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On the Jordan model of C_0 operators. II

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The existence of the Jordan model for operators of class C_0 was established in [9] and [10] for operators of finite multiplicity, in [4] for operators acting on separable Hilbert spaces and in [2] for operators acting on nonseparable spaces. In Sec. 2 of this note we give a common description of these three types of Jordan models. We also find a direct definition of the inner functions appearing in the Jordan model.

B. Sz.-Nagy and C. Foiaş have shown in [9], Sec. 7, that the space $\mathfrak{H}$ on which an operator T of class $C_0(N)$ is acting admits a decomposition into an approximate sum of invariant subspaces $\mathfrak{H}_j$ for T such that $T|\mathfrak{H}_j$ is multiplicity-free. In Sec. 3 of this note we extend this result to operators of class C_0 of arbitrary multiplicity. In fact we prove the existence of an almost-direct decomposition (cf. Theorem 3.4). Moreover, in the case of weak contractions (which contains the case discussed in [9]) we show that there exists a quasi-direct decomposition (cf. [7], ch. III). The main ingredient in Sec. 3 is a generalization of [4], Proposition 2.

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1. Preliminaries

We begin with some known facts about cardinal and ordinal numbers (cf. [12]). Here 0 is considered as ordinal number so that each ordinal α is the ordering type of the well-ordered set of ordinals $\{\beta: \beta<\alpha\}$. An ordinal number is a limit ordinal if it has no predecessor. Each ordinal number is of the form $\alpha+n$ with α a limit ordinal and $n<\omega$, where ω is the first transfinite ordinal. For each ordinal number α we denote by $\bar{\alpha}$ the associated cardinal number.

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Lemma 1.1. *For each cardinal number* $\aleph$ *we have* $\aleph=\text{card}\,\{\alpha\colon\ \bar{\alpha}<\aleph\}$.

Proof. Let us denote $A=\{\alpha\colon\ \bar{\alpha}<\aleph\}$ and let β be the ordinal number corresponding to A. Then $\bar{\beta}=\text{card}\,A$ and $\beta\notin A$ so that $\bar{\beta}=\text{card}\,A\geqq\aleph$. Now let γ be the first ordinal number such that $\bar{\gamma}=\aleph$; then $\gamma\notin A$ so that $\gamma\geqq\beta$ and therefore $\aleph=\bar{\gamma}\geqq\bar{\beta}=\text{card}\,A$. The Lemma follows by the Cantor—Bernstein theorem.

Remark 1.2. The preceding proof shows that $\beta=\gamma=$the first ordinal with $\bar{\beta}=\aleph$.

Corollary 1.3. *If* $\aleph_1<\aleph_2$ *are cardinal numbers and* $\aleph_2$ *is transfinite, we have* $\aleph_2=\text{card}\,\{\alpha\colon\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}$.

Proof. By Lemma 1.1 we have $\aleph_2=\text{card}\,\{\alpha\colon\ \bar{\alpha}<\aleph_2\}=\text{card}\,\{\alpha\colon\ \bar{\alpha}<\aleph_1\}+$ $+\text{card}\,\{\alpha\colon\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}=\aleph_1+\aleph$, where $\aleph=\text{card}\,\{\alpha\colon\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}$. Because $\aleph_2$ is transfinite $\aleph_1$ or $\aleph$ must be transfinite and we have $\aleph_2=\max\,\{\aleph_1,\aleph\}=\aleph$ because $\aleph_1\neq\aleph_2$. The Corollary is proved.

Corollary 1.4. *If* $\aleph$ *is a transfinite cardinal number then* $\aleph'=\text{card}\,\{\alpha\colon\ \bar{\alpha}=\aleph\}$ *is the first cardinal greater than* $\aleph$.

Proof. We have only to apply the preceding Corollary for $\aleph_1=\aleph$ and $\aleph_2=$ $=$the successor of $\aleph$ in the series of cardinal numbers.

Now let us recall that the multiplicity μ_T of the operator T acting on the Hilbert space $\mathfrak{H}$ is the minimum dimension of a subspace $\mathfrak{M}\subset\mathfrak{H}$ such that $\mathfrak{H}=\bigvee_{n\geqq 0}T^n\mathfrak{M}$. It is obvious that

$$\mu_T\leqq\dim\mathfrak{H}\leqq\aleph_0\cdot\mu_T \tag{1.1}$$

so that the equality

$$\mu_T=\dim\mathfrak{H} \tag{1.2}$$

holds whenever $\dim\mathfrak{H}>\aleph_0$ or $\mu_T\geqq\aleph_0$.

Lemma 1.5. *We have* $\mu_T=\mu_{T^*}$ *for any operator* T *of class* C_0.

Proof. For $\mu_T<\aleph_0$ see [10], Theorem 3. Therefore if $\mu_T\geqq\aleph_0$ we also have $\mu_{T^*}\geqq\aleph_0$ and the equality $\mu_T=\mu_{T^*}$ follows from (1.2).

Let us recall that the operator T can be injected into T' ($T\overset{i}{\prec}T'$) if there exists an injection X such that $T'X=XT$. If there exists a quasi-affinity X such that $T'X=XT$ we say that T is a quasi-affine transform of T' ($T\prec T'$).

Lemma 1.6. *If* T *and* T' *are two operators of class* C_0 *and* $T\overset{i}{\prec}T'$, *we have* $\mu_T\leqq\mu_{T'}$. *If* $T\prec T'$ *then* $\mu_T=\mu_{T'}$.

Proof. Let T, T' be acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let X be any injection such that $T'X=XT$. Then X^* has dense range; if $\mathfrak{M}\subset\mathfrak{H}'$ is such that $\bigvee_{n\geqq 0} T'^{*n}\mathfrak{M}=\mathfrak{H}'$ we have $\bigvee_{n\geqq 0} T^{*n}X^*\mathfrak{M}=\mathfrak{H}$ and obviously $\dim (X^*\mathfrak{M})^-\leqq\dim\mathfrak{M}$. Therefore $\mu_{T*}\leqq\mu_{T'*}$ so that $\mu_T\leqq\mu_{T'}$ by Lemma 1.5. If $T\prec T'$, we may assume X has dense range so that $\mu_{T'}\leqq\mu_T$ obviously also follows. The Lemma is proved.

