of finite multiplicity, we infer by [3], Corollary 2.6, and Lemma 7.3,

$$\gamma(\ker X_j) = \gamma(\ker X_j^*). \tag{7.8}$$

Because obviously $Xm_j(T)|\ker X=0$, we have $\ker X_j\supset(m_j(T)\ker X)^-$ and, as in the proof of Theorem 4.1, we infer $\ker X=\bigvee_{j\geq 0}\ker X_j$. Therefore, by Lemma 7.4 applied to $T|\ker X$ it follows that

$$\gamma(\ker X)=\bigvee_{j\geq 0}\gamma(\ker X_j). \tag{7.9}$$

We have $X_j^* P_{\mathfrak{H}_j}|\ker X^*=P_{\mathfrak{H}_j}X^*P_{\mathfrak{H}_j}|\ker X^*=P_{\mathfrak{H}_j}X^*|\ker X^*=0$ so that $P_{\mathfrak{H}_j}(\ker X^*)\subset\ker X_j^*$. Because $P_{\mathfrak{H}_j}T^*=T^*_{\mathfrak{H}_j}P_{\mathfrak{H}_j}$ we shall have $P_{\mathfrak{H}_j}T^*|\ker X^*=$ $=(T^*_{\mathfrak{H}_j}|\ker X_j^*)P_{\mathfrak{H}_j}|\ker X^*$. This relation implies that $(T^*|\ker X^*)_{\mathfrak{R}_j}$, where

$$\mathfrak{R}_j=(\ker(P_{\mathfrak{H}_j}|\ker X^*))^{\perp}=\ker X^*\ominus(\ker X^*\cap\mathfrak{H}_j^{\perp})\in\mathrm{Lat}\,(T_{\ker X^*}),$$

is quasisimilar to some restriction of $T^*_{\mathfrak{H}_j}|\ker X_j^*$ and therefore

$$\gamma(\mathfrak{R}_j)\leq\gamma(\ker X_j^*). \tag{7.10}$$

Now $\bigvee_{j\geq 0}\mathfrak{R}_j=\ker X^*\ominus(\ker X^*\cap(\bigcap_{j\geq 0}\mathfrak{H}_j^{\perp}))=\ker X^*$ so that from (7.8—10) and Lemma 7.4 applied to $T_{\ker X^*}$ we infer $\gamma(\ker X^*)=\bigvee_{j\geq 0}\gamma(\mathfrak{R}_j)\leq\bigvee_{j\geq 0}\gamma(\ker X_j^*)=$ $=\bigvee_{j\geq 0}\gamma(\ker X_j)=\gamma(\ker X)$.

By the same argument applied to T^* instead of T we infer $\gamma(\ker X)\leq\gamma(\ker X^*)$. The Theorem follows.

Corollary 7.10. *If $T\in\mathcal{P}$ and $\mathfrak{H}\in\mathrm{Lat}\,(T)$ then $\gamma_T=\gamma_T(\mathfrak{H})+\gamma_T(\mathfrak{H}^{\perp})$.*

Proof. Obviously follows from Corollary 7.6 and Theorem 7.9.

Corollary 7.11. *Let $T\in\mathcal{P}$ be acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\mathrm{Lat}\,(T)$ be such that $\mathfrak{H}_0=\mathfrak{H}$, $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}$ $(0\leq j<\infty)$ and $\bigcap_{j\geq 0}\mathfrak{H}_j=\{0\}$. Then $\gamma_T=\sum_{j=0}^{\infty}\gamma_T(\mathfrak{R}_j)$, where $\mathfrak{R}_j=$ $=\mathfrak{H}_j\ominus\mathfrak{H}_{j+1}$ $(0\leq j<\infty)$.*

Proof. By Lemma 7.4 and Remark 7.5 we have $\gamma_T=\bigvee_{j\geq 0}\gamma_T(\mathfrak{H}_j^{\perp})$. Because $\mathfrak{H}_{j+1}^{\perp}=\mathfrak{H}_j^{\perp}\oplus\mathfrak{R}_j$ and $\mathfrak{R}_j\in\mathrm{Lat}\,(T_{\mathfrak{H}_{j+1}^{\perp}})$ we have $\gamma_T(\mathfrak{H}_{j+1}^{\perp})=\gamma_T(\mathfrak{H}_j^{\perp})+\gamma_T(\mathfrak{R}_j)$ by the Corollary 7.10. By induction it follows that $\gamma_T(\mathfrak{H}_{j+1}^{\perp})=\sum_{n=0}^{j}\gamma_T(\mathfrak{R}_n)$. Corollary 7.11 follows.

