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Finite homogeneous algebras. I

BÉLA CSÁKÁNY and TAT'JANA GAVALCOVÁ

1. Preliminaries. Following MARCZEWSKI [7], an operation $f\colon A^k \to A$ is called *homogeneous* if $h(f(x_1, ..., x_k)) = f(h(x_1), ..., h(x_k))$ for every permutation h and any elements $x_1, ..., x_k$ of A. An algebra $\langle A; F\rangle$ is said to be *homogeneous* if each operation $f \in F$ is homogeneous.

In this paper, we shall describe all finite homogeneous algebras up to equivalence. This is the same as determining all clones of homogeneous operations on finite sets. In the present Part I we shall

(1) list all minimal clones consisting of homogeneous operations (it turns out that this list contains at most three items on any finite set, and the dual discriminator function d, introduced by E. Fried and A. F. Pixley, always generates such a minimal clone);

(2) determine all clones of homogeneous operations containing the minimal clone generated by the dual discriminator.

Let us start with notions and notations. The symbol $\mathbf{n}$ means the set $\{0, 1, ..., n-1\}$. For the sake of simplicity, we shall consider algebras of the form $\langle \mathbf{n}; F\rangle$ only. The following description of homogeneous operations was given by MARCZEWSKI [7]: for a homogeneous k-ary operation f on $\mathbf{n}$, $f(a_1, ..., a_k) = a_i$ where $1 \leqq i \leqq k$, or, possibly, $f(a_1, ..., a_k) = a_{k+1}$ if a_{k+1} is the unique element of $\mathbf{n}$ distinct from $a_1, ..., a_k$, in such a way that the index of the value of $f(a_1, ..., a_k)$ depends upon the pattern of equalities in the sequence $\langle a_1, ..., a_k\rangle$ only. A homogeneous operation f is called a *pattern function* provided $f(a_1, ..., a_k)$ always belongs to $\{a_1, ..., a_k\}$.

Several kinds of homogeneous operations will play an important role in the sequel: Pixley's ternary discriminator p, the dual discriminator d, the switching function s, the k-ary near-projection l_k where $k \geqq 3$ (they are defined on any set); further, the $(n-1)$-ary operation r_n, defined on $\mathbf{n}$ for $n \geqq 2$, and Świerczkowski's ternary function f_0, defined on $\mathbf{4}$. Let us recall their definitions:

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$p(a, b, c) = c$ if $a = b$, and $p(a, b, c) = a$ otherwise;
$d(a, b, c) = a$ if $a = b$, and $d(a, b, c) = c$ otherwise;
$s(a, b, c) = c$ if $a = b$, $s(a, b, c) = b$ if $a = c$ and $s(a, b, c) = a$ otherwise;
$l_k(a_1, \ldots, a_k) = a_1$ if $a_1, \ldots, a_k$ are pairwise distinct and $l_k(a_1, \ldots, a_k) = a_k$ otherwise;
$r_n(a_1, \ldots, a_{n-1}) = a_n$ if $\{a_1, \ldots, a_{n-1}, a_n\} = \mathbf{n}$ and $r_n(a_1, \ldots, a_{n-1}) = a_1$ otherwise;

finally,

$$f_0(1, 2, 3) = f_0(0, 1, 1) = f_0(1, 0, 1) = f_0(1, 1, 0) = f_0(0, 0, 0) = 0$$

(see [8], [7], [9], [3], [2]).

A set of operations on a set $\mathbf{n}$ is called a *clone* if it contains all trivial operations (i.e., all projections) and it is closed under superposition. For any set F of operations on $\mathbf{n}$, we say that F *produces* the operation g and we use the symbol $F \to g$ if g can be obtained from operations in F and the projections by superposition (in this case, one can also say that g is a term function of the algebra $\langle \mathbf{n}; F \rangle$). In the case $F = \{f\}$ we write $f \to g$. Obviously, the relation $\to$ is transitive. For the negation of $F \to g$ we write $F \nrightarrow g$. An algebra $\langle \mathbf{n}; F \rangle$ is *functionally complete* if the set $F \cup \{0, 1, \ldots, n-1\}$ (i.e., F together with the constant nullary operations) produces each possible operation on $\mathbf{n}$. The *clone* $[F]$ *generated by* F is the set of all operations F produces. We write $[f_1, f_2, \ldots]$ instead of $[\{f_1, f_2, \ldots\}]$. The algebras $\langle \mathbf{n}; F \rangle$ and $\langle \mathbf{n}; G \rangle$ are said to be *equivalent* if $[F] = [G]$. A clone T is called *minimal* if the clone of all projections is the unique one which is contained in T properly; this means that T contains a non-projection, and any non-projection in T produces every other non-projection.