If T is an operator of class C_0 we shall use the notation

$$\mu_T(m)=\mu_{T|(\operatorname{ran} m(T))^-},\quad m\in H_i^\infty \tag{1.3}$$

where H_i^∞ denotes the set of inner functions in H^∞. We shall consider the set H_i^∞ (pre)ordered as in [2]. Namely, we write $m_1\leqq m_2$ if m_1 divides m_2 or, equivalently, if $|m_1(z)|\geqq|m_2(z)|$ for $|z|<1$.

The following Lemma also follows from [8], Theorem III.6.3; we prove it for the sake of completeness.

Lemma 1.7. *If T is an operator of class C_0 and $m_1, m_2\leqq m_T$, then $(\operatorname{ran} m_1(T))^-\subset(\operatorname{ran} m_2(T))^-$ if and only if $m_1\geqq m_2$.*

Proof. If $m_1\geqq m_2$, we have $m_1=m_2m_3$ so that obviously $\operatorname{ran} m_1(T)\subset\operatorname{ran} m_2(T)$. Conversely, if $(\operatorname{ran} m_1(T))^-\subset(\operatorname{ran} m_2(T))^-$, we have $(m_T/m_2)(T)m_1(T)=0$ and therefore $m_T\leqq(m_T/m_2)m_1$. The Lemma follows.

Corollary 1.8. *The function μ_T is decreasing on H_i^∞.*

Proof. Obviously follows from Lemma 1.6 and the proof of Lemma 1.7.

Corollary 1.9. *If T and T' are operators of class C_0 and $T\overset{i}{\prec}T'$, we have $\mu_T(m)\leqq\mu_{T'}(m)$, $m\in H_i^\infty$. If $T\prec T'$, we have $\mu_T(m)=\mu_{T'}(m)$, $m\in H_i^\infty$.*

Proof. If X is any injection such that $T'X=XT$, we also have $m(T')X=Xm(T)$, $m\in H_i^\infty$, and therefore $T|(\operatorname{ran} m(T))^-\overset{i}{\prec}T'|(\operatorname{ran} m(T'))^-$. If X is a quasi-affinity we have $(X\operatorname{ran} m(T))^-=(\operatorname{ran} m(T'))^-$ so that $T|(\operatorname{ran} m(T))^-\prec T'|(\operatorname{ran} m(T'))^-$. The Corollary follows by Lemma 1.6.

We shall see that the converse of Corollary 1.9 is also true.

Let us recall that for an operator T of class C_0 acting on $\mathfrak{H}$ and for $f\in\mathfrak{H}$, m_f stands for the minimal function of $T|\mathfrak{H}_f$, where

$$\mathfrak{H}_f=\bigvee_{n\geqq 0} T^n f. \tag{1.4}$$

The following result is proved in [4], Proposition 1.

Proposition 1.10. *The set $\{f\colon m_f=m_T\}$ is dense in $\mathfrak{H}$.*

In fact, from the proof of [10], Theorem 1, it follows that $\{f: m_f = m_T\}$ is a dense G_δ.

Finally let us recall the definition of approximate sums and quasi-direct sums (cf. [6] and [5], ch. III). Let $\mathfrak{H}$ be a Hilbert space and $\{\mathfrak{H}_j\}_{j\in J}$ be a family of subspaces of $\mathfrak{H}$ such that

$$\mathfrak{H} = \bigvee_{j\in J} \mathfrak{H}_j. \tag{1.5}$$

We say that $\mathfrak{H}$ is the *approximate sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if for each subset $K\subset J$ we have

$$(\bigvee_{j\in K} \mathfrak{H}_j)\cap(\bigvee_{j\notin K} \mathfrak{H}_j) = \{0\}. \tag{1.6}$$

We say that H is the *quasi-direct sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if for each family $\{K_a\}_{a\in A}$ of subsets of J we have

$$\bigcap_{a\in A} (\bigvee_{j\in K_a} \mathfrak{H}_j) = \bigvee_{j\in K} \mathfrak{H}_j, \quad K = \bigcap_{a\in A} K_a. \tag{1.7}$$

We shall introduce an intermediate notion. Namely, we shall say that $\mathfrak{H}$ is the *almost-direct sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if the relation (1.7) holds whenever $K=\emptyset$.

Lemma 1.11. *Let $\{\mathfrak{H}_j\}_{j\in J}$ be a family of subspaces of $\mathfrak{H}$ such that* (1.5) *holds. $\mathfrak{H}$ is the almost-direct sum of $\{\mathfrak{H}_j\}_{j\in J}$ if and only if we have*

$$\mathfrak{H} = \bigvee_{j\in J} \mathfrak{H}_j^*, \quad \textit{where} \quad \mathfrak{H}_j^* = (\bigvee_{k\neq j} \mathfrak{H}_k)^\perp, \quad j\in J. \tag{1.8}$$

Proof. If $\mathfrak{H}$ is the almost-direct sum of $\{\mathfrak{H}_j\}_{j\in J}$, we have

$$\bigvee_{j\in J} \mathfrak{H}_j^* = \bigvee_{j\in J} (\bigvee_{k\neq j} \mathfrak{H}_k)^\perp \supset (\bigcap_{j\in J} (\bigvee_{k\neq j} \mathfrak{H}_k))^\perp = (\{0\})^\perp = \mathfrak{H}.$$

Conversely, if (1.8) holds and $\{K_a\}_{a\in A}$ are such that $\bigcap_{a\in A} K_a=\emptyset$, then

$$(\bigcap_{a\in A} (\bigvee_{j\in K_a} \mathfrak{H}_j))^\perp \supset \bigvee_{a\in A} (\bigvee_{j\in K_a} \mathfrak{H}_j)^\perp \supset \bigvee_{a\in A} (\bigvee_{j\notin K_a} \mathfrak{H}_j^*)$$

and because $\bigcup_{a\in A} \{j: j\notin K_a\}=J$, we have $\bigvee_{a\in A} (\bigvee_{j\notin K_a} \mathfrak{H}_j^*)=\bigvee_{j\in J} \mathfrak{H}_j^*=\mathfrak{H}$. The Lemma follows.