Corollary 7.12. *Let $T\in\mathcal{P}$ be acting on $\mathfrak{H}$. Then $T\in\mathcal{P}_0$ if and only if $\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)=0$ for each decreasing sequence $\{\mathfrak{H}_m\}_{m=0}^{\infty}\subset\mathrm{Lat}\,(T)$ such that $\bigcap_{j\geq 0}\mathfrak{H}_j=$ $=\{0\}$.*

Proof. Let us assume $T\in\mathcal{P}_0$. By Corollary 7.10 we have $\gamma_T=\gamma_T(\mathfrak{H}_j)+\gamma_T(\mathfrak{H}_j^{\perp})$ so that by Lemma 7.4 we infer $\gamma_T=\gamma_T+\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)$. Because $\gamma_T\in\tilde{\Gamma}_0$ it follows that $0=\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)$.

Conversely, if $T\notin\mathscr{P}_0$, let $S(M)$ be the Jordan model of T. By the proof of [5], Theorem 1, there exist $\mathfrak{H}_j\in\text{Lat}(T)$ such that $\mathfrak{H}_{j+1}\subset\mathfrak{H}_j$, $\bigcap_{j\geq 0}\mathfrak{H}_j=0$ and the Jordan model of $T|\mathfrak{H}_j$ is $\bigoplus_{k\geq j} S(m_k)$. Because $\gamma_T(\mathfrak{H}_j^\perp)=\sum_{k<j}\gamma(m_k)\in\Gamma$, from the relation $\gamma_T=\gamma_T(\mathfrak{H}_j^\perp)+\gamma_T(\mathfrak{H}_j)$ we infer $(\gamma_T)_\infty=(\gamma_T(\mathfrak{H}_j))_\infty$ and therefore $\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)\geq(\gamma_T)_\infty\neq 0$. Corollary 7.12 is proved.

We shall prove now a partial converse of Theorem 7.9.

Theorem 7.13. (i) *If* $T, T'\in\mathscr{P}$ *are weak contractions and* $\gamma_T=\gamma_{T'}$, *then* $T\varrho T'$.

(ii) *If* $T, T'\in\mathscr{P}$ *are such that* $\gamma_T=\gamma_{T'}$ *then there exists* $S\in\mathscr{P}$ *such that* $T\varrho S$ *and* $S\varrho T'$.

Proof. Let $S(M)$ and $S(M')$ be the Jordan models of T and T', respectively. The condition $\gamma_T=\gamma_{T'}$. is equivalent to $d_T=d_{T'}$; let us denote $d=d_T=d_{T'}$. If we denote $d_j=d/m_0m_1\dots m_{j-1}$, $d_{-j}=d/m_0'm_1'\dots m_{j-1}'$ for $1\leq j<\infty$ and $d_0=d$, we have $\bigwedge_{j\geq 0} d_j=\bigwedge_{j\geq 0} d_{-j}=1$ and by Theorem 4.1 and Proposition 4.4 the operator

$$K=\bigoplus_{j=-\infty}^{+\infty} S(d_j) \tag{7.11}$$

has property (P), that is, $K\in\mathscr{P}$. We define now an operator $X\in\{K\}'$ by $X(\bigoplus_{j=-\infty}^{+\infty} h_j)=$ $=\bigoplus_{j=-\infty}^{+\infty} k_j$ where

$$\begin{cases} k_j=P_{\mathfrak{H}(d_j)}h_{j-1} & \text{if } j\geq 1,\\ \quad =(d_j/d_{j-1})h_{j-1} & \text{if } j\leq 0.\end{cases} \tag{7.12}$$

It is easy to see that $\ker X=\bigoplus_{j=0}^{+\infty}\ker(X|\mathfrak{H}(d_j))$ and $\ker X^*=\bigoplus_{j=0}^{-\infty}\ker(X^*|\mathfrak{H}(d_j))$. For $j\geq 0$

$$\ker(X|\mathfrak{H}(d_j))=d_{j+1}H^2\ominus d_jH^2$$

so that $S(d_j)|\ker(X|\mathfrak{H}(d_j))$ is unitarily equivalent to $S(d_j/d_{j+1})=S(m_j)$ and therefore $K|\ker X$ is unitarily equivalent to $S(M)$. We can analogously verify that $K_{\ker X^*}$ is unitarily equivalent to $S(M')$.

Let us remark that the minimal function of K coincides with the common determinant function of T and T'.