In the next lemma we collect the basic facts about how the above-mentioned homogeneous operations produce each other:

Lemma 1. *On a finite set* $\mathbf{n}$, *the following hold:*

(1) $p \to f$ *for any pattern function* f.
(2) $l_j \to l_k$ *for* $j \leqq k$.
(3) $r_n \to l_{n-1}$ *for* $n > 3$.
(4) $l_k \nrightarrow d$ *for* $n > 1$.
(5) $d \nrightarrow l_k$ *for* $n > 2$, $n \geqq k$.
(6) $l_j \nrightarrow l_k$ *for* $j > k$, $n \geqq k$.

Proof. (1) is a result in [4].
(2). It is sufficient to establish $l_j \to l_{j+1}$, and this is given by the identity $l_{j+1}(x_1, \ldots, x_j, x_{j+1}) = l_j(l_j(x_1, x_3, \ldots, x_j, x_{j+1}), l_j(x_2, x_3, \ldots, x_j, x_{j+1}), x_4, \ldots, x_{j+1})$.
(3). $l_{n-1}(x_1, \ldots, x_{n-1}) = r_n(x_{n-1}, \ldots, x_3, x_2, r_n(x_{n-1}, \ldots, x_2, x_1))$.

To prove (4)—(6), we use the following fact. Let f, g be operations on $\mathbf{n}$ and $f \to g$; then, for any natural number t, the subalgebras of $\langle \mathbf{n}; f \rangle^t$ are closed under the (componentwise performed) operation g.

(4). Observe that $\sigma = \{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$ is a subalgebra of $\langle \mathbf{n}; l_k \rangle^3$ but $d(\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle) = \langle 0, 0, 0 \rangle \notin \sigma$. Hence $l_k \to d$ is impossible.

Concerning (5) and (6), we present the crucial subalgebras only:

(5) $$\{\langle k-1, 0 \rangle, \ldots, \langle 2, 0 \rangle, \langle 1, 0 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle\} \subset \langle \mathbf{n};\ d \rangle^2,$$

(6) $$\{\langle j-2, 0 \rangle, \ldots, \langle 2, 0 \rangle, \langle 1, 0 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle\} \subset \langle \mathbf{n};\ l_j \rangle^2.$$

2. Minimal clones of homogeneous operations. In this section, our main tool is the following fact:

Lemma 2. *For $n \geqq 3$, every non-trivial pattern function on $\mathbf{n}$ produces d or some l_k with $k \leqq n$.*

Proof. It was proved in [2] (see the proof of Lemma 5 there) that any non-trivial pattern function on $\mathbf{n}$ produces d or an l_k which is non-trivial; but l_k is trivial if $k > n$.

The clones in the title of this paragraph are given by

Theorem 1. *The minimal clones consisting of homogeneous operations on a finite set $\mathbf{n}$ $(n > 1)$ are the following:*

$[l_n]$ and $[d]$, if $n \geqq 5$;
$[l_4]$, $[d]$ and $[f_0]$, if $n = 4$;
$[l_3]$, $[d]$ and $[r_3]$, if $n = 3$;
$[s]$, $[d]$ and $[r_2]$, if $n = 2$.

Proof. First we prove that, for $n \geqq 3$, $[l_n]$ is minimal on $\mathbf{n}$. Take a non-trivial f with $l_n \to f$; it is sufficient to show $f \to l_n$. As pattern functions produce pattern functions only, by Lemma 2 we have $f \to d$ or $f \to l_k$ for a suitable $k \leqq n$. From $f \to d$ it follows $l_n \to d$, contradicting Lemma 1(4); therefore $f \to l_k$ holds. Now $k < n$ is impossible by Lemma 1(6), i.e., $f \to l_n$, which was needed.