2. Jordan models

Definition 2.1. A *model function* is a function M which associates with every ordinal number α an inner function $M(\alpha)$ such that

(i) $M(\beta) \leqq M(\alpha)$ whenever $\bar{\alpha} \leqq \bar{\beta}$;

(ii) $M(\alpha) = M(\beta)$ whenever $\bar{\alpha} = \bar{\beta}$;

(iii) $M(\alpha) = 1$ for some α.

If M is a model function, the *operator* $S(M)$ acting on $\mathfrak{H}(M)$ is defined as

$$S(M) = \bigoplus_{\alpha} S(m_\alpha), \quad m_\alpha = M(\alpha). \tag{2.1}$$

Lemma 2.2. *Let* $\{m_a\}_{a\in A}\subset H_i^\infty$ *be a totally ordered family of nonconstant functions. Then the multiplicity of* $T=\bigoplus_{a\in A} S(m_a)$ *equals* card A.

Proof. If A is finite, the assertion follows from [9]. If A is infinite, it follows from the inequality $\mu_{T'\oplus T''}\geqq\mu_{T'}$ that μ_T is also infinite so that $\mu_T=\dim\left(\bigoplus_{a\in A}\mathfrak{H}(m_a)\right)$ by (1.2). Therefore, card $A\leqq\mu_T\leqq$card $A\cdot\aleph_0=$card A. The Lemma follows.

Corollary 2.3. *If M is a model function, we have* $\mu_{S(M)}=\bar{\alpha}$, *where* α *is the first ordinal number such that* $m_\alpha=1$.

Proof. If α is the first ordinal number with $m_\alpha=1$, it follows from Definition 2.1 (ii) that $\{\beta\colon m_\beta\neq 1\}=\{\beta\colon \bar{\beta}<\bar{\alpha}\}$ so that the Corollary follows by Lemmas 1.1 and 2.2.

Definition 2.4. For any operator T of class C_0 we define

$$M_T(\alpha) = m_\alpha[T] = \wedge\{m\colon \mu_T(m)\leqq\bar{\alpha}\} \tag{2.2}$$

where "$\wedge$" stands for the greatest common inner divisor.

Let us remark that $M_T(0)=m_0[T]$ coincides with the minimal function of T. M_T is a model function. Indeed, the conditions (i) and (ii) of Definition 2.1 are obviously satisfied while (iii) is satisfied because $M_T(\alpha)=1$ whenever $\bar{\alpha}=\dim\mathfrak{H}$ ($\mu_T(1)=\mu_T\leqq\dim\mathfrak{H}$ by (1.1)). It is also clear by Corollary 1.9 that M_T is invariant with respect to quasi-affine transforms.

Proposition 2.5. *If M is a model function we have* $M_{S(M)}=M$.

Proof. Let us put $T=S(M)$, $M'=M_T$, $m_\alpha=M(\alpha)$ and $m'_\alpha=M'(\alpha)$. Let us assume $m\geqq m_\beta$. Because $m(S(m'))=0$ if and only if $m\geqq m'$ (moreover, $S(m')|(\operatorname{ran} m(S(m')))^-$ is quasisimilar to $S(m'/m\wedge m')$), by Lemma 2.2 we have

$$\mu_T(m)\leqq\mu_T(m_\beta)\leqq\text{card}\,\{\alpha;\ \bar{\alpha}<\beta\}=\bar{\beta}.$$

Conversely, let us assume m not $\geqq m_\beta$. Then $\mu_T(m)\geqq$card $\{\alpha\colon\bar{\alpha}\leqq\bar{\beta}\}>\bar{\beta}$. By (2.2) we infer $m'_\beta=m_\beta$ and the Proposition is proved.

Now let us recall the definition of a Jordan operator (cf. [2]). If $\aleph$ is a cardinal number and T is an operator, $T^{(\aleph)}$ denotes the direct sum of $\aleph$ copies of T.

Definition 2.6. A *Jordan operator*, is an operator of the form

$$T = \bigoplus_{m\in H_i^\infty} S(m)^{(h(m))} \tag{2.3}$$

where h is a cardinal number valued function on H_i^∞ such that
(i) $A=\{m: h(m)\neq 0\}$ is a well anti-ordered set;
(ii) $\{m\in A: h(m)<\aleph_0\}$ is a decreasing (possibly finite or empty) sequence;
(iii) $h(m)>\sum_{m'>m} h(m')$ whenever $\sum_{m'>m} h(m')\geqq\aleph_0$.

Our condition (iii) slightly differs from condition (b) of [2], Definition 1. If we analyse the proof of [2], Theorem 1, we remark that the Jordan model obtained there satisfies the actual condition (iii). Indeed, if $h(m)=\sum_{m'>m} h(m')$ it is easy to see that (with the notation of [2]) m is not a saltus point for f.

Let us remark that, by Lemma 2.2, we have

$$\mu_T(u)=\sum_{u \text{ not } \geqq m} h(m), \quad u\in H_i^\infty \tag{2.4}$$

if T is the operator given by (2.3).

Theorem 2.7. *Each operator T of class C_0 is quasisimilar to $S(M_T)$.*

Proof. From Corollary 1.9 it follows that M_T is a quasisimilarity invariant Therefore, by [2], Theorem 1, it is enough to prove that for T a Jordan operator in the sense of Definition 2.6, T and $S(M_T)$ are unitarily equivalent. So, let T be given by (2.3) and denote $m_\alpha=M_T(\alpha)$. It is enough to prove that

$$\operatorname{card}\{\alpha;\ m_\alpha=m\}=h(m), \quad m\in H_i^\infty. \tag{2.5}$$

Let us assume firstly that $h(m)=0$. There exists a last $m^1\in A=\{m': h(m')\neq 0\}$ such that $m^1\geqq m\wedge m_T$. Thus for $m'\in A$ we have $m(S(m'))=0$ if and only if $m^1(S(m'))=0$. By Lemma 2.2 we infer $\mu_T(m)=\mu_T(m^1)$ so that by (2.2) there is no α such that $m_\alpha=m$ and (2.5) is proved in this case.