(ii) Let $S(M)$ and $S(M')$ be the Jordan models of T and T', respectively. The equality $\gamma_T=\gamma_{T'}$ is equivalent to $\sum_{j=0}^{\infty}\gamma(m_j)=\sum_{j=0}^{\infty}\gamma(m_j')$. By Proposition 6.7 we can find $\gamma_{ij}\in\tilde{\Gamma}$ such that $\sum_{j=0}^{\infty}\gamma_{ij}=\gamma(m_i)$ and $\sum_{i=0}^{\infty}\gamma_{ij}=\gamma(m_j')$, $0\leq i,j<\infty$. Because $\gamma_{ij}\leq$

$\leqq\gamma(m_i)$ we have $\gamma_{ij}\in\Gamma$ and therefore $\gamma_{ij}=\gamma(m_{ij})$ for $m_{ij}=\delta(\gamma_{ij})\in H_i^\infty$. We define the operator

$$S=\bigoplus_{i=0}^{\infty}\left(\bigoplus_{j=0}^{\infty}S(m_{ij})\right)=\bigoplus_{i=0}^{\infty}S_i,\quad S_i=\bigoplus_{j=0}^{\infty}S(m_{ij}),\quad 0\leqq i<\infty. \tag{7.13}$$

Because $\gamma(m_i)=\sum_{j=0}^{\infty}\gamma(m_{ij})$, the operator S_1 is a weak contraction and $\gamma_{S_i}=\gamma_{S(m_i)}$, $0\leqq i<\infty$ (cf. Lemma 7.3). By the proof of (i) we can find operators $K^i\in\mathcal{P}$ acting on $\mathfrak{H}_i$ and contractions $X_i\in\{K^i\}'$ such that

$$m_0[K^i]=m_i,\quad 0\leqq i<\infty, \tag{7.14}$$

$K^i|\ker X_i$ and $K^i_{\ker X_i^*}$ are unitarily equivalent to $S(m_i)$ and S_i, respectively. The operator $K=\bigoplus_{i=0}^{\infty}K^i$ is of class C_0, $X=\bigoplus_{i=0}^{\infty}X_i\in\{K\}'$ and $K|\ker X$, $K_{\ker X^*}$ are unitarily equivalent to $S(M)$, S, respectively.

Let us show that $K\in\mathcal{P}$. The spaces $\mathfrak{K}_i=\mathfrak{H}_0\oplus\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_i$ are invariant for T, $\bigvee_{i\geqq0}\mathfrak{K}_i=\bigoplus_{i=0}^{\infty}\mathfrak{H}_i$ and $m_0[K|\mathfrak{K}_i^\perp]=m_{i+1}$, $0\leqq i<\infty$. Because $T\in\mathcal{P}$ we have $\bigwedge_{i\geqq0}m_{i+1}=1$ and by Proposition 4.6 it follows that $K\in\mathcal{P}$. In particular S also has the property (P) by Proposition 4.4 and therefore we proved that $T\varrho S$. The relation $S\varrho T'$ is proved analogously. The Theorem follows.

Remark 7.14. If T and T' have finite multiplicities, then the operator K used for the proof of (i) also has finite multiplicity. Thus we obtain a new proof of Proposition 3.2 of [3].

8. C_0-Fredholm operators

The results of sec. 7 suggest the following generalization of [3], Definition 2.2.

Definition 8.1. Let T and T' be operators of class C_0 and let $X\in\mathcal{I}(T',T)$. Then X is called a *(T',T)-semi-Fredholm* operator if $X|(\ker X)^\perp$ is a $(T'|(\operatorname{ran}X)^-,\ T_{(\ker X)^\perp})$*-lattice-isomorphism* and either $T|\ker X\in\mathcal{P}$ or $T'_{\ker X^*}\in\mathcal{P}$ holds. A (T',T)-semi-Fredholm operator X is *(T',T)-Fredholm* if both $T|\ker X$ and $T'_{\ker X^*}$ have property (P). If X is (T',T)-Fredholm, its index is defined as

$$\operatorname{ind}(X)=(\gamma_T(\ker X),\ \gamma_{T'}(\ker X^*))\in\tilde{\Gamma}\times\tilde{\Gamma}. \tag{8.1}$$

If X is (T',T)-semi-Fredholm but not (T',T)-Fredholm, we define

$$\begin{aligned}\operatorname{ind}(X)&=+\infty\quad \textit{if}\quad T|\ker X\notin\mathcal{P};\\ &=-\infty\quad \textit{if}\quad T'_{\ker X^*}\notin\mathcal{P}.\end{aligned} \tag{8.2}$$

Let us remark that for $T|\ker X \in \mathcal{P}_0$ and $T'_{\ker X^*} \in \mathcal{P}_0$, ind (X) is uniquely determined (modulo the relation "$\sim$") by the element $\gamma_T(\ker X) - \gamma_{T'}(\ker X^*) \in \mathcal{G}_0$ (cf. sec. 6).