For $n \geqq 3$, the minimality of $[d]$ can be proved by an analogous argument; here we have to apply Lemma 1(5) instead of (4).

For $n \geqq 5$, there is no other minimal clone of operations on $\mathbf{n}$. In order to show this, we shall verify that each non-trivial homogeneous operation g on $\mathbf{n}$ produces l_n or d. There are two possibilities:

a) $g \to r_n$. Then, by Lemma 1(3) and (2), we have $g \to l_n$.

b) $g \not\to r_n$. If, in addition, g is a pattern function, then Lemma 2 applies in the above manner. If g is not a pattern function, then we can identify variables of g (if necessary) so that we obtain an $(n-1)$-ary g' satisfying $g'(a_1, \ldots, a_{n-1}) = a_n$, whenever $\{a_1, \ldots, a_{n-1}, a_n\} = \mathbf{n}$, i.e., a_n is the unique element of $\mathbf{n}$ distinct from $a_1, \ldots, a_{n-1}$. Now, if there exist two variables of g' whose identification furnishes a non-trivial pattern function, then, applying Lemma 2 for g' again, our claim follows. Suppose that g' turns into a projection by identifying any two of its variables. By a result of Świerczkowski, g' always turns into the same projection ([8]; see also [5], pp. 206—207; note that g' is at least quaternary). Hence g' equals r_n up to permutation of variables, implying $g \to r_n$, contrary to the hypothesis.

Next we prove that $[f_0]$ is minimal on $\mathbf{4}$. Let $f_0 \to f$ and suppose $f \not\to f_0$. Then $\langle \mathbf{4}; f_0\rangle$ and $\langle \mathbf{4}; f\rangle$ are not equivalent. A homogeneous non-trivial algebra $\langle \mathbf{4}; F\rangle$ is not functionally complete iff it is equivalent to $\langle \mathbf{4}; f_0\rangle$ (see [2]); therefore, $\langle \mathbf{4}; f\rangle$ is functionally complete. Now, $\langle \mathbf{4}; f_0\rangle$ is functionally complete a fortiori, a contradiction.

Similarly, a non-trivial homogeneous functionally incomplete algebra $\langle \mathbf{3}; F\rangle$ is equivalent to $\langle \mathbf{3}; r_3\rangle$ (see [2]), hence the minimality of $[r_3]$ on $\mathbf{3}$ follows.

Furthermore, every non-trivial homogeneous operation g on $\mathbf{4}$ produces one of l_4, d and f_0, showing that there are no other minimal clones of homogeneous operations on $\mathbf{4}$. Indeed, if g is a pattern function, Lemma 2 applies. If g fails to be a pattern function, then an appropriate identification of variables of g leads to a ternary g' satisfying $g'(a_1, a_2, a_3) = a_4$, whenever $\{a_1, \ldots, a_4\} = \mathbf{4}$. As we have $g'(a_1, a_2, a_3) = a_i$ $(1 \leqq i \leqq 3)$ if card $\{a_1, a_2, a_3\} < 3$, and the pattern of equalities in $\langle a_1, a_2, a_3\rangle$ determines the value of i, the operation g' is defined uniquely by the sequence $\langle g'(0,1,1), g'(1,0,1), g'(1,1,0)\rangle$ (of course, $g'(0,0,0)=0$ always). Let us denote g' by f_k $(k=0, 1, \ldots, 7)$ if this sequence is the dyadic form of k (i.e., $4g'(0,1,1)+2g'(1,0,1)+g'(1,1,0)=k$). This notation is consistent with the original definition of f_0. We have to verify that every f_k produces one of l_4, d and f_0.