Now let us assume $0<h(m)<\aleph_0$. Then the sum

$$k=\sum_{m'>m} h(m') \tag{2.6}$$

is finite by Definition 2.6 (iii). It is clear that $\mu_T(u)\leqq k$ if and only if $u\geqq m$ and therefore if and only if $\mu_T(u)\leqq k+n-1$, $n=h(m)$. We obtain

$$m_k=m_{k+1}=\ldots=m_{k+n-1}=m.$$

Analogously we obtain $m_{k+n}=m'$ where m' is the predecessor of m in A; thus $\{\alpha: m_\alpha=m\}=\{k, k+1, \ldots, k+n-1\}$ and (2.5) is proved in this case also.

Finally let us assume $h(m)\geqq\aleph_0$. If $k\leqq\bar{\alpha}<h(m)$, where k is defined by (2.6), we have $\mu_T(u)\leqq\bar{\alpha}$ if and only if $u\geqq m$. Indeed, if u not $\geqq m$, we have $\mu_T(u)\geqq h(m)$ by Lemma 2.2. Therefore

$$m_\alpha=m \quad \text{whenever} \quad k\leqq\bar{\alpha}<h(m). \tag{2.7}$$

If $\bar{\alpha} \geqq h(m)$ and m' is the predecessor of m in A (if m is the first element of A we take $m'=1$) then, again by Lemma 2.2, $\mu_T(m') = \sum_{m''>m'} h(m'') = \sum_{m''>m} h(m'') + h(m) = h(m)$ so that $m_\alpha \neq m$. Therefore

$$\{\alpha;\ m_\alpha = m\} = \{\alpha;\ k \leqq \bar{\alpha} < h(m)\}$$

and (2.5) follows by Corollary 1.4 in this case. The Theorem is proved.

Let us recall that $f^{\sim}(z) = \overline{f(\bar{z})}$ for $f \in H^\infty$.

Corollary 2.8. *For each operator T of class C_0 we have $\mu_T(m) = \mu_{T^*}(m^{\sim})$, $m \in H_i^\infty$ and $m_\alpha[T^*] = m_\alpha[T]^{\sim}$ for each ordinal number α.*

Proof. Since $\mu_T(m)$ is a quasisimilarity invariant it is enough to prove the Corollary for $T = S(M)$ and in this case the assertions of the Corollary become obvious.

We are now able to prove the converse of Corollary 1.9.

Corollary 2.9. *For two operators T, T' of class C_0 the following assertions are equivalent:*

(i) $T \overset{i}{\prec} T'$;

(i)$_*$ $T^* \overset{i}{\prec} T'^*$;

(ii) $\mu_T(m) \leqq \mu_{T'}(m)$, $m \in H_i^\infty$;

(iii) $m_\alpha[T] \leqq m_\alpha[T']$ *for each ordinal number* α.

Proof. (i)$\Rightarrow$(ii) by Corollary 1.9. (ii)$\Rightarrow$(iii) by Definition 2.4.

(iii)$\Rightarrow$(i). Let us denote $m_\alpha = m_\alpha[T]$, $m'_\alpha = m_\alpha[T']$. There exist (cf. [9]) isometries R_α: $\mathfrak{H}(m_\alpha) \to \mathfrak{H}(m'_\alpha)$ such that $S(m'_\alpha) R_\alpha = R_\alpha S(m_\alpha)$. If X and Y are two quasi-affinities such that $T'X = XS(M_{T'})$ and $S(M_T)Y = YT$, the operator $Z = X(\bigoplus_\alpha R_\alpha)Y$ is an injection and $T'Z = ZT$.

Finally, the condition $m_\alpha[T] \leqq m_\alpha[T']$ is equivalent to $m_\alpha[T^*] \leqq m_\alpha[T'^*]$ by Corollary 2.8; it follows that the condition (i)$_*$ is equivalent with (i)—(iii). The Corollary is proved.

The following Corollary gives in particular a new proof of [11], Theorem 1.

Corollary 2.10. *For two operators T, T' of class C_0 the following assertions are equivalent:*

(i) $T \prec T'$;

(ii) $T \overset{i}{\prec} T'$ *and* $T' \overset{i}{\prec} T$;

(iii) $\mu_T(m) = \mu_{T'}(m)$, $m \in H_i^\infty$;

(iv) *T and T' are quasisimilar.*

Proof. (i)⇒(iii) and (ii)⇒(iii) by Corollary 1.9. (iii)⇒(iv). By Definition 2.4 we infer $m_\alpha[T]=m_\alpha[T']$ so that T and T' are quasisimilar having the same Jordan model. (iv)⇒(i) and (iv)⇒(ii) are obvious.

Corollary 2.11. *If T is an operator of class C_0 on the Hilbert space $\mathfrak{H}$ then each invariant subspace $\mathfrak{M}$ of T is of the form $\mathfrak{M}=(X\mathfrak{H})^-=\ker Y$ for some $X, Y\in\{T\}'$.*

Proof. Let us denote by T' the restriction $T|\mathfrak{M}$ and by J the inclusion of $\mathfrak{M}$ into $\mathfrak{H}$. By Corollary 2.9 we have $T'^* \overset{i}{\prec} T^*$ so that there exists an injection $Z\colon \mathfrak{M}\to\mathfrak{H}$ such that $T^*Z=ZT'^*$. Then $X=JZ^*\in\{T\}'$ and $(X\mathfrak{H})^-=J(Z^*\mathfrak{H})^-= =J\mathfrak{M}=\mathfrak{M}$. Analogously $\mathfrak{M}^\perp=(Y^*\mathfrak{H})^-$ for some $Y^*\in\{T^*\}'$ so that $\mathfrak{M}=\ker Y$. The Corollary follows.