In order to distinguish the operator introduced by Definition 8.1 from the operators considered in [3] we shall denote by $\Phi(T', T)$ and $\sigma\Phi(T', T)$ the set of (T', T)-Fredholm and (T', T)-semi-Fredholm operators, respectively. If $T' = T$ we write $\Phi(T)$, and $\sigma\Phi(T)$ instead of $\Phi(T, T)$, $\sigma\Phi(T, T)$, respectively.

Obviously $\mathcal{F}(T', T) \subset \Phi(T', T)$ and for $X \in \mathcal{F}(T', T)$ we have

(8.3) $$\operatorname{ind}(X) = \gamma(j(X))$$

if ind (X) is interpreted as an element of $\mathcal{G}_0$ and

$$\gamma(m/n) = \gamma(m) - \gamma(n) \quad \text{for} \quad m, n \in H_i^\infty.$$

The following Proposition extends [3], Corollary 2.6 and Remark 2.7.

Proposition 8.2. (i) *If* $T, T' \in \mathcal{P}$ *then* $\Phi(T', T) = \mathcal{I}(T', T)$ *and*

(8.4) $$\operatorname{ind}(X) \sim (\gamma_T, \gamma_{T'}) \quad \textit{for} \quad X \in \mathcal{I}(T', T).$$

(ii) *If exactly one of the operators T and T' has property* (P) *then* $\Phi(T', T) = \emptyset$, $\sigma\Phi(T', T) = \mathcal{I}(T', T)$, *and for* $X \in \mathcal{I}(T', T)$,

$$\begin{aligned} \operatorname{ind}(X) &= +\infty \quad \textit{if} \quad T \notin \mathcal{P}, \\ &= -\infty \quad \textit{if} \quad T' \notin \mathcal{P}. \end{aligned}$$

Proof. (i) because $T_{(\ker X)^\perp}$ and $T'|(\operatorname{ran} X)^-$ are quasisimilar and have the property (P) for any $X \in \mathcal{I}(T', T)$ (cf. Corollary 4.5 and Lemma 1.1) it follows that $X|(\ker X)^\perp$ is a lattice-isomorphism by Proposition 4.8 (i). In particular $\gamma_T((\ker X)^\perp) = \gamma_{T'}((\operatorname{ran} X)^-)$. By Corollary 7.10 it follows that $\gamma_T = \gamma_T(\ker X) + \gamma_T((\ker X)^\perp)$ and $\gamma_{T'}(\ker X^*) + \gamma_{T'}((\operatorname{ran} X)^-) = \gamma_{T'}$ so that

$$\gamma_T + \gamma_{T'}(\ker X^*) + \gamma = \gamma_{T'} + \gamma_T(\ker X) + \gamma$$

where $\gamma = \gamma_T((\ker X)^\perp) = \gamma_{T'}((\operatorname{ran} X)^-)$. Because

$$\gamma \leqq \gamma_T \wedge \gamma_{T'}$$

we infer by Lemma 6.5 (ii):

$$\gamma_T + \gamma_{T'}(\ker X^*) = \gamma_{T'} + \gamma_T(\ker X);$$

this means exactly ind $(X) \sim (\gamma_T, \gamma_{T'})$.

(ii) As in the preceding proof $T_{(\ker X)^\perp}$ and $T'|(\operatorname{ran} X)^-$ are quasisimilar and one of them must have the property (P) by Corollary 4.5. Then Corollary 4.3 and Proposition 4.8 (i) show that $X|(\ker X)^\perp$ is a lattice-isomorphism. To end the proof it is enough to show that $\Phi(T', T) = \emptyset$. Assume by example $T' \notin \mathcal{P}$; then for any $X\in\mathcal{I}(T',T)$, $T'|(\operatorname{ran}X)^-\in\mathcal{P}$ so that $T'_{\ker X^*}\notin\mathcal{P}$ by Proposition 4.4. The case $T\notin\mathcal{P}$ is treated analogously. The Proposition is proved.