One can check the following identities:

(a) $f_3(x, y, z) = r_4(x, y, z)$;

(b) $f_5(y, x, z) = f_6(z, y, x) = f_3(x, y, z)$;

(c) $f_1(y, z, f_1(z, y, x)) = f_4(y, f_4(z, x, y), z) = p(x, y, z)$;

(d) $f_2(y, f_2(y, z, x), x) = f_7(y, f_7(y, z, x), x) = d(x, y, z)$.

From (a) and Lemma 1(3) and (2), it follows $f_3 \to l_4$. From (b), $f_5 \to l_4$ and $f_6 \to l_4$. Further, (c) together with Lemma 1(1) implies $f_1 \to d$ and $f_4 \to d$; finally, (d) shows $f_2 \to d$ and $f_7 \to d$. The case $n=4$ is settled.

In the case $n=3$ we can proceed similarly. Any non-trivial homogeneous function g on $\mathbf{3}$ is either a pattern function — then we use Lemma 2 — or not. In the latter case g produces a binary g' in the usual way such that $g'(a_1, a_2) = a_3$ whenever $\{a_1, a_2, a_3\} = \mathbf{3}$, and $g'(a, a) = a$. Clearly, $g' = r_3$, hence $g \to r_3$, as required.

All minimal clones we have found are distinct. This is implied by Lemma 1(4) and the fact that pattern functions produce merely pattern functions.

The case $n=2$ of Theorem 1 can be realized by casting a glance at the diagram of the lattice of all clones on **2**, due to Post (see, e.g., [6]; note that $r_2(x)=$ $=x+1 \bmod 2$ and $d(x, y, z)=xy+xz+yz \bmod 2$ on **2**).

3. Homogeneous dual discriminator algebras. After Werner [9], an algebra $\langle \mathbf{n}; F\rangle$ is said to be a discriminator algebra (or quasi-primal algebra) if $p\in[F]$. Analogously, an algebra $\langle \mathbf{n}; F\rangle$ will be called a *dual discriminator algebra* if $d\in[F]$. In this paragraph we determine all homogeneous dual discriminator algebras up to equivalence, i.e., for any $\mathbf{n}$, we determine all clones of homogeneous operations on $\mathbf{n}$ containing d. From now on, n is fixed and $n\geqq 3$.

Call a ternary operation m on $\mathbf{n}$ a *majority operation* if, for any $x, y\in \mathbf{n}$, $m(x, x, y)=m(x, y, x)=m(y, x, x)=x$ holds. The dual discriminator is a majority operation. The following theorem of Baker and Pixley [1; Corollary 5.1] is basic for our considerations (see also [9]):

Let $\langle \mathbf{n}; F\rangle$ be a finite algebra such that F produces a majority operation and let g be an arbitrary operation on $\mathbf{n}$. If every subalgebra of $\langle \mathbf{n}; F\rangle^2$ is closed under the (componentwise performed) operation g, then F produces g.

For a clone T on $\mathbf{n}$, let $\mathrm{S}T$ stand for the set consisting of base sets of all subalgebras of $\langle \mathbf{n}; T\rangle^2$. Let $\mathscr{F}$ be the set of all clones on the set $\mathbf{n}$ containing d. We call a set P of subsets of $\mathbf{n}^2$ *complete* if there exists a clone $T\in\mathscr{F}$ such that $P=\mathrm{S}T$ (i.e., if there exists a dual discriminator algebra on $\mathbf{n}$ such that P is the set of all subalgebras of the direct square of this algebra). Denote by $\mathscr{S}$ the set of all complete sets.

Lemma 3. S *is an inclusion-reversing one-to-one mapping of* $\mathscr{F}$ *onto* $\mathscr{S}$.

Proof. The unique non-trivial part of this assertion is that S is one-to-one. Suppose $T_1, T_2\in\mathscr{F}$ and $\mathrm{S}T_1=\mathrm{S}T_2$. If $f\in T_2$ then every set in $\mathrm{S}T_1(=\mathrm{S}T_2)$ is closed under f, hence, by the Baker—Pixley theorem, $T_1\to f$ follows. This means $f\in T_1$ as T_1 is a clone. Therefore, $T_2\subseteqq T_1$ (and by symmetry, $T_1\subseteqq T_2$). We get $T_1=T_2$, which was needed.