As shown by Proposition 2.5 and Theorem 2.7 the operators of the form $S(M)$ with M a model function form a complete system of representants for the class C_0 with respect to the relation of quasisimilarity. Sometimes it is more convenient to use Jordan operators as given by Definition 2.6.

Proposition 2.12. *If M is a model function and*

(2.8) $$h(m) = \text{card}\,\{\alpha;\, m_\alpha = m\},\quad m\in H_i^\infty,$$

then the function h satisfies the conditions (i)—(iii) *of Definition* 2.6.

Proof. (i) $A=\{m\colon h(m)\neq 0\}$ is the range of the decreasing function M defined on a well-ordered set so that obviously A is well anti-ordered.

(ii) If $h(m)<\aleph_0$ we infer $m\neq m_\alpha$ for $\alpha\geqq\omega$. Therefore $\{m\colon 0<h(m)<\aleph_0\}$ is the range of the function M on a segment of the natural numbers.

(iii) Let us assume $h(m)\geqq\aleph_0$ and let α be the first ordinal number such that $m_\alpha=m$. By Lemma 1.1 $\bar\alpha=\sum_{m'>m} h(m')$. If α is a finite number, the relation $h(m)>\bar\alpha$ is obvious. If α is transfinite we infer by Corollary 1.4 and Definition 2.1 (ii)

$$h(m)\geqq \text{card}\,\{\beta;\, \bar\beta=\bar\alpha\}=\bar\alpha'>\bar\alpha=\sum_{m'>m} h(m'),$$

where $\bar\alpha'$ is the successor of $\bar\alpha$ in the series of cardinal numbers. The Proposition is proved.

From now on we shall call *Jordan operators* the operators $S(M)$ with M a model function and $S(M_T)$ will be called the *Jordan model of the operator T* of class C_0.

Remark 2.13. *For any operator T of class C_0 we have*

(2.9) $$\mu_T(m_\alpha[T])\leqq\bar\alpha.$$

Indeed, we have only to verify (2.9) for $T=S(M)$ and in this case (2.9) is obvious.

3. Decomposition theorems

The following Lemma is essentially contained in [9], sec. 2. We prove it for the sake of completeness. Let us remark that Lemma 3.1 also follows from [11], Theorem 2.

Lemma 3.1. *Let T and T' be operators of class C_0, both quasisimilar to $S(m)$ ($m\in H_i^\infty$) and let A be such that $T'A=AT$. Then A is one-to-one if and only if it has dense range.*

Proof. Let X and Y be two quasi-affinities such that $TX=XS(m)$ and $S(m)Y=YT'$. The operator YAX commutes with $S(m)$ so that $YAX=u(S(m))$ for some $u\in H^\infty$ by Sarason's Theorem [7]. If A is one to one or has dense range then so does $u(S(m))$ and therefore $u\wedge m=1$. Now

$$XYAXY = Xu(S(m))Y = u(T)XY = XYu(T')$$

so that $XYA=u(T)$ and $AXY=u(T')$. $u(T)$ and $u(T')$ are quasi-affinities because $u\wedge m=1$ and $\operatorname{ran} A\supset\operatorname{ran} u(T')$, $\ker A\subset\ker u(T)$ so that A is a quasi-affinity in both cases.

The following result is a generalisation of [4], Proposition 2.

Proposition 3.2. *Let T and T' be two operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, X be a quasi-affinity such that $T'X=XT$, $f\in\mathfrak{H}$ be such that $m_f=m_T$ and $\varepsilon>0$. Then there exist subspaces $\mathfrak{H}_1$, $\mathfrak{M}_1$ invariant for T and $\mathfrak{H}_1^*$, $\mathfrak{M}_1^*$ invariant for T'^* such that:*

(i) $\mathfrak{H}_1=\mathfrak{H}_f$;

(ii) $\|P_{\mathfrak{H}_1^*}Xf-Xf\|<\varepsilon$;

(iii) $\mathfrak{M}_1=(X^*\mathfrak{H}_1^*)^\perp$, $\mathfrak{M}_1^*=(X\mathfrak{H}_1)^\perp$;

(iv) $\mathfrak{H}_1\vee\mathfrak{M}_1=\mathfrak{H}$, $\mathfrak{H}_1\cap\mathfrak{M}_1=\{0\}$, $\mathfrak{H}_1^*\vee\mathfrak{M}_1^*=\mathfrak{H}'$, $\mathfrak{H}_1^*\cap\mathfrak{M}_1^*=\{0\}$;

(v) $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ *and* $P_{\mathfrak{M}_1^*}X|\mathfrak{M}_1$ *are quasi-affinities.*

Proof. The conditions (i)—(v) are not independent. Indeed, let us assume that (i) and (iii) are verified and $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity. It follows that $T'|(X\mathfrak{H}_1)^-$ and $(T^*|\mathfrak{H}_1^*)^*$ are both quasisimilar to $S(m_T)$ and $P_{\mathfrak{H}_1^*}|(X\mathfrak{H}_1)^-$ has dense range; by Lemma 3.1 $P_{(X\mathfrak{H}_1)^-}|\mathfrak{H}_1^*$ also has dense range, that is $(X\mathfrak{H}_1)^-=(P_{(X\mathfrak{H}_1)^-}\mathfrak{H}_1^*)^-$. Then $\mathfrak{M}_1=\ker P_{\mathfrak{H}_1^*}X$ so that $\mathfrak{H}_1\cap\mathfrak{M}_1=\ker P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1=\{0\}$. Analogously $\mathfrak{H}_1^*\cap\mathfrak{M}_1^*=\{0\}$. Now $\mathfrak{H}'=(X\mathfrak{H}_1)^-\oplus\mathfrak{M}_1^*=(P_{(X\mathfrak{H}_1)^-}\mathfrak{H}_1^*)\vee\mathfrak{M}_1^*=\mathfrak{H}_1^*\vee\mathfrak{M}_1^*$ and analogously $\mathfrak{H}_1\vee\mathfrak{M}_1=\mathfrak{H}$. Obviously $\mathfrak{M}_1^*=(P_{\mathfrak{M}_1^*}X\mathfrak{H})^-=(P_{\mathfrak{M}_1^*}X\mathfrak{M}_1)^-$ and $\mathfrak{M}_1=(P_{\mathfrak{M}_1}X^*\mathfrak{H}')^-=(P_{\mathfrak{M}_1}X^*\mathfrak{M}_1^*)^-$ and it follows that $P_{\mathfrak{M}_1^*}X|\mathfrak{M}_1$ is a quasi-affinity.