Example 8.3. The relation $\operatorname{ind}(X)\sim(\gamma_T,\gamma_{T'})$ obtained in Proposition 8.2 cannot be improved. By example, if $\gamma_T=\gamma_{T'}$ it does not follow that $\gamma_T(\ker X)=\gamma_{T'}(\ker X^*)$ for each $X\in\mathcal{I}(T',T)$. Indeed, let us take $T'=S(M)\in\mathcal{P}$ such that $\gamma_{T'}=(0,\mu)$, $\mu\in\mathcal{M}_\infty$, and $T=\bigoplus_{j\geq1}S(m_j)$. Then $\gamma_{T'}=\gamma_T+\gamma(m_0)$ so that $\gamma_T=\gamma_{T'}$ by the choice of γ_T. The inclusion $X\colon\bigoplus_{j\geq1}\mathfrak{H}(m_j)\to\bigoplus_{j\geq0}\mathfrak{H}(m_j)$ is one-to-one and $\gamma_{T'}(\ker X^*)=\gamma(m_0)\neq0$.

Lemma 8.4. *For any two contractions T and T' of class C_0 we have $\sigma\Phi(T,T')^*=\sigma\Phi(T'^*,T^*)$, $\Phi(T,T')^*=\Phi(T'^*,T^*)$ and*

$$\operatorname{ind}(X^*)=-\operatorname{ind}(X)^\sim,\quad X\in\sigma\Phi(T,T') \tag{8.5}$$

(*here* $-(\gamma,\gamma')^\sim=(\gamma'^\sim,\gamma^\sim)$).

Proof. Cf. the proof of [3], Lemma 2.10.

The following Theorem extends [3], Theorem 2.11 to this more general setting.

Theorem 8.5. *Let T, T', T'' be operators of class C_0, $A\in\sigma\Phi(T',T)$, $B\in\sigma\Phi(T'',T')$. If* $\operatorname{ind}(A)+\operatorname{ind}(B)$ *makes sense we have $BA\in\sigma\Phi(T'',T')$ and*

$$\operatorname{ind}(BA)\sim\operatorname{ind}(A)+\operatorname{ind}(B). \tag{8.6}$$

Proof. We have to follow the proof of [3], Theorem 2.11, replacing weak contractions by contractions having property (P) and using Proposition 4.10 instead of [3], Proposition 2.3. Only relation (8.6) needs some comments if A and B are C_0-Fredholm. With the notation of the proof of [3], Theorem 2.11 we have

$$\gamma_T(\ker BA)=\gamma_T(\ker A)+\gamma_{T'}(\mathfrak{H}_1)\quad\text{([3], relation (2.18))}, \tag{8.7}$$

$$\gamma_{T'}(\mathfrak{H}_2)=\gamma_{T'}(\mathfrak{H}_2^*)\quad\text{([3] relation (2.20))}, \tag{8.8}$$

$$\gamma_{T''}(\ker(BA)^*)=\gamma_{T''}(\ker B^*)+\gamma_{T'}(\mathfrak{H}_1^*)\quad\text{(relation (2.18)*)}, \tag{8.9}$$

and

$$\ker B=\mathfrak{H}_1\oplus\mathfrak{H}_2,\quad \ker A^*=\mathfrak{H}_1^*\oplus\mathfrak{H}_2^*\quad\text{(relation (2.19))}. \tag{8.10}$$

We infer, with the notation $\gamma=\gamma_{T'}(\mathfrak{H}_2)=\gamma_{T'}(\mathfrak{H}_2^*)$, that

$$\gamma_T(\ker BA)+\gamma=\gamma_T(\ker A)+\gamma_{T'}(\mathfrak{H}_1)+\gamma=\gamma_T(\ker A)+\gamma_{T'}(\ker B)$$

and

$$\gamma_{T''}(\ker(BA)^*)+\gamma=\gamma_{T''}(\ker B^*)+\gamma_{T'}(\mathfrak{H}_1^*)+\gamma=\gamma_{T'}(\ker A^*)+\gamma_{T''}(\ker B^*).$$

By addition we obtain

$$\gamma_T(\ker BA)+\gamma_{T'}(\ker A^*)+\gamma_{T''}(\ker B^*)+\gamma=$$
$$=\gamma_{T''}(\ker(BA)^*)+\gamma_T(\ker A)+\gamma_{T'}(\ker B)+\gamma$$

and since $\gamma \leqq \gamma_{T'}(\ker B) \wedge \gamma_{T'}(\ker A^*)$, Lemma 6.5 (ii) implies

$$\gamma_T(\ker BA)+\gamma_{T'}(\ker A^*)+\gamma_{T''}(\ker B^*) =$$
$$= \gamma_{T''}(\ker (BA)^*)+\gamma_T(\ker A)+\gamma_{T'}(\ker B).$$

The last relation is equivalent to (8.6). The Theorem follows.