By virtue of Lemma 3, we can investigate complete sets instead of clones. First we establish some properties of complete sets. Subsets of $\mathbf{n}^2$ may be considered as binary relations on $\mathbf{n}$. The following lemma is familiar:

Lemma 4. *Any complete set contains the complete relation; furthermore, it is closed under relation product, intersection and forming the inverse relation.*

For convenience, several kinds of subsets of $\mathbf{n}^2$ will bear special names. A set of form $K\times L$ with $K, L\subseteqq \mathbf{n}$, card $K=k$, card $L=l$ is a *block of size* (k, l). A set of form $\{\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle\}$, where $i_1, \ldots, i_k$ are pairwise distinct as well as $j_1, \ldots, j_k$, is a *string of size* k. A set of form $\{\langle i_k, j_1\rangle, \ldots, \langle i_2, j_1\rangle, \langle i_1, j_1\rangle, \langle i_1, j_2\rangle, \ldots, \langle i_1, j_l\rangle\}$

$(k, l \geqq 2)$ is called a *cross of size* (k, l). Essentially, a string of size k is a partial permutation with a k-element domain and a cross of size (k, l) is the union of two blocks of size $(k, 1)$ and $(1, l)$ with a non-empty intersection. *Block of size m* means a block of size (m, l) or (k, m); similarly for crosses.

Lemma 5. *Any complete set consists of blocks, strings and crosses; in particular,* S$[d]$ *consists of all blocks, strings and crosses.*

Proof. A complete set consists of subsets of $\mathbf{n}^2$ preserved by d, and, by result of Fried and Pixley [3; Theorem 2.4], d preserves a subset σ of $\mathbf{n}^2$ iff σ is *p-rectangular*, i.e.,

$$\langle i, j_1\rangle, \langle i, j_2\rangle, \langle k, l\rangle \in \sigma \quad \text{implies} \quad \langle i, l\rangle \in \sigma \quad \text{for} \quad j_1 \neq j_2$$

and

$$\langle i_1, j\rangle, \langle i_2, j\rangle, \langle k, l\rangle \in \sigma \quad \text{implies} \quad \langle k, j\rangle \in \sigma \quad \text{for} \quad i_1 \neq i_2.$$

Clearly, blocks, strings and crosses are p-rectangular and the converse can also be checked without trouble.

From now on, we shall use the following notations: B is the set of all blocks and B' is the set of all blocks of size (k, l) with $k, l \neq n-1$. The set of strings and crosses S, S' and C, C', resp., are defined analogously. Finally, let C_m be the set of all crosses of size (k, l) with $k, l \leqq m$. Now Lemma 5 can be reformulated as follows:

For any complete set P, the inclusion $P \subseteq B \cup S \cup C$ holds; in particular, S$[d] = B \cup S \cup C$.

Next we clear up the structure of several further complete sets:

Lemma 6. (1) S$[d, l_{m+1}] = B \cup S \cup C_m$ for $m = 2, \ldots, n-1$.

(2) S$[p] = B \cup S$.

(3) S$[d, l_{m+1}, r_n] = B' \cup S' \cup C_m$ for $m = 2, \ldots, n-2$.

(4) S$[p, r_n] = B' \cup S'$.

Proof. (1) The following inclusions are obvious: $B \cup S \cup C_m \subseteq$ S$[d, l_{m+1}] \subseteq$ S$[d] = B \cup S \cup C$. Take a set from $C \setminus C_m$, i.e., a cross of form $\{\langle i_k, j_1\rangle, \ldots, \langle i_1, j_1\rangle, \ldots, \langle i_1, j_l\rangle\}$ with $k > m$ (the case $l > m$ can be settled similarly). Then $l_{m+1}(\langle i_{m+1}, j_1\rangle, \ldots$ $\ldots, \langle i_2, j_1\rangle, \langle i_1, j_2\rangle) = \langle i_{m+1}, j_2\rangle$ showing that our cross is not closed under l_{m+1}. Thus, the set of all subalgebras of $\langle \mathbf{n}; d, l_{m+1}\rangle^2$ is $B \cup S \cup C_m$, as asserted.