It follows by the preceding remark that it will be enough to define $\mathfrak{H}_1$ by (i), to find $\mathfrak{H}_1^*$ satisfying (ii) and such that $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity and then to define $\mathfrak{M}_1$, $\mathfrak{M}_1^*$ by (iii).

The operator $T'|(X\mathfrak{H}_1)^-$ has the cyclic vector Xf so that by [10], Theorem 2, $(T'|(X\mathfrak{H}_1)^-)^*$ has a cyclic vector k. Moreover, by Proposition 1.10, the set of cyclic vectors of $(T'|(X\mathfrak{H}_1)^-)^*$ is dense in $(X\mathfrak{H}_1)^-$ so that we may assume

$$\|k - Xf\| < \varepsilon. \tag{3.1}$$

We define $\mathfrak{H}_1^* = \bigvee_{n\geq 0} T'^{*n}k$ so that $k \in \mathfrak{H}_1^*$ and (ii) is verified by (3.1). Let us compute the minimal function m of $(T'^*|\mathfrak{H}_1^*)^*$. Obviously m divides $m_{T'} = m_T$. Now the operator $Y = P_{(X\mathfrak{H}_1)^-}|\mathfrak{H}_1^*$ satisfies the relation

$$(T'|(X\mathfrak{H}_1)^-)^* Y = YT'^*|\mathfrak{H}_1^* \tag{3.2}$$

and $\operatorname{ran} Y \ni k$; it follows that Y has dense range and from (3.2) we infer $m^\sim((T'|(X\mathfrak{H}_1)^-)^*)Y = Ym^\sim(T^*|\mathfrak{H}_1^*) = 0$ so that $m_{T'|(X\mathfrak{H}_1)^-} = m_T$ divides m. Because $(T'|(X\mathfrak{H}_1)^-)^*$ and $T'^*|\mathfrak{H}_1^*$ are both quasisimilar to $S(m_T)$ we infer by Lemma 3.1 that Y is a quasi-affinity. In particular, $Y^*X|\mathfrak{H}_1 = P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity. Proposition 3.3 follows.

Lemma 3.3. *Let T be an operator of class C_0 acting on $\mathfrak{H}$, let $S(M)$ be the Jordan model of T and let $\mathfrak{H}'(\subset\mathfrak{H})$ be a separable space. Then there exists a reducing subspace $\mathfrak{H}_0$ for T such that $T|\mathfrak{H}_0$ is quasisimilar to $\bigoplus_{j<\omega} S(m_j)$ $(m_j = M(j))$ and $\mathfrak{H}_0 \supset \mathfrak{H}'$.*

Proof. Let X be any quasi-affinity such that

$$TX = XS(M). \tag{3.3}$$

We shall denote by $\mathfrak{H}_0$ the least reducing subspace of T containing $\mathfrak{H}'$ and $X(\bigoplus_{j<\omega} \mathfrak{H}(m_j))$. The space $\mathfrak{H}_0$ is separable; let $\bigoplus_{j<\omega} S(m_j')$ be the Jordan model of $T|\mathfrak{H}_0$. We have $m_j' \leq m_j$ by Corollary 2.9. Because $\mathfrak{H}_0 \supset (X(\bigoplus_{j<\omega} \mathfrak{H}(m_j)))^-$ we have:

$$\bigoplus_{j<\omega} S(m_j) \overset{i}{\prec} T|\mathfrak{H}_0 \tag{3.4}$$

and therefore $m_j \leq m_j'$ again by Corollary 2.9. Therefore $m_j = m_j'$ and the Lemma follows.

Theorem 3.4. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. We can associate with each limit ordinal α a reducing subspace $\mathfrak{H}_\alpha$ for T such that:*

(i) $\mathfrak{H} = \bigoplus_{\alpha} \mathfrak{H}_\alpha$;

(ii) *$T|\mathfrak{H}_\alpha$ is quasisimilar to* $\bigoplus_{j<\omega} S(m_{\alpha+j})$.

Proof. Let X be as in the preceding proof. We shall construct by transfinite induction reducing subspaces $\mathfrak{H}_\alpha$ for each limit ordinal α such that:

(3.5) $$\bigoplus_{\alpha<\beta} H_\alpha \supset X(\bigoplus_{\alpha<\beta}\bigoplus_{j<\omega} \mathfrak{H}(m_{\alpha+j}));$$

(3.6) $$T|\mathfrak{H}_\alpha \text{ is quasisimilar to } \bigoplus_{j<\omega} S(m_{\alpha+j}).$$

Let $\mathfrak{H}_0$ be given by Lemma 3.3 (with $\mathfrak{H}'=(X(\bigoplus_{j<\omega}\mathfrak{H}(m_j))^-)$ and assume $\mathfrak{H}_\alpha$ are defined for $\alpha<\beta$. Let us denote:

(3.7) $$\mathfrak{L}=\bigoplus_{\alpha<\beta}\mathfrak{H}_\alpha, \quad \mathfrak{R}=\mathfrak{H}\ominus\mathfrak{L}.$$

Then $\mathfrak{R}$ reduces T; let us denote by $S(M')$ the Jordan model of $T|\mathfrak{R}$. From the condition (3.5) we infer $X^*(\mathfrak{R})\subset\bigoplus_{\gamma\geqq\beta}\mathfrak{H}(m_\gamma)$ and therefore:

(3.8) $$T^*|\mathfrak{R} \overset{i}{\prec} \bigoplus_{\gamma} S(m_{\beta+\gamma})^*.$$

By Corollary 2.9 we infer:

(3.9) $$M'(\gamma)\leqq m_{\beta+\gamma}.$$

By Theorem 2.7 and Definition 2.2 we have for any ordinal γ:

(3.10) $$m_{\beta+\gamma}=\bigwedge\{m\colon \mu_T(m)\leqq\overline{\beta+\gamma}\}=$$
$$=\bigwedge\{m\colon \mu_{(T|\mathfrak{R})\oplus(T|\mathfrak{L})}(m)\leqq\overline{\beta+\gamma}\}.$$

Now,

(3.11) $$\mu_{(T|\mathfrak{R})\oplus(T|\mathfrak{L})}(m)\leqq\mu_{T|\mathfrak{R}}(m)+\mu_{T|\mathfrak{L}}\leqq$$
$$\leqq\mu_{T|\mathfrak{R}}(m)+\bar\beta\cdot\aleph_0=\mu_{T|\mathfrak{R}}(m)+\bar\beta$$

since β is transfinite. Because: $\overline{\beta+\gamma}=\bar\beta+\bar\gamma$, we infer:

(3.12) $$m_{\beta+\gamma}\leqq\bigwedge\{m\colon \mu_{T|\mathfrak{R}}(m)\leqq\bar\gamma\}=M'(\gamma).$$

From (3.9) and (3.12) it follows that $M'(\gamma)=m_{\beta+\gamma}$. An application of Lemma 3.3 to $T|\mathfrak{R}$ shows the existence of a reducing subspace $\mathfrak{H}_\beta\subset\mathfrak{R}$ such that:

(3.13) $$T|\mathfrak{H}_\beta \text{ is quasisimilar to } \bigoplus_{j<\omega} S(m_{\beta+j})$$

and

(3.14) $$H_\beta\supset P_{\mathfrak{R}}X(\bigoplus_{j<\omega}\mathfrak{H}(m_{\beta+j})).$$

Conditions (3.5—6) are obviously conserved. Theorem 3.4 follows now because from (3.5) we infer $\mathfrak{H}=\bigoplus_\alpha\mathfrak{H}_\alpha$.

The proof of the following theorem is a refinement of the proof of [4], Theorem 1.

Theorem 3.5. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. There exists a decomposition of $\mathfrak{H}$ into an almost-direct sum*

$$\mathfrak{H} = \bigvee_{\alpha} \mathfrak{H}_\alpha \tag{3.15}$$

of invariant subspaces of T such that:

(i) *$T|\mathfrak{H}_\alpha$ is quasisimilar to $S(m_\alpha)$ for each ordinal α;*

(ii) *$\mathfrak{H}_{\alpha+n} \perp \mathfrak{H}_{\beta+m}$ if α, β are different limit ordinals and $m, n < \omega$.*

Proof. Theorem 3.4 allows us to consider only the case where $\mathfrak{H}$ is separable. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence of vectors dense in $\mathfrak{H}$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be a sequence in which each ψ_k appears infinitely many times. We shall construct inductively subspaces $\mathfrak{H}_0, \mathfrak{H}_1, \ldots, \mathfrak{H}_n$, $\mathfrak{M}_n$ invariant for T and $\mathfrak{H}_0^*, \mathfrak{H}_1^*, \ldots, \mathfrak{H}_n^*$, $\mathfrak{M}_n^*$ invariant for T^* such that

$$\mathfrak{H}_n = \mathfrak{H}_{f_n}, \quad f_n \in \mathfrak{M}_{n-1} \quad \text{and} \quad m_{f_n} = m_{T|\mathfrak{M}_{n-1}}; \quad \mathfrak{H}_n^* \subset \mathfrak{M}_{n-1}^*; \tag{3.16}$$

$$(\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n)^\perp = \mathfrak{M}_n^*, \quad (\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*)^\perp = \mathfrak{M}_n; \tag{3.17}$$

$$P_{\mathfrak{M}_n^*} | \mathfrak{M}_n \quad \text{is a quasi-affinity;} \tag{3.18}$$

$$\begin{cases} \|P_{\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n} \varphi_k - \varphi_k\| < 2^{-n}, & k = n/2 \text{ if } n \text{ is even,} \\ \|P_{\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*} \varphi_k - \varphi_k\| < 2^{-n}, & k = (n-1)/2 \text{ if } n \text{ is odd.} \end{cases} \tag{3.19}$$

To begin we put $\mathfrak{M}_{-1} = \mathfrak{M}_{-1}^* = \mathfrak{H}$; the conditions (3.16—19) are obviously satisfied for $n = -1$. Let us assume that the spaces $\mathfrak{H}_j$, $\mathfrak{H}_j^*$, $\mathfrak{M}_j$, $\mathfrak{M}_j^*$ have been constructed for $0 \leqq j \leqq n-1$. From (3.17) and (3.18) we infer

$$\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1} \vee \mathfrak{M}_{n-1} = (\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1}) \oplus (P_{\mathfrak{M}_{n-1}^*} \mathfrak{M}_{n-1})^- = \mathfrak{H}$$

and analogously $\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_{n-1}^* \vee \mathfrak{M}_{n-1}^* = \mathfrak{H}$. Therefore there exist $u \in \mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1}$, $v \in \mathfrak{M}_{n-1}$ and $u^* \in \mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_{n-1}^*$, $v^* \in \mathfrak{M}_{n-1}^*$ such that

$$\begin{cases} \|\varphi_k - u - v\| < 2^{-n-1}, & k = n/2 \text{ if } n \text{ is even,} \\ \|\varphi_k - u^* - v^*\| < 2^{-n-1}, & k = (n-1)/2 \text{ if } n \text{ is odd.} \end{cases} \tag{3.20}$$

By Proposition 1.10 we can choose $f_n \in \mathfrak{M}_{n-1}$ with $m_{f_n} = m_{T|\mathfrak{M}_{n-1}}$ and such that

$$\begin{cases} \|f_n - v\| < 2^{-n-1} & \text{if } n \text{ is even,} \\ \|P_{\mathfrak{M}_{n-1}^*} f_n - v^*\| < 2^{-n-2} & \text{if } n \text{ is odd.} \end{cases} \tag{3.21}$$