The proof of [3], Theorem 2.12 is easily extended to the general setting.

Proposition 8.6. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $X \in \{T\}'$ be such that $T|(X\mathfrak{H})^- \in \mathcal{P}$. Then $Y=I+X \in \Phi(T)$ and $(T|\ker Y)\varrho T_{\ker Y^*}$. In particular* $\operatorname{ind}(Y) \sim (0, 0)$.

Proof. We have shown in the proof of [3], Theorem 2.12 that $\ker Y = \ker (Y|\mathfrak{U})$, $\mathfrak{U}=(X\mathfrak{H})^-$, and that $(T|\mathfrak{U})_{\ker(Y|\mathfrak{U})^*}$ and $T_{\ker Y^*}$ are similar. This shows that $(T|\ker Y)\varrho T_{\ker Y^*}$.

In fact we shall prove a more general perturbation theorem.

Theorem 8.7. *Let T, T' be two operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let us take $X \in \sigma\Phi(T', T)$, $Y \in \mathcal{I}(T', T)$. If $T'|(Y\mathfrak{H})^- \in \mathcal{P}$, we have $X+Y \in \sigma\Phi(T', T)$ and*

$$\operatorname{ind}(X+Y) \sim \operatorname{ind}(X)+(\gamma, \gamma), \quad \gamma = \gamma_{T'}((Y\mathfrak{H})^-). \tag{8.11}$$

Proof. We shall prove firstly that $(X+Y)(\mathfrak{H})$ is dense in each cyclic subspace of T' contained in $((X+Y)\mathfrak{H})^-$. The same argument applied to $(X+Y)^*$ will show, via [3], Lemma 1.4, that $(X+Y)|(\ker (X+Y))^\perp$ is a lattice-isomorphism.

In proving this we may assume that $\mathfrak{H}'=X\mathfrak{H} \vee Y\mathfrak{H}$ so that $\ker X^*=(P_{\ker X^*}Y\mathfrak{H})^-$; it follows that $T'_{\ker X^*} \overset{i}{\prec} T'|(Y\mathfrak{H})^-$ so that necessarily $T'_{\ker X^*} \in \mathcal{P}$ (cf. Corollary 4.5). Analogously we may assume that $T|\ker X \in \mathcal{P}$ so that X is C_0-Fredholm.

The injection J: $\ker Y \to \mathfrak{H}$ is C_0-Fredholm, $J \in \Phi(T, T|\ker Y)$ by the assumption of the Theorem, and therefore, by Theorem 8.5, $XJ \in \Phi(T', T|\ker Y)$; in particular $T'_{\ker(XJ)^*}=T'_{\mathfrak{U}} \in \mathcal{P}$ where $\mathfrak{U}=\ker (XJ)^*=(X(\ker Y))^\perp$.

Let us take $f \in ((X+Y)\mathfrak{H})^-$ and denote $\mathfrak{H}'_f = \bigvee_{j\geqq 0} T'^j f$. Because

$$P_{\mathfrak{U}}|\mathfrak{H}'_f \in \mathcal{I}(T'_{\mathfrak{U}}, T'|\mathfrak{H}'_f)$$

and $P_{\mathfrak{U}}(X+Y) \in \mathcal{I}(T'_{\mathfrak{U}}, T)$ are such that $\operatorname{ran}(P_{\mathfrak{U}}|\mathfrak{H}'_f) \subset (\operatorname{ran} P_{\mathfrak{U}}(X+Y))^-$ we infer by Corollary 4.11 the existence of a cyclic vector g of $T'|\mathfrak{H}'_f$ such that $P_{\mathfrak{U}}g = P_{\mathfrak{U}}(X+Y)h$ for some $h \in \mathfrak{H}$. Then the difference $g'=g-(X+Y)h \in (\operatorname{ran} XJ)^- = (X(\ker Y))^-$ and because XJ is a C_0-Fredholm operator we infer the existence of $h' \in \ker Y$ such that Xh' is cyclic for $T'|\mathfrak{H}'_{g'}$. Let us denote

$$\mathfrak{H}_0 = \mathfrak{H}_h \vee \mathfrak{H}_{h'} \quad \text{and} \quad Z = (X+Y)|\mathfrak{H}_0 \in \mathcal{I}(T', T|\mathfrak{H}_0).$$

Then $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_f$; indeed, because $h' \in \ker Y$, we have $Zh' = Xh'$ and therefore $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_{Xh'} = \mathfrak{H}'_{g'}$, in particular $g' \in (Z\mathfrak{H}_0)^-$. Now $g = g' + Zh \in (Z\mathfrak{H}_0)^-$ so that $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_g = \mathfrak{H}'_f$. By Proposition 8.2 (ii) $Z \in \sigma\Phi(T', T|\mathfrak{H}_0)$ so that $\mathfrak{H}_f = (Z\mathfrak{R})^- = ((X+Y)\mathfrak{R})^-$ for some $\mathfrak{R} \in \mathrm{Lat}\,(T|\mathfrak{H}_0) \subset \mathrm{Lat}\,(T)$. The first part of the proof is done.