(2)—(4) can be verified in an analogous manner observing that no cross is closed under p, because we have $p(\langle i_2, j_1\rangle, \langle i_1, j_1\rangle, \langle i_1, j_2\rangle) = \langle i_2, j_2\rangle$; furthermore, no block, string and cross, each of size $n-1$, is closed under r_n. Indeed, take, e.g., a block $\{i_1, \ldots, i_{n-1}\} \times L$ of size $n-1$ and a $j \in L$; then $\langle i_1, j\rangle, \ldots, \langle i_{n-1}, j\rangle$ belong to this block but $r_n(\langle i_1, j\rangle, \ldots, \langle i_{n-1}, j\rangle)$ does not.

Lemma 7. *For the clone H of all homogeneous operations on* $\mathbf{n}$, $SH=B'\cup S'$.

Proof. By (4) of the previous lemma, $SH\subseteqq B'\cup S'$. On the other hand, SH contains all permutations of $\mathbf{n}$, i.e. all strings of size n, since for any operation f homogeneity means that each permutation is a subalgebra of $\langle \mathbf{n};f\rangle^2$. Now we can apply Lemma 4 in order to obtain all sets in $B'\cup S'$. Namely, every string of size less than $n-1$ is the intersection of two permutations, every block of size (k,n) is the (relation) product of a string of size k and the complete relation, every block of size (n,l) is the inverse of a block of size (l,n), and every block of size (k,l) is the intersection of blocks of size (k,n) and (n,l).

In view of Lemmas 5 and 7, our task is reduced to determining all complete sets between $B'\cup S'$ and $B\cup S\cup C$.

Lemma 8. *All complete sets containing $B'\cup S'$ and contained in $B\cup S\cup C$ are those listed in Lemma* 6.

Proof. It is sufficient to prove the following two propositions:

(a) If a complete set contains $B'\cup S'$ and a block, or a string, or a cross, any of them of size $n-1$, then it contains $B\cup S$.

(b) If a complete set contains $B'\cup S'$ and a cross of size m, then it contains C_m; moreover, if $m\geqq n-1$, it contains even $B\cup S$.

Indeed, suppose (a) and (b) are fulfilled, and let P be a complete set with $B'\cup S'\subseteqq P\subseteqq B\cup S\cup C$. If P contains no crosses, then (a) implies $P=B'\cup S'$ or $P=B\cup S$. Otherwise, let m be the maximum of the sizes of crosses in P. If there is a block or a string of size $n-1$ in P, then in virtue of (a), (b) and the maximality of m we have $P=B\cup S\cup C_m$. In the opposite case, $P=B'\cup S'\cup C_m$ by the same reason.

It remains to prove (a) and (b). As for (a), one can check easily that all blocks and strings of size $n-1$ can be obtained from sets in $B'\cup S'$ and an arbitrary *fixed* block or string or cross, any of them of size $n-1$, by product, intersection and formation of inverse relation. Applying Lemma 4, the assertion (a) follows.

(b) First let R be a complete set containing $B'\cup S'$ and an arbitrary cross ζ of size (m,l) where $2\leqq l<m\leqq n-1$. Then any cross of the same size (m,l) can be obtained in the form $\pi_1\zeta\pi_2$ with appropriate strings π_1, π_2 of size n; crosses of size (l,m) arise as inverses of the previous ones; crosses of size (m,m) can be represented as $\zeta_1\pi\zeta_2$ where ζ_1 and ζ_2 are crosses of size (m,l) and (l,m), respectively, and π is a string of size n; finally, an arbitrary cross of size (k_1,k_2) with $k_1,k_2\leqq m$ is the intersection of a cross of size (m,m) and an appropriate block of size (k_1,k_2). Thus, $C_m\subseteqq R$, as required. In the case $m=n-1$, the second part of (b) is a consequence of (a).