Proposition 3.2 allows us to construct the subspaces $\mathfrak{H}_n = \mathfrak{H}_{f_n}$, $\mathfrak{H}_n^*$, $\mathfrak{M}_n$ and $\mathfrak{M}_n^*$ such that

$$\|P_{\mathfrak{H}_n^*} P_{\mathfrak{M}_{n-1}^*} f_n - P_{\mathfrak{M}_{n-1}^*} f_n\| < 2^{-n-2}; \tag{3.22}$$

$$\mathfrak{M}_n^* = \mathfrak{M}_{n-1}^* \ominus (P_{\mathfrak{M}_{n-1}^*} \mathfrak{H}_n)^-, \quad \mathfrak{M}_n = \mathfrak{M}_{n-1} \ominus (P_{\mathfrak{M}_{n-1}} \mathfrak{H}_n^*)^-; \tag{3.23}$$

$$P_{\mathfrak{M}_n^*} | \mathfrak{M}_n \quad \text{is quasi-affinity.} \tag{3.24}$$

Let us show that the conditions (3.16—19) are verified. (3.16) is obvious and (3.18) coincides with (3.24). For (3.17) we have $(\mathfrak{H}_0\vee\mathfrak{H}_1\vee\ldots\vee\mathfrak{H}_n)^\perp=$ $=(\mathfrak{H}_0\vee\mathfrak{H}_1\vee\ldots\vee\mathfrak{H}_{n-1})^\perp\cap\mathfrak{H}_n^\perp=\mathfrak{M}_{n-1}^*\cap\mathfrak{H}_n^\perp=\mathfrak{M}_{n-1}^*\ominus(P_{\mathfrak{M}_{n-1}^*}\mathfrak{H}_n)^-=\mathfrak{M}_n^*$ by (3.23) and analogously $(\mathfrak{H}_0^*\vee\mathfrak{H}_1^*\vee\ldots\vee\mathfrak{H}_n^*)^\perp=\mathfrak{M}_n$. If n is even we have

$$\|P_{\mathfrak{H}_0\vee\mathfrak{H}_1\vee\ldots\vee\mathfrak{H}_n}\varphi_k-\varphi_k\|\leqq\|u+f_n-\varphi_k\|\leqq\|u+v-\varphi_k\|+\|v-f_n\|<2^{-n},$$

by (3.20) and (3.21). If n is odd we have

$$\|P_{\mathfrak{H}_0^*\vee\mathfrak{H}_1^*\vee\ldots\vee\mathfrak{H}_n^*}\varphi_k-\varphi_k\|\leqq\|u^*+P_{\mathfrak{H}_n^*}P_{\mathfrak{M}_{n-1}^*}f_n-\varphi_k\|<$$
$$<\|u^*+v^*-\varphi_k\|+\|v^*-P_{\mathfrak{M}_{n-1}^*}f_n\|+\|P_{\mathfrak{M}_{n-1}^*}f_n-P_{\mathfrak{H}_n^*}P_{\mathfrak{M}_{n-1}^*}f_n\|<2^{-n} \text{ by}$$

(3.20—22); thus (3.19) is also verified.

From (3.19) we infer

$$\mathfrak{H}=\bigvee_{j<\omega}\mathfrak{H}_j=\bigvee_{j<\omega}\mathfrak{H}_j^*. \tag{3.25}$$

If $i\neq j$ (say $i<j$ by example) we have $\mathfrak{H}_i\perp\mathfrak{M}_i^*$ and $\mathfrak{H}_j^*\subset\mathfrak{M}_i^*$ by (3.16), so that $\mathfrak{H}_i\perp\mathfrak{H}_j^*$. Therefore $\mathfrak{H}_j^*\subset(\bigvee_{i\neq j}\mathfrak{H}_i)^\perp$ and (3.25) shows, by Lemma 1.11, that the decomposition $\mathfrak{H}=\bigvee_{j<\omega}\mathfrak{H}_j$ is almost direct. To finish the proof let us remark that $\mathfrak{M}_{n+1}=(\mathfrak{H}_0^*\vee\mathfrak{H}_1^*\vee\ldots\vee\mathfrak{H}_{n+1}^*)^\perp\subset\mathfrak{M}_n$ by (3.17), so that $m_{f_{n+1}}$ divides m_{f_n}. As in [4], Theorem 1, it follows that the Jordan model of T is $\bigoplus_{j<\omega}S(m_j)$, where $m_j=m_{f_j}$. Theorem 3.5 is proved.

In the case of weak contractions the result of Theorem 3.5 can be improved.

Proposition 3.6. *Let T be a weak contraction of class C_0 acting on the (necessarily separable) Hilbert space $\mathfrak{H}$ and let $\bigoplus_{j<\omega}S(m_j)$ be the Jordan model of T. There exists a decomposition*

$$\mathfrak{H}=\bigvee_{j<\omega}\mathfrak{H}_j \tag{3.26}$$

of $\mathfrak{H}$ into a quasi-direct sum of invariant subspaces of T such that $T|\mathfrak{H}_j$ is quasi-similar to $S(m_j)$.

Proof. Let X be a quasi-affinity such that $TX=X(\bigoplus_{j<\omega}S(m_j))$ and define $\mathfrak{H}_j=(X\mathfrak{H}(m_j))^-$. Let $\{K_a\}_{a\in A}$ be a family of subsets of the natural numbers and denote $K=\bigcap_{a\in A}K_a$. Because the mapping $\mathfrak{M}\mapsto(X\mathfrak{M})^-$ is an isomorphism of the lattice of invariant subspaces of $\bigoplus_{j<\omega}S(m_j)$ onto the lattice of invariant subspaces of T (cf. [3], Corollary 2.4) we have

$$\bigcap_{a\in A}(\bigvee_{j\in K_a}\mathfrak{H}_j)=(X(\bigcap_{a\in A}(\bigoplus_{j\in K_a}\mathfrak{H}(m_j))))^-=(X(\bigoplus_{j\in K}\mathfrak{H}(m_j)))^-=\bigvee_{j\in K}\mathfrak{H}_j.$$

Proposition 3.6 follows.

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