Let us assume that $T|\ker X \in \mathscr{P}$. Then $\ker (X+Y) \subset X^{-1}(Y\mathfrak{H})$ and

$$T|X^{-1}((Y\mathfrak{H})^-) = \begin{bmatrix} T|\ker X & * \\ 0 & T_1 \end{bmatrix}$$

where $T_1 \overset{i}{\prec} T'|(Y\mathfrak{H})^-$ so that T_1 has the property (P) (cf. Corollary 4.5). By Proposition 4.4, $T|X^{-1}((Y\mathfrak{H})^-) \in \mathscr{P}$ and therefore $T|\ker (X+Y) \in \mathscr{P}$. Analogously $T'_{\ker (X+Y)^*} \in \mathscr{P}$ if $T'_{\ker X^*} \in \mathscr{P}$ so that in any case $X+Y \in \sigma\Phi(T', T)$. Conversely, because $X = (X+Y) - Y$, $T|\ker X \in \mathscr{P}$ whenever $T|\ker (X+Y) \in \mathscr{P}$ and $T'_{\ker X^*} \in \mathscr{P}$ whenever $T'_{\ker (X+Y)^*} \in \mathscr{P}$. Therefore $\mathrm{ind}\,(X) \in \{+\infty, -\infty\}$ if and only if

$$\mathrm{ind}\,(X+Y) \in \{+\infty, -\infty\}$$

and in this case $\mathrm{ind}\,(X) = \mathrm{ind}\,(X+Y)$.

It remains to prove that (8.11) holds whenever $X \in \Phi(T', T)$. To do this let us remark that $P_{(Y\mathfrak{H})^\perp} \in \Phi(T'_{(Y\mathfrak{H})^\perp}, T')$ and $\mathrm{ind}\,(P_{(Y\mathfrak{H})^\perp}) = (\gamma, 0)$, where $\gamma = \gamma_{T'}((Y\mathfrak{H})^-)$. Because obviously $P_{(Y\mathfrak{H})^\perp}(X+Y) = P_{(Y\mathfrak{H})^\perp} X$ we infer by Theorem 8.5

$$\mathrm{ind}\,(X+Y) + (\gamma, 0) \sim \mathrm{ind}\,(P_{(Y\mathfrak{H})^\perp} X) \sim \mathrm{ind}\,(X) + (\gamma, 0) \tag{8.12}$$

so that

$$\gamma_T(\ker (X+Y)) + \gamma + \gamma_{T'}(\ker (P_{(Y\mathfrak{H})^\perp} X)^*) = \\ = \gamma_{T'}(\ker (X+Y)^*) + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp} X)$$

and

$$\gamma_T(\ker P_{(Y\mathfrak{H})^\perp} X) + \gamma_{T'}(\ker X^*) = \\ = \gamma_{T'}(\ker (P_{(Y\mathfrak{H})^\perp} X)^*) + \gamma_T(\ker X) + \gamma.$$

By addition we obtain

$$\begin{cases} \gamma_T(\ker (X+Y)) + \gamma_{T'}(\ker X^*) + \gamma + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp} X) + \gamma_{T'}(\ker (P_{(Y\mathfrak{H})^\perp} X)^*) = \\ = \gamma_{T'}(\ker (X+Y)^*) + \gamma_T(\ker X) + \gamma + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp} X) + \gamma_{T'}(\ker (P_{(Y\mathfrak{H})^\perp} X)^*). \end{cases} \tag{8.13}$$

As shown in the proof of Theorem 8.5 (cf. relations (8.8—10)) we have

$$\gamma_T(\ker P_{(Y\mathfrak{H})^\perp} X) \leqq \gamma_T(\ker X) + \gamma_{T'}((Y\mathfrak{H})^-) = \gamma_T(\ker X) + \gamma$$

and

$$\gamma_{T'}(\ker (P_{(Y\mathfrak{H})^\perp} X)^*) \leqq \gamma_{T'}(\ker X^*) + \gamma.$$

Moreover, as shown in the first part of this proof, we have $\gamma_T(\ker (X+Y)) \leqq \gamma_T(X^{-1}((Y\mathfrak{H})^-)) \leqq \gamma_T(\ker X) + \gamma$ and analogously $\gamma_{T'}(\ker X^*) \leqq \gamma_{T'}(\ker (X+Y)^*) + \gamma$.