Secondly, let R be complete with $R\supseteqq B'\cup S'$ and let R contain a cross of size n. The preceding considerations show that we have two possibilities only, namely, $R=B'\cup S'\cup C'$ or $R=B\cup S\cup C$. The proof will be complete if we deduce that $B'\cup S'\cup C'$ is not a complete set. Assume $SF=B'\cup S'\cup C'$ for some homogeneous dual discriminator algebra $\langle \mathbf{n}; F\rangle$. As SF is closed under r_n, we have $F\to r_n$ by the Baker—Pixley theorem, hence, according to Lemma 1(3) and (2), $F\to l_n$ follows. However, as we have seen in the proof of Lemma 6(1), our cross of size n is not closed under l_n, a contradiction.

Now we are ready to formulate the main result of this paragraph.

Theorem 2. *The finite homogeneous dual discriminator algebras with more than one element are the following (up to equivalence):*

$\langle \mathbf{2}; d\rangle$, $\langle \mathbf{2}; p\rangle$, $\langle \mathbf{2}; p, r_2\rangle$;

$\langle \mathbf{3}; d\rangle$, $\langle \mathbf{3}; p\rangle$, $\langle \mathbf{3}; p, r_3\rangle$, $\langle \mathbf{3}; d, l_3\rangle$;

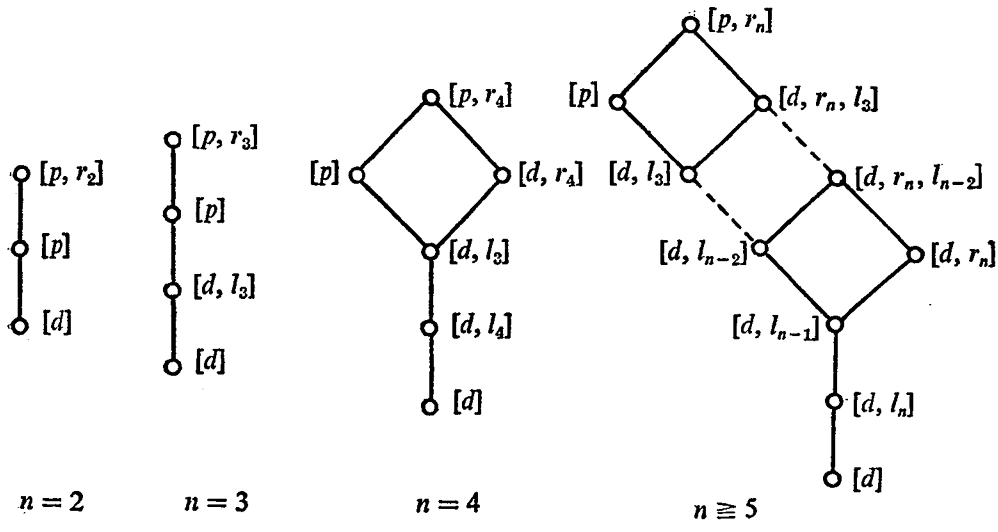
$\langle \mathbf{4}; d\rangle$, $\langle \mathbf{4}; p\rangle$, $\langle \mathbf{4}; p, r_4\rangle$, $\langle \mathbf{4}; d, l_3\rangle$, $\langle \mathbf{4}; d, l_4\rangle$, $\langle \mathbf{4}; d, r_4\rangle$

and for $n\geqq 5$

$\langle \mathbf{n}; d\rangle$, $\langle \mathbf{n}; p\rangle$, $\langle \mathbf{n}; p, r_n\rangle$, $\langle \mathbf{n}; d, l_k\rangle$ $(k=3, \ldots, n)$,

$\langle \mathbf{n}; d, r_n\rangle$, $\langle \mathbf{n}; d, r_n, l_k\rangle$ $(k=3, \ldots, n-2)$.

The interval of clones between $[d]$ and $H=[p, r_n]$ on $\mathbf{n}$ is the lattice with the diagram presented below:



Proof. For $n>2$, this follows immediately from Lemmas 6, 7 and 8. The case $n=2$ can be found in Post's work ([6], pp. 72—76).

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BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

KATEDRA MATEMATIKY
STROJNÍCKA FAKULTA VŠT
ŠVERMOVA 9
041 87 KOŠICE, CZECHOSLOVAKIA