All these relations show, via Lemma 6.5 (ii), that from (8.13) we may infer

$$\gamma_T(\ker(X+Y))+\gamma_{T'}(\ker X^*)+\gamma = \gamma_{T'}(\ker(X+Y)^*)+\gamma_T(\ker X)+\gamma.$$

The last relation is equivalent to (8.11). Theorem 8.7 is proved.

We shall prove now a partial converse of Theorem 8.5. For simplifying notations we shall consider the case of a single operator T of class C_0.

Proposition 8.8. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $A\in\{T\}'$. If there exist $B, C\in\{T\}'$ such that $AB, CA\in\Phi(T)$, we have $A\in\Phi(T)$.*

Proof. Because $\ker A\subset\ker CA$ and $\ker A^*\subset\ker(AB)^*$ we obviously have $T|\ker A$, $T_{\ker A^*}\in\mathscr{P}$. We shall now prove that the mapping $\mathfrak{K}\to(A\mathfrak{K})^-$ is onto $\mathrm{Lat}\,(T|(A\mathfrak{H})^-)$. As in the first part of the proof of Theorem 8.7 we take $f\in(A\mathfrak{H})^-$ and remark that

$$P_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}|\mathfrak{H}_f\in\mathscr{I}(T_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}, T|\mathfrak{H}_f),$$

$$P_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}A\in\mathscr{I}(T_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}, T);$$

an application of Corollary 4.11 proves the existence of a cyclic $g\in\mathfrak{H}_f$ and of a vector $h\in\mathfrak{H}$ such that $g-Ah\in(AB\mathfrak{H})^-$. Because $AB\in\Phi(T)$ we find h' such that ABh' is cyclic for $T|\mathfrak{H}_{g-Ah}$. If $\mathfrak{H}_0=\mathfrak{H}_h\vee\mathfrak{H}_{Bh'}$ we obtain as in the proof of Theorem 8.7 $(A\mathfrak{H}_0)^-\supset\mathfrak{H}_f$ and therefore $\mathfrak{H}_f=(A\mathfrak{K})^-$ for some $\mathfrak{K}\in\mathrm{Lat}\,(T|\mathfrak{H}_0)\subset\mathrm{Lat}\,(T)$.

Analogously we can show, using the operator $A^*C^*\in\Phi(T^*)$, that the mapping $\mathfrak{K}\to(A^*\mathfrak{K})^-$ is onto $\mathrm{Lat}\,(T^*|(A^*\mathfrak{H})^-)$. By [3], Lemma 1.4, Proposition 8.8 follows.

Example 8.9. *For each pair $(\gamma,\gamma')\in\tilde{\Gamma}\times\tilde{\Gamma}$ there exist a C_0-operator T and $X\in\Phi(T)$ such that $\mathrm{ind}\,(X)=(\gamma,\gamma')$.*

Proof. As in the proof of [3], Proposition 3.1, we take operators $K, K'\in\mathscr{P}$ such that $\gamma_K=\gamma$, $\gamma_{K'}=\gamma'$ and we define $T=(K\otimes I)\oplus(K'\otimes I)$, where I denotes the identity on H^2. If U_+ denotes the unilateral shift on H^2, the required C_0-Fredholm operator is given by

$$X=(I\otimes U_+^*)\oplus(I\otimes U_+).$$

The proof of [3], Proposition 3.4, can be applied to obtain the following result.

Proposition 8.10. *For each operator T of class C_0 we have $\sigma\Phi(T)\cap\{T\}''=\Phi(T)\cap\{T\}''$ and $\mathrm{ind}\,(X)\sim(0,0)$ for $X\in\Phi(T)\cap\{T\}''$.*

The operators X_n, X defined in the proof of [3], Proposition 3.6, are such that $X_n\notin\sigma\Phi(T)$, $X\in\Phi(T)$, and $\lim_{n\to\infty}\|X_n-X\|=0$. Thus we have the following result.

Proposition 8.11. *The sets $\sigma\Phi(T)$, $\Phi(T)$ are not generally open subsets of $\{T\}'$, for T an operator of class C_0.*